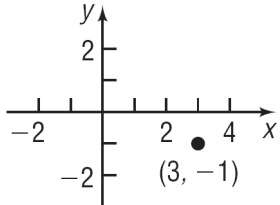# Chapter 9

Polar Coordinates; Vectors

Section 9.1

1.

The point lies in quadrant IV.

2. $\left(\dfrac{6}{2}\right)^2 = 9$

3. $\dfrac{b}{a}$

4. $-\dfrac{\pi}{4}$

5. pole, polar axis

6. True

7. False

8. $r\cos\theta \; ; r\sin\theta$

9. A

10. B

11. C

12. C

13. B

14. D

15. A

16. D

17.

18.

19.

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31.
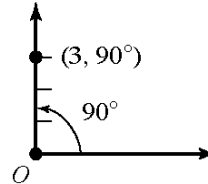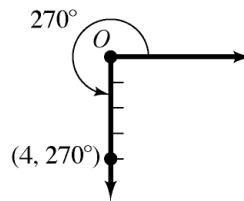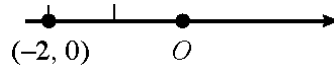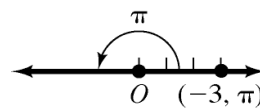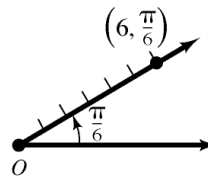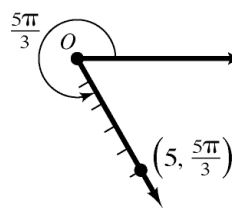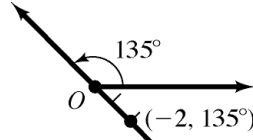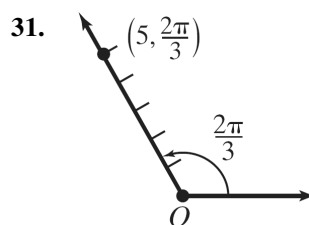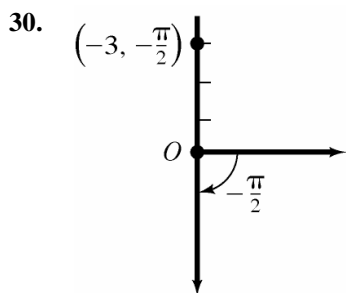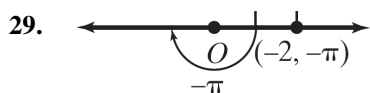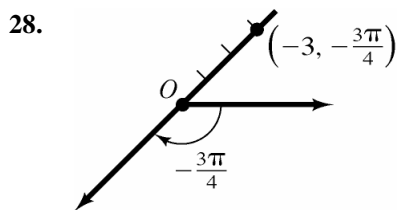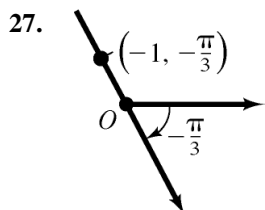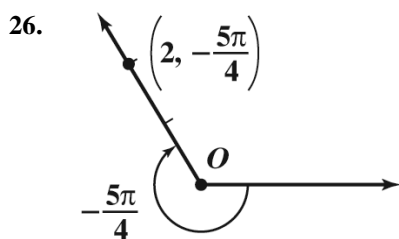
a. $r > 0, \ -2\pi \le \theta < 0$ $\left(5, -\dfrac{4\pi}{3}\right)$

b. $r < 0, \ 0 \le \theta < 2\pi$ $\left(-5, \dfrac{5\pi}{3}\right)$

c. $r > 0, \ 2\pi \le \theta < 4\pi$ $\left(5, \dfrac{8\pi}{3}\right)$

32.

a. $r > 0, \ -2\pi \le \theta < 0$ $\left(4, -\dfrac{5\pi}{4}\right)$

b. $r < 0, \ 0 \le \theta < 2\pi$ $\left(-4, \dfrac{7\pi}{4}\right)$

c. $r > 0, \ 2\pi \le \theta < 4\pi$ $\left(4, \dfrac{11\pi}{4}\right)$

33.

a. $r > 0, \ -2\pi \le \theta < 0$ $(2, -2\pi)$

b. $r < 0, \ 0 \le \theta < 2\pi$ $(-2, \pi)$

c. $r > 0, \ 2\pi \le \theta < 4\pi$ $(2, 2\pi)$

34.

a. $r > 0, \ -2\pi \le \theta < 0$ $(3, -\pi)$

b. $r < 0, \ 0 \le \theta < 2\pi$ $(-3, 0)$

c. $r > 0, \ 2\pi \le \theta < 4\pi$ $(3, 3\pi)$

35.
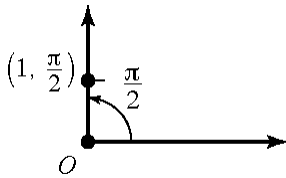
a. $r > 0,\ -2\pi \le \theta < 0$ $\quad \left(1, -\dfrac{3\pi}{2}\right)$

b. $r < 0,\ 0 \le \theta < 2\pi$ $\quad \left(-1, \dfrac{3\pi}{2}\right)$

c. $r > 0,\ 2\pi \le \theta < 4\pi$ $\quad \left(1, \dfrac{5\pi}{2}\right)$

36.
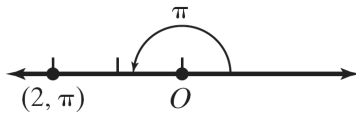
a. $r > 0,\ -2\pi \le \theta < 0$ $\quad (2, -\pi)$

b. $r < 0,\ 0 \le \theta < 2\pi$ $\quad (-2, 0)$

c. $r > 0,\ 2\pi \le \theta < 4\pi$ $\quad (2, 3\pi)$

37.
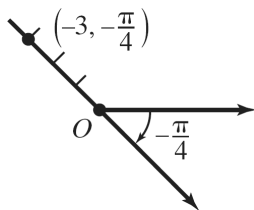
a. $r > 0,\ -2\pi \le \theta < 0$ $\quad \left(3, -\dfrac{5\pi}{4}\right)$

b. $r < 0,\ 0 \le \theta < 2\pi$ $\quad \left(-3, \dfrac{7\pi}{4}\right)$

c. $(r > 0,\ 2\pi \le \theta < 4\pi$ $\quad \left(3, \dfrac{11\pi}{4}\right)$

38.
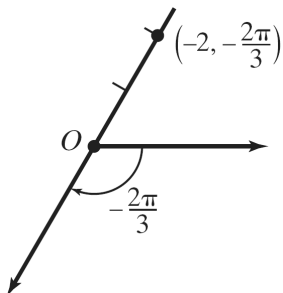
a. $r > 0,\ -2\pi \le \theta < 0$ $\quad \left(2, -\dfrac{5\pi}{3}\right)$

b. $r < 0,\ 0 \le \theta < 2\pi$ $\quad \left(-2, \dfrac{4\pi}{3}\right)$

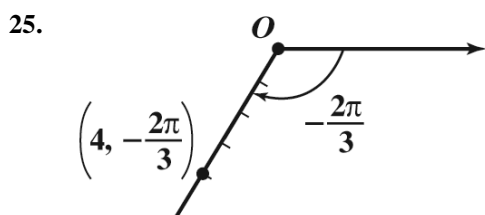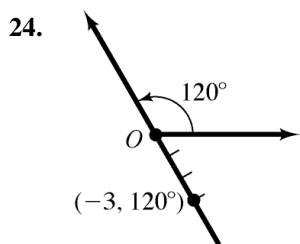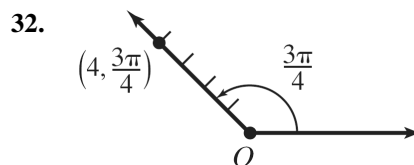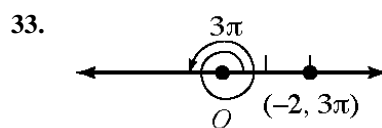**c.** $r > 0,\ 2\pi \le \theta < 4\pi$ $\quad \left(2, \dfrac{7\pi}{3}\right)$

39. $x = r\cos\theta = 3\cos\dfrac{\pi}{2} = 3 \cdot 0 = 0$

$y = r\sin\theta = 3\sin\dfrac{\pi}{2} = 3 \cdot 1 = 3$

Rectangular coordinates of the point $\left(3, \dfrac{\pi}{2}\right)$ are

$(0, 3)$.

40. $x = r\cos\theta = 4\cos\dfrac{3\pi}{2} = 4 \cdot 0 = 0$

$y = r\sin\theta = 4\sin\dfrac{3\pi}{2} = 4 \cdot (-1) = -4$

Rectangular coordinates of the point $\left(4, \dfrac{3\pi}{2}\right)$ are

$(0, -4)$.

41. $x = r\cos\theta = -2\cos 0 = -2 \cdot 1 = -2$

$y = r\sin\theta = -2\sin 0 = -2 \cdot 0 = 0$

Rectangular coordinates of the point $(-2, 0)$ are

$(-2, 0)$.

42. $x = r\cos\theta = -3\cos\pi = -3(-1) = 3$

$y = r\sin\theta = -3\sin\pi = -3 \cdot 0 = 0$

Rectangular coordinates of the point $(-3, \pi)$ are

$(3, 0)$.

43. $x = r\cos\theta = 6\cos 150° = 6\left(-\dfrac{\sqrt{3}}{2}\right) = -3\sqrt{3}$

$y = r\sin\theta = 6\sin 150° = 6 \cdot \dfrac{1}{2} = 3$

Rectangular coordinates of the point $(6, 150°)$

are $\left(-3\sqrt{3}, 3\right)$.

44. $x = r\cos\theta = 5\cos 300° = 5 \cdot \dfrac{1}{2} = \dfrac{5}{2}$

$y = r\sin\theta = 5\sin 300° = 5\left(-\dfrac{\sqrt{3}}{2}\right) = -\dfrac{5\sqrt{3}}{2}$

Rectangular coordinates of the point $(5, 300°)$

are $\left(\dfrac{5}{2}, -\dfrac{5\sqrt{3}}{2}\right)$.

45. $x = r\cos\theta = -2\cos\dfrac{3\pi}{4} = -2\left(-\dfrac{\sqrt{2}}{2}\right) = \sqrt{2}$

$y = r\sin\theta = -2\sin\dfrac{3\pi}{4} = -2 \cdot \dfrac{\sqrt{2}}{2} = -\sqrt{2}$

Rectangular coordinates of the point $\left(-2, \dfrac{3\pi}{4}\right)$

are $\left(\sqrt{2}, -\sqrt{2}\right)$.

46. $x = r\cos\theta = -2\cos\dfrac{2\pi}{3} = -2\left(-\dfrac{1}{2}\right) = 1$

$y = r\sin\theta = -2\sin\dfrac{2\pi}{3} = -2 \cdot \dfrac{\sqrt{3}}{2} = -\sqrt{3}$

Rectangular coordinates of the point $\left(-2, \dfrac{2\pi}{3}\right)$

are $\left(1, -\sqrt{3}\right)$.

47. $x = r\cos\theta = -1\cos\left(-\dfrac{\pi}{3}\right) = -1 \cdot \dfrac{1}{2} = -\dfrac{1}{2}$

$y = r\sin\theta = -1\sin\left(-\dfrac{\pi}{3}\right) = -1\left(-\dfrac{\sqrt{3}}{2}\right) = \dfrac{\sqrt{3}}{2}$

Rectangular coordinates of the point $\left(-1, -\dfrac{\pi}{3}\right)$

are $\left(-\dfrac{1}{2}, \dfrac{\sqrt{3}}{2}\right)$.

48. $x = r\cos\theta = -3\cos\left(-\dfrac{3\pi}{4}\right) = -3\left(-\dfrac{\sqrt{2}}{2}\right) = \dfrac{3\sqrt{2}}{2}$

$y = r\sin\theta = -3\sin\left(-\dfrac{3\pi}{4}\right) = -3\left(-\dfrac{\sqrt{2}}{2}\right) = \dfrac{3\sqrt{2}}{2}$

Rectangular coordinates of the point $\left(-3, -\dfrac{3\pi}{4}\right)$

are $\left(\dfrac{3\sqrt{2}}{2}, \dfrac{3\sqrt{2}}{2}\right)$.

49. $x = r\cos\theta = -2\cos(-180°) = -2(-1) = 2$

$y = r\sin\theta = -2\sin(-180°) = -2 \cdot 0 = 0$

Rectangular coordinates of the point
$(-2, -180°)$ are $(2, 0)$.

50. $x = r\cos\theta = -3\cos(-90°) = -3 \cdot 0 = 0$

$y = r\sin\theta = -3\sin(-90°) = -3(-1) = 3$

Rectangular coordinates of the point $(-3, -90°)$

are $(0, 3)$.

51. $x = r\cos\theta = 7.5\cos110° \approx 7.5(-0.3420) \approx -2.57$

$y = r\sin\theta = 7.5\sin110° \approx 7.5(0.9397) \approx 7.05$

Rectangular coordinates of the point $(7.5, 110°)$

are about $(-2.57, 7.05)$.

52. $x = r\cos\theta = -3.1\cos182° \approx -3.1(-0.9994) \approx 3.10$

$y = r\sin\theta = -3.1\sin182° \approx -3.1(-0.0349) \approx 0.11$

Rectangular coordinates of the point $(-3.1, 182°)$

are about $(3.10, 0.11)$.

53. $x = r\cos\theta = 6.3\cos(3.8) \approx 6.3(-0.7910) \approx -4.98$

$y = r\sin\theta = 6.3\sin(3.8) \approx 6.3(-0.6119) \approx -3.85$

Rectangular coordinates of the point $(6.3, 3.8)$ are

about $(-4.98, -3.85)$.

54. $x = r\cos\theta = 8.1\cos(5.2) \approx 8.1(0.4685) \approx 3.79$

$y = r\sin\theta = 8.1\sin(5.2) \approx 8.1(-0.8835) \approx -7.16$

Rectangular coordinates of the point $(8.1, 5.2)$

are about $(3.79, -7.16)$.

55. $r = \sqrt{x^2 + y^2} = \sqrt{3^2 + 0^2} = \sqrt{9} = 3$

$\theta = \tan^{-1}\left(\dfrac{y}{x}\right) = \tan^{-1}\left(\dfrac{0}{3}\right) = \tan^{-1}0 = 0$

Polar coordinates of the point $(3, 0)$ are $(3, 0)$.

56. $r = \sqrt{x^2 + y^2} = \sqrt{0^2 + 2^2} = \sqrt{4} = 2$

$\theta = \tan^{-1}\left(\dfrac{y}{x}\right) = \tan^{-1}\left(\dfrac{2}{0}\right)$

Since $\dfrac{2}{0}$ is undefined, $\theta = \dfrac{\pi}{2}$

Polar coordinates of the point $(0, 2)$ are $\left(2, \dfrac{\pi}{2}\right)$.

57. $r = \sqrt{x^2 + y^2} = \sqrt{(-1)^2 + 0^2} = \sqrt{1} = 1$

$\theta = \tan^{-1}\left(\dfrac{y}{x}\right) = \tan^{-1}\left(\dfrac{0}{-1}\right) = \tan^{-1} 0 = 0$

The point lies on the negative *x*-axis, so $\theta = \pi$.
Polar coordinates of the point $(-1,\ 0)$ are $(1, \pi)$.

58. $r = \sqrt{x^2 + y^2} = \sqrt{0^2 + (-2)^2} = \sqrt{4} = 2$

$\theta = \tan^{-1}\left(\dfrac{y}{x}\right) = \tan^{-1}\left(\dfrac{-2}{0}\right)$

Since $\dfrac{-2}{0}$ is undefined, $\theta = \dfrac{\pi}{2}$.

The point lies on the negative *y*-axis, so $\theta = -\dfrac{\pi}{2}$.

Polar coordinates of the point $(0, -2)$ are $\left(2, -\dfrac{\pi}{2}\right)$.

59. The point $(1, -1)$ lies in quadrant IV.

$r = \sqrt{x^2 + y^2} = \sqrt{1^2 + (-1)^2} = \sqrt{2}$

$\theta = \tan^{-1}\left(\dfrac{y}{x}\right) = \tan^{-1}\left(\dfrac{-1}{1}\right) = \tan^{-1}(-1) = -\dfrac{\pi}{4}$

Polar coordinates of the point $(1, -1)$ are

$\left(\sqrt{2}, -\dfrac{\pi}{4}\right)$.

60. The point $(-3,\ 3)$ lies in quadrant II.

$r = \sqrt{x^2 + y^2} = \sqrt{(-3)^2 + 3^2} = 3\sqrt{2}$

$\theta = \tan^{-1}\left(\dfrac{y}{x}\right) = \tan^{-1}\left(\dfrac{3}{-3}\right) = \tan^{-1}(-1) = -\dfrac{\pi}{4}$

Polar coordinates of the point $(-3,\ 3)$ are

$\left(3\sqrt{2}, \dfrac{3\pi}{4}\right)$.

61. The point $\left(\sqrt{3}, 1\right)$ lies in quadrant I.

$r = \sqrt{x^2 + y^2} = \sqrt{\left(\sqrt{3}\right)^2 + 1^2} = \sqrt{4} = 2$

$\theta = \tan^{-1}\left(\dfrac{y}{x}\right) = \tan^{-1}\dfrac{1}{\sqrt{3}} = \dfrac{\pi}{6}$

Polar coordinates of the point $\left(\sqrt{3}, 1\right)$ are $\left(2, \dfrac{\pi}{6}\right)$.

62. The point $\left(-2, -2\sqrt{3}\right)$ lies in quadrant III.

$r = \sqrt{x^2 + y^2} = \sqrt{(-2)^2 + \left(-2\sqrt{3}\right)^2} = \sqrt{16} = 4$

$\theta = \tan^{-1}\left(\dfrac{y}{x}\right) = \tan^{-1}\left(\dfrac{-2\sqrt{3}}{-2}\right) = \tan^{-1}\sqrt{3} = \dfrac{\pi}{3}$

The point lies in quadrant III, so $\theta = \dfrac{\pi}{3} - \pi = -\dfrac{2\pi}{3}$

Polar coordinates of the point $\left(-2, -2\sqrt{3}\right)$ are

$\left(4, -\dfrac{2\pi}{3}\right)$.

63. The point $(1.3, -2.1)$ lies in quadrant IV.

$r = \sqrt{x^2 + y^2} = \sqrt{1.3^2 + (-2.1)^2} = \sqrt{6.1} \approx 2.47$

$\theta = \tan^{-1}\left(\dfrac{y}{x}\right) = \tan^{-1}\left(\dfrac{-2.1}{1.3}\right) \approx -1.02$

The polar coordinates of the point $(1.3, -2.1)$ are

$(2.47, -1.02)$.

64. The point $(-0.8, -2.1)$ lies in quadrant III.

$r = \sqrt{x^2 + y^2} = \sqrt{(-0.8)^2 + (-2.1)^2} = \sqrt{5.05} = 2.25$

$\theta = \tan^{-1}\left(\dfrac{y}{x}\right) = \tan^{-1}\left(\dfrac{-2.1}{-0.8}\right) \approx 1.21$

Since the point lies in quadrant III,
$\theta = 1.21 - \pi \approx -1.93$.
The polar coordinates of the point $(-0.8, -2.1)$ are

$(2.25, -1.93)$.

65. The point $(8.3,\ 4.2)$ lies in quadrant I.

$r = \sqrt{x^2 + y^2} = \sqrt{8.3^2 + 4.2^2} = \sqrt{86.53} \approx 9.30$

$\theta = \tan^{-1}\left(\dfrac{y}{x}\right) = \tan^{-1}\left(\dfrac{4.2}{8.3}\right) \approx 0.47$

The polar coordinates of the point $(8.3,\ 4.2)$ are

$(9.30,\ 0.47)$.

66. The point $(-2.3,\ 0.2)$ lies in quadrant II.

$r = \sqrt{x^2 + y^2} = \sqrt{(-2.3)^2 + 0.2^2} = \sqrt{5.33} \approx 2.31$

$\theta = \tan^{-1}\left(\dfrac{y}{x}\right) = \tan^{-1}\left(\dfrac{0.2}{-2.3}\right) \approx -0.09$

Since the point lies in quadrant II,
$\theta = \pi - 0.09 \approx 3.05$.

The polar coordinates of the point $(-2.3, 0.2)$ are $(2.31, 3.05)$.

67. $2x^2 + 2y^2 = 3$

$2(x^2 + y^2) = 3$

$2r^2 = 3$

$r^2 = \dfrac{3}{2}$ or $r = \sqrt{\dfrac{3}{2}} = \dfrac{\sqrt{6}}{2}$

68. $x^2 + y^2 = x$

$r^2 = r\cos\theta$

$r = \cos\theta$

69. $x^2 = 4y$

$(r\cos\theta)^2 = 4r\sin\theta$

$r^2\cos^2\theta - 4r\sin\theta = 0$

70. $y^2 = 2x$

$(r\sin\theta)^2 = 2r\cos\theta$

$r^2\sin^2\theta - 2r\cos\theta = 0$

71. $2xy = 1$

$2(r\cos\theta)(r\sin\theta) = 1$

$r^2(2\sin\theta\cos\theta) = 1$

$r^2\sin 2\theta = 1$

72. $4x^2 y = 1$

$4(r\cos\theta)^2 r\sin\theta = 1$

$4r^2\cos^2\theta\, r\sin\theta = 1$

$r^3\cos^2\theta\sin\theta = \dfrac{1}{4}$

73. $x = 4$

$r\cos\theta = 4$

74. $y = -3$

$r\sin\theta = -3$

75. $r = \cos\theta$

$r^2 = r\cos\theta$

$x^2 + y^2 = x$

$x^2 - x + y^2 = 0$

$x^2 - x + \dfrac{1}{4} + y^2 = \dfrac{1}{4}$

$\left(x - \dfrac{1}{2}\right)^2 + y^2 = \dfrac{1}{4}$

76. $r = \sin\theta + 1$

$r^2 = r\sin\theta + r$

$x^2 + y^2 = y + \sqrt{x^2 + y^2}$

77. $r^2 = \cos\theta$

$r^3 = r\cos\theta$

$\left(r^2\right)^{3/2} = r\cos\theta$

$\left(x^2 + y^2\right)^{3/2} = x$

$\left(x^2 + y^2\right)^{3/2} - x = 0$

78. $r = \sin\theta - \cos\theta$

$r^2 = r\sin\theta - r\cos\theta$

$x^2 + y^2 = y - x$

$x^2 + x + y^2 - y = 0$

$x^2 + x + \dfrac{1}{4} + y^2 - y + \dfrac{1}{4} = \dfrac{1}{4} + \dfrac{1}{4}$

$\left(x + \dfrac{1}{2}\right)^2 + \left(y - \dfrac{1}{2}\right)^2 = \dfrac{1}{2}$

79. $r = 2$

$r^2 = 4$

$x^2 + y^2 = 4$

80. $r = 4$

$r^2 = 16$

$x^2 + y^2 = 16$

81.
$$r = \frac{4}{1 - \cos\theta}$$
$$r(1 - \cos\theta) = 4$$
$$r - r\cos\theta = 4$$
$$\sqrt{x^2 + y^2} - x = 4$$
$$\sqrt{x^2 + y^2} = x + 4$$
$$x^2 + y^2 = x^2 + 8x + 16$$
$$y^2 = 8(x + 2)$$

82.
$$r = \frac{3}{3 - \cos\theta}$$
$$r(3 - \cos\theta) = 3$$
$$3r - r\cos\theta = 3$$
$$3\sqrt{x^2 + y^2} - x = 3$$
$$3\sqrt{x^2 + y^2} = x + 3$$
$$9(x^2 + y^2) = x^2 + 6x + 9$$
$$9x^2 + 9y^2 = x^2 + 6x + 9$$
$$8x^2 - 6x + 9y^2 - 9 = 0$$
$$64x^2 - 48x + 72y^2 - 72 = 0$$
$$64\left(x^2 - \frac{3}{4}x\right) + 72y^2 = 72$$
$$64\left(x^2 - \frac{3}{4}x + \frac{9}{64}\right) + 72y^2 = 72 + 64\left(\frac{9}{64}\right)$$
$$64\left(x - \frac{3}{8}\right)^2 + 72y^2 = 81$$

83. a. For this application, west is a negative direction and north is positive. Therefore, the rectangular coordinate is $(-10, 36)$.

   b. The distance r from the origin to $(-10, 36)$ is $r = \sqrt{x^2 + y^2} = \sqrt{(-10)^2 + (36)^2} = \sqrt{1396} = 2\sqrt{349} \approx 37.36$.

   Since the point $(-10, 36)$ lies in quadrant II, we use $\theta = 180° + \tan^{-1}\left(\frac{y}{x}\right)$. Thus,

   $$\theta = 180° + \tan^{-1}\left(\frac{36}{-10}\right) = 180° + \tan^{-1}\left(-\frac{18}{5}\right) \approx 105.5°.$$

   The polar coordinate of the point is $\left(2\sqrt{349}, 180° + \tan^{-1}\left(-\frac{18}{5}\right)\right) \approx (37.36, 105.5°)$.

   c. For this application, west is a negative direction and south is also negative. Therefore, the rectangular coordinate is $(-3, -35)$.

   d. The distance r from the origin to $(-3, -35)$ is $r = \sqrt{x^2 + y^2} = \sqrt{(-3)^2 + (-35)^2} = \sqrt{1234} \approx 35.13$.

   Since the point $(-3, -35)$ lies in quadrant III, we use $\theta = 180° + \tan^{-1}\left(\frac{y}{x}\right)$. Thus,

   $$\theta = 180° + \tan^{-1}\left(\frac{-35}{-3}\right) = 180° + \tan^{-1}\left(\frac{35}{3}\right) \approx 265.1°$$

   The polar coordinate of the point is $\left(\sqrt{1234}, 180° + \tan^{-1}\left(\frac{35}{3}\right)\right) \approx (35.13, 265.1°)$.

84. Rewrite the polar coordinates in rectangular form:
   Since $P_1 = (r_1, \theta_1)$ and $P_2 = (r_2, \theta_2)$, we have that $(x_1, y_1) = (r_1\cos\theta_1, r_1\sin\theta_1)$ and

$$\left(x_1, y_1\right) = \left(r_2 \cos\theta_2, \ r_2 \sin\theta_2\right).$$

$$d = \sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2}$$

$$= \sqrt{\left(r_2 \cos\theta_2 - r_1 \cos\theta_1\right)^2 + \left(r_2 \sin\theta_2 - r_1 \sin\theta_1\right)^2}$$

$$= \sqrt{r_2^2 \cos^2\theta_2 - 2r_1 r_2 \cos\theta_2 \cos\theta_1 + r_1^2 \cos^2\theta_1 + r_2^2 \sin^2\theta_2 - 2r_1 r_2 \sin\theta_2 \sin\theta_1 + r_1^2 \sin^2\theta_1}$$

$$= \sqrt{r_1^2\left(\cos^2\theta_1 + \sin^2\theta_1\right) + r_2^2\left(\cos^2\theta_2 + \sin^2\theta_2\right) - 2r_1 r_2\left(\cos\theta_2 \cos\theta_1 + \sin\theta_2 \sin\theta_1\right)}$$

$$= \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos\left(\theta_2 - \theta_1\right)}$$

85. $x = r\cos\theta$ and $y = r\sin\theta$

86 – 87. Answers will vary.

Section 9.2

1. $\left(-4, 6\right)$

2. $\cos A \cos B + \sin A \sin B$

3. $\left(x - (-2)\right)^2 + \left(y - 5\right)^2 = 3^2$
$\left(x + 2\right)^2 + \left(y - 5\right)^2 = 9$

4. odd, since $\sin(-x) = -\sin x$.

5. $-\dfrac{\sqrt{2}}{2}$

6. $-\dfrac{1}{2}$

7. polar equation

8. False. They are sufficient but not necessary.

9. $-\theta$

10. $\pi - \theta$

11. True

12. 2n ; n

13. $r = 4$
The equation is of the form $r = a, \ a > 0$. It is a circle, center at the pole and radius 4. Transform

to rectangular form:
$$r = 4$$
$$r^2 = 16$$
$$x^2 + y^2 = 16$$
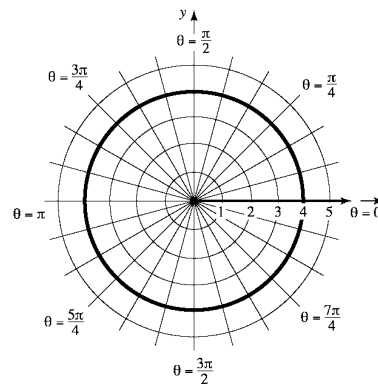
14. $r = 2$
The equation is of the form $r = a, \ a > 0$. It is a circle, center at the pole and radius 2. Transform to rectangular form:
$$r = 2$$
$$r^2 = 4$$
$$x^2 + y^2 = 4$$
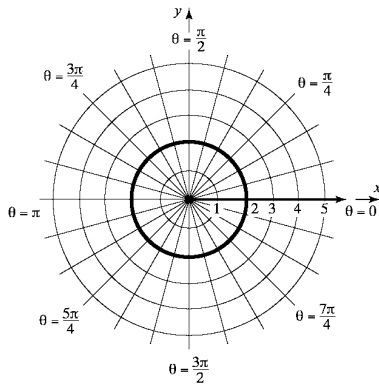
15. $\theta = \dfrac{\pi}{3}$

The equation is of the form $\theta = \alpha$. It is a line,

passing through the pole making an angle of $\dfrac{\pi}{3}$

with the polar axis. Transform to rectangular
form:

$$\theta = \frac{\pi}{3}$$

$$\tan \theta = \tan \frac{\pi}{3}$$

$$\frac{y}{x} = \sqrt{3}$$

$$y = \sqrt{3}x$$
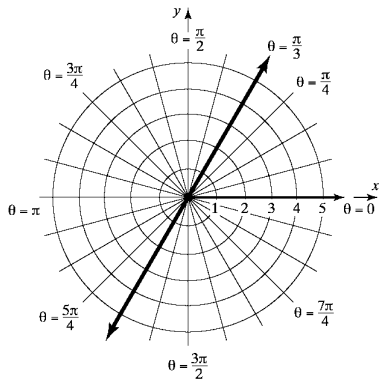
16. $\theta = -\dfrac{\pi}{4}$

The equation is of the form $\theta = \alpha$. It is a line,
passing through the pole making an angle of

$-\dfrac{\pi}{4}\left(\text{or } \dfrac{3\pi}{4}\right)$ with the polar axis. Transform to

rectangular form:

$$\theta = -\frac{\pi}{4}$$

$$\tan \theta = \tan\left(-\frac{\pi}{4}\right)$$

$$\frac{y}{x} = -1$$

$$y = -x$$
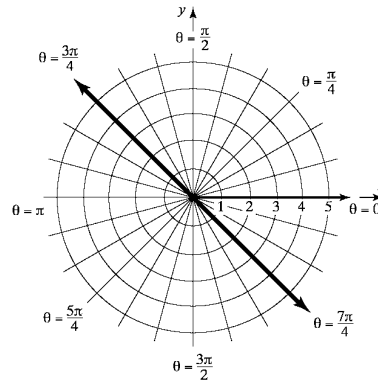
17. $r \sin \theta = 4$

The equation is of the form $r \sin \theta = b$. It is a
horizontal line, 4 units above the pole.
Transform to rectangular form:

$$r \sin \theta = 4$$

$$y = 4$$
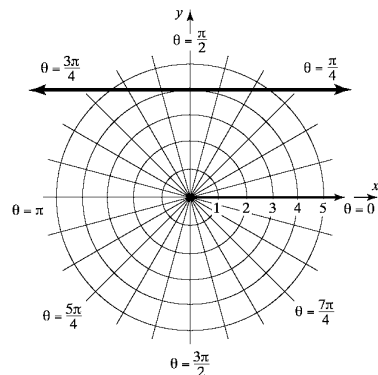
18. $r\cos\theta = 4$

The equation is of the form $r\cos\theta = a$. It is a vertical line, 4 units to the right of the pole. Transform to rectangular form:

$$r\cos\theta = 4$$
$$x = 4$$
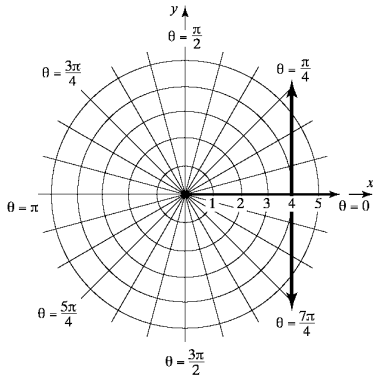
19. $r\cos\theta = -2$

The equation is of the form $r\cos\theta = a$. It is a vertical line, 2 units to the left of the pole. Transform to rectangular form:

$$r\cos\theta = -2$$
$$x = -2$$
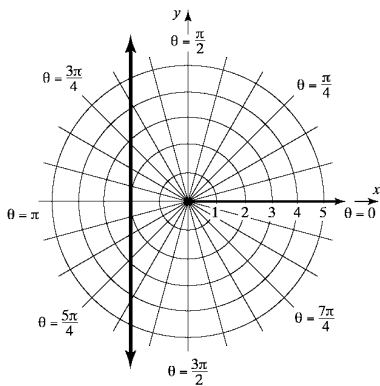
20. $r\sin\theta = -2$

The equation is of the form $r\sin\theta = b$. It is a horizontal line, 2 units below the pole. Transform to rectangular form:

$$r\sin\theta = -2$$
$$y = -2$$
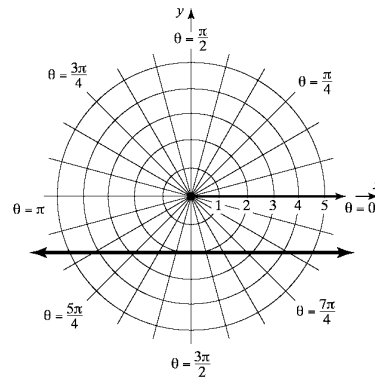
21. $r = 2\cos\theta$

The equation is of the form $r = \pm 2a\cos\theta,\ a > 0$. It is a circle, passing through the pole, and center on the polar axis. Transform to rectangular form:

$$r = 2\cos\theta$$
$$r^2 = 2r\cos\theta$$
$$x^2 + y^2 = 2x$$
$$x^2 - 2x + y^2 = 0$$
$$(x-1)^2 + y^2 = 1$$

center $(1,\ 0)$; radius 1

22. $r = 2\sin\theta$

The equation is of the form $r = \pm 2a\sin\theta,\ a > 0$. It is a circle, passing through the pole, and center on the line $\theta = \dfrac{\pi}{2}$. Transform to rectangular form:

$$r = 2\sin\theta$$
$$r^2 = 2r\sin\theta$$
$$x^2 + y^2 = 2y$$
$$x^2 + y^2 - 2y = 0$$
$$x^2 + (y-1)^2 = 1$$

center $(0,\ 1)$; radius 1
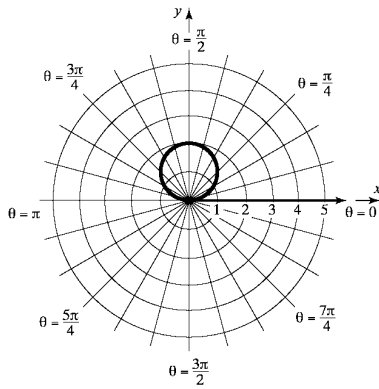
23. $r = -4\sin\theta$

The equation is of the form $r = \pm 2a\sin\theta, \; a > 0$.
It is a circle, passing through the pole, and center

on the line $\theta = \dfrac{\pi}{2}$. Transform to rectangular form:

$$r = -4\sin\theta$$
$$r^2 = -4r\sin\theta$$
$$x^2 + y^2 = -4y$$
$$x^2 + y^2 + 4y = 0$$
$$x^2 + (y+2)^2 = 4$$

center $(0,-2)$; radius 2
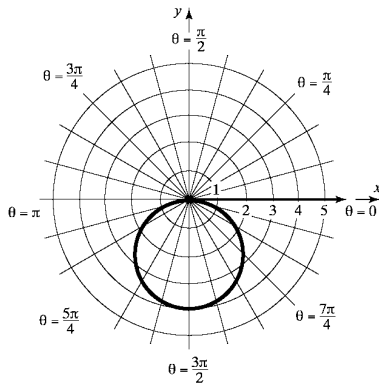
24. $r = -4\cos\theta$

The equation is of the form $r = 2a\cos\theta, \; a > 0$.
It is a circle, passing through the pole, and center
on the polar axis. Transform to rectangular form:
$$r = -4\cos\theta$$
$$r^2 = -4r\cos\theta$$
$$x^2 + y^2 = -4x$$
$$x^2 + 4x + y^2 = 0$$
$$(x+2)^2 + y^2 = 4$$
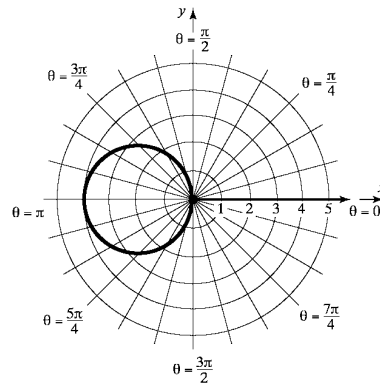
center $(-2,\ 0)$; radius 2

25. $r\sec\theta = 4$

The equation is a circle, passing through the
pole, center on the polar axis and radius 2.
Transform to rectangular form:
$$r\sec\theta = 4$$
$$r \cdot \frac{1}{\cos\theta} = 4$$
$$r = 4\cos\theta$$
$$r^2 = 4r\cos\theta$$
$$x^2 + y^2 = 4x$$
$$x^2 - 4x + y^2 = 0$$
$$(x-2)^2 + y^2 = 4$$

center $(2,\ 0)$; radius 2
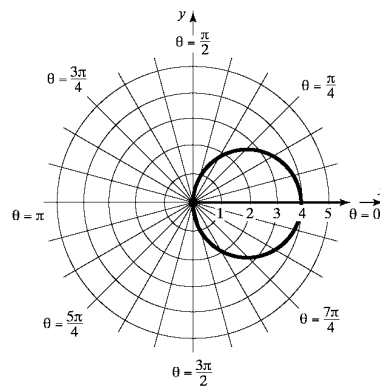
26. $r\csc\theta = 8$

The equation is a circle, passing through the

pole, center on the line $\theta = \dfrac{\pi}{2}$ and radius 4.

Transform to rectangular form:
$$r\csc\theta = 8$$
$$r \cdot \frac{1}{\sin\theta} = 8$$
$$r = 8\sin\theta$$
$$r^2 = 8r\sin\theta$$

$$x^2 + y^2 = 8y$$
$$x^2 + y^2 - 8y = 0$$
$$x^2 + (y-4)^2 = 16$$

center $(0,\ 4)$; radius 4
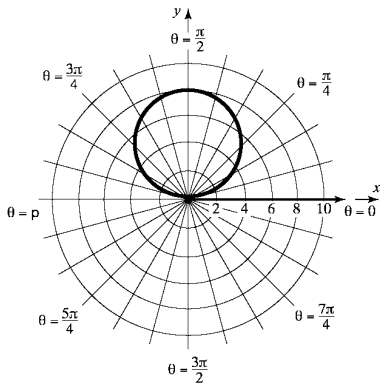
27. $r \csc \theta = -2$

The equation is a circle, passing through the

pole, center on the line $\theta = \dfrac{\pi}{2}$ and radius 1.

Transform to rectangular form:

$$r \csc \theta = -2$$
$$r \cdot \frac{1}{\sin \theta} = -2$$
$$r = -2 \sin \theta$$
$$r^2 = -2r \sin \theta$$
$$x^2 + y^2 = -2y$$
$$x^2 + y^2 + 2y = 0$$
$$x^2 + (y+1)^2 = 1$$

center $(0, -1)$; radius 1
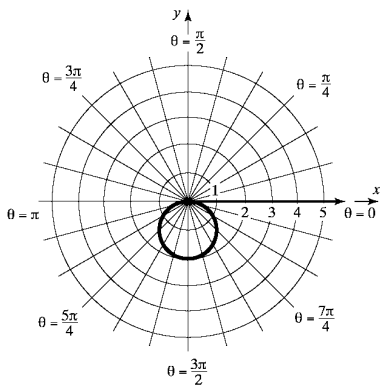
28. $r \sec \theta = -4$

The equation is a circle, passing through the
pole, center on the polar axis and radius 2.
Transform to rectangular form:

$$r \sec \theta = -4$$
$$r \cdot \frac{1}{\cos \theta} = -4$$
$$r = -4 \cos \theta$$
$$r^2 = -4r \cos \theta$$
$$x^2 + y^2 = -4x$$
$$x^2 + 4x + y^2 = 0$$
$$(x+2)^2 + y^2 = 4$$

center $(-2,\ 0)$; radius 2

29. E

30. A

31. F

32. B

33. H

34. G

35. D

36. C

37. $r = 2 + 2\cos\theta$

The graph will be a cardioid. Check for symmetry:

Polar axis: Replace θ by $-\theta$. The result is

$r = 2 + 2\cos(-\theta) = 2 + 2\cos\theta$.

The graph is symmetric with respect to the polar axis.

The line $\theta = \dfrac{\pi}{2}$: Replace θ by $\pi - \theta$.

$r = 2 + 2\cos(\pi - \theta)$

$\quad = 2 + 2[\cos(\pi)\cos\theta + \sin(\pi)\sin\theta]$

$\quad = 2 + 2(-\cos\theta + 0)$

$\quad = 2 - 2\cos\theta$

The test fails.

The pole: Replace r by $-r$.

$-r = 2 + 2\cos\theta$. The test fails.

Due to symmetry with respect to the polar axis, assign values to θ from 0 to π.

θ	$r = 2 + 2\cos\theta$
0	4
$\dfrac{\pi}{6}$	$2 + \sqrt{3} \approx 3.7$
$\dfrac{\pi}{3}$	3
$\dfrac{\pi}{2}$	2
$\dfrac{2\pi}{3}$	1
$\dfrac{5\pi}{6}$	$2 - \sqrt{3} \approx 0.3$
π	0

38. $r = 1 + \sin\theta$

The graph will be a cardioid. Check for symmetry:

Polar axis: Replace θ by $-\theta$. The result is

$r = 1 + \sin(-\theta) = 1 - \sin\theta$. The test fails.

The line $\theta = \dfrac{\pi}{2}$: Replace θ by $\pi - \theta$.

$r = 1 + \sin(\pi - \theta)$

$\quad = 1 + [\sin(\pi)\cos\theta - \cos(\pi)\sin\theta]$

$\quad = 1 + (0 + \sin\theta)$

$\quad = 1 + \sin\theta$

The graph is symmetric with respect to the line $\theta = \dfrac{\pi}{2}$.

The pole: Replace r by $-r$. $-r = 1 + \sin\theta$. The test fails.

Due to symmetry with respect to the line $\theta = \dfrac{\pi}{2}$, assign values to θ from $-\dfrac{\pi}{2}$ to $\dfrac{\pi}{2}$.

θ	$r = 1 + \sin\theta$
$-\dfrac{\pi}{2}$	0
$-\dfrac{\pi}{3}$	$1 - \dfrac{\sqrt{3}}{2} \approx 0.1$
$-\dfrac{\pi}{6}$	$\dfrac{1}{2}$
0	1
$\dfrac{\pi}{6}$	$\dfrac{3}{2}$
$\dfrac{\pi}{3}$	$1 + \dfrac{\sqrt{3}}{2} \approx 1.9$
$\dfrac{\pi}{2}$	2

39. $r = 3 - 3\sin\theta$

The graph will be a cardioid. Check for symmetry:
Polar axis: Replace θ by $-\theta$. The result is

$r = 3 - 3\sin(-\theta) = 3 + 3\sin\theta$. The test fails.

The line $\theta = \dfrac{\pi}{2}$: Replace θ by $\pi - \theta$.

$r = 3 - 3\sin(\pi - \theta)$
$\quad = 3 - 3\left[\sin(\pi)\cos\theta - \cos(\pi)\sin\theta\right]$
$\quad = 3 - 3(0 + \sin\theta)$
$\quad = 3 - 3\sin\theta$

The graph is symmetric with respect to the line

$\theta = \dfrac{\pi}{2}$.

The pole: Replace r by $-r$. $-r = 3 - 3\sin\theta$.
The test fails.

Due to symmetry with respect to the line $\theta = \dfrac{\pi}{2}$,

assign values to θ from $-\dfrac{\pi}{2}$ to $\dfrac{\pi}{2}$.

θ	$r = 3 - 3\sin\theta$
$-\dfrac{\pi}{2}$	6
$-\dfrac{\pi}{3}$	$3 + \dfrac{3\sqrt{3}}{2} \approx 5.6$
$-\dfrac{\pi}{6}$	$\dfrac{9}{2}$
0	3
$\dfrac{\pi}{6}$	$\dfrac{3}{2}$
$\dfrac{\pi}{3}$	$3 - \dfrac{3\sqrt{3}}{2} \approx 0.4$
$\dfrac{\pi}{2}$	0

40. $r = 2 - 2\cos\theta$

The graph will be a cardioid. Check for symmetry:

Polar axis: Replace θ by $-\theta$. The result is

$r = 2 - 2\cos(-\theta) = 2 - 2\cos\theta$. The graph is symmetric with respect to the polar axis.

The line $\theta = \dfrac{\pi}{2}$: Replace θ by $\pi - \theta$.

$r = 2 - 2\cos(\pi - \theta)$
$\quad = 2 - 2\left[\cos(\pi)\cos\theta + \sin(\pi)\sin\theta\right]$
$\quad = 2 - 2(-\cos\theta + 0)$
$\quad = 2 + 2\cos\theta$

The test fails.

The pole: Replace r by $-r$. $-r = 2 - 2\cos\theta$.
The test fails.

Due to symmetry with respect to the polar axis,
assign values to θ from 0 to π.

θ	$r = 2 - 2\cos\theta$
0	0
$\dfrac{\pi}{6}$	$2 - \sqrt{3} \approx 0.3$
$\dfrac{\pi}{3}$	1
$\dfrac{\pi}{2}$	2
$\dfrac{2\pi}{3}$	3
$\dfrac{5\pi}{6}$	$2 + \sqrt{3} \approx 3.7$
π	4

41. $r = 2 + \sin\theta$

The graph will be a limaçon without an inner loop. Check for symmetry:

Polar axis: Replace θ by $-\theta$. The result is $r = 2 + \sin(-\theta) = 2 - \sin\theta$. The test fails.

The line $\theta = \dfrac{\pi}{2}$: Replace θ by $\pi - \theta$.

$r = 2 + \sin(\pi - \theta)$
$\quad = 2 + \left[\sin(\pi)\cos\theta - \cos(\pi)\sin\theta\right]$
$\quad = 2 + (0 + \sin\theta) = 2 + \sin\theta$

The graph is symmetric with respect to the line $\theta = \dfrac{\pi}{2}$.

The pole: Replace r by $-r$. $-r = 2 + \sin\theta$. The test fails.

Due to symmetry with respect to the line $\theta = \dfrac{\pi}{2}$, assign values to θ from $-\dfrac{\pi}{2}$ to $\dfrac{\pi}{2}$.

θ	$r = 2 + \sin\theta$
$-\dfrac{\pi}{2}$	1
$-\dfrac{\pi}{3}$	$2 - \dfrac{\sqrt{3}}{2} \approx 1.1$
$-\dfrac{\pi}{6}$	$\dfrac{3}{2}$
0	2
$\dfrac{\pi}{6}$	$\dfrac{5}{2}$
$\dfrac{\pi}{3}$	$2 + \dfrac{\sqrt{3}}{2} \approx 2.9$
$\dfrac{\pi}{2}$	3

42. $r = 2 - \cos\theta$

The graph will be a limaçon without an inner loop. Check for symmetry:

Polar axis: Replace θ by $-\theta$. The result is $r = 2 - \cos(-\theta) = 2 - \cos\theta$. The graph is symmetric with respect to the polar axis.

The line $\theta = \dfrac{\pi}{2}$: Replace θ by $\pi - \theta$.

$r = 2 - \cos(\pi - \theta)$
$\quad = 2 - \left[\cos(\pi)\cos\theta + \sin(\pi)\sin\theta\right]$
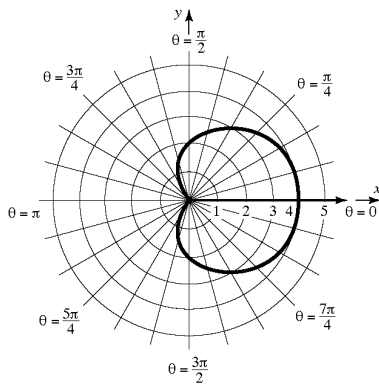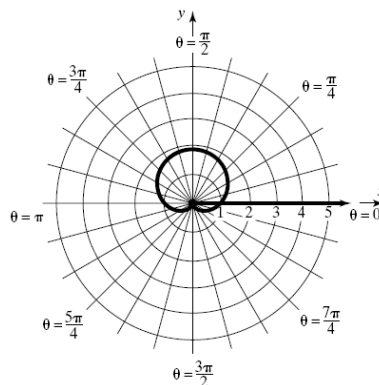$\quad = 2 - (-\cos\theta + 0)$
$\quad = 2 + \cos\theta$

The test fails.

The pole: Replace r by $-r$. $-r = 2 - \cos\theta$. The test fails.

Due to symmetry with respect to the polar axis, assign values to θ from 0 to π.

θ	$r = 2 - \cos\theta$
0	1
$\dfrac{\pi}{6}$	$2 - \dfrac{\sqrt{3}}{2} \approx 1.1$
$\dfrac{\pi}{3}$	$\dfrac{3}{2}$
$\dfrac{\pi}{2}$	2
$\dfrac{2\pi}{3}$	$\dfrac{5}{2}$
$\dfrac{5\pi}{6}$	$2 + \dfrac{\sqrt{3}}{2} \approx 2.9$
π	3
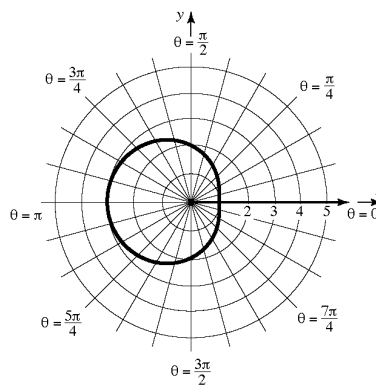
43. $r = 4 - 2\cos\theta$

The graph will be a limaçon without an inner loop. Check for symmetry:

Polar axis: Replace θ by $-\theta$. The result is $r = 4 - 2\cos(-\theta) = 4 - 2\cos\theta$. The graph is symmetric with respect to the polar axis.

The line $\theta = \dfrac{\pi}{2}$: Replace θ by $\pi - \theta$.

$r = 4 - 2\cos(\pi - \theta)$
$\quad = 4 - 2\left[\cos(\pi)\cos\theta + \sin(\pi)\sin\theta\right]$
$\quad = 4 - 2(-\cos\theta + 0)$
$\quad = 4 + 2\cos\theta$

The test fails.

The pole: Replace r by $-r$. $-r = 4 - 2\cos\theta$. The test fails.

Due to symmetry with respect to the polar axis, assign values to θ from 0 to π.

θ	$r = 4 - 2\cos\theta$
0	2
$\dfrac{\pi}{6}$	$4 - \sqrt{3} \approx 2.3$
$\dfrac{\pi}{3}$	3
$\dfrac{\pi}{2}$	4
$\dfrac{2\pi}{3}$	5
$\dfrac{5\pi}{6}$	$4 + \sqrt{3} \approx 5.7$
π	6

44. $r = 4 + 2\sin\theta$

The graph will be a limaçon without an inner loop. Check for symmetry:

Polar axis: Replace θ by $-\theta$. The result is $r = 4 + 2\sin(-\theta) = 4 - 2\sin\theta$. The test fails.

The line $\theta = \dfrac{\pi}{2}$: Replace θ by $\pi - \theta$.

$r = 4 + 2\sin(\pi - \theta)$
$\quad = 4 + 2\left[\sin(\pi)\cos\theta - \cos(\pi)\sin\theta\right]$
$\quad = 4 + 2(0 + \sin\theta)$
$\quad = 4 + 2\sin\theta$

The graph is symmetric with respect to the line $\theta = \dfrac{\pi}{2}$.
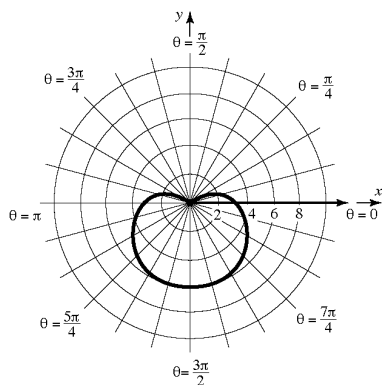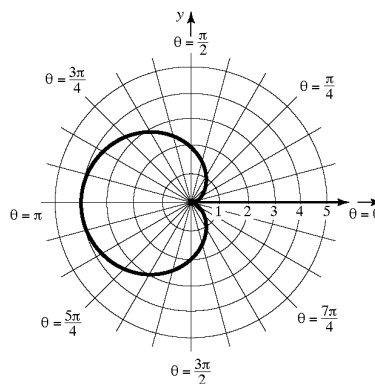
The pole: Replace r by $-r$. $-r = 4 + 2\sin\theta$. The test fails.

Due to symmetry with respect to the line $\theta = \dfrac{\pi}{2}$, assign values to θ from $-\dfrac{\pi}{2}$ to $\dfrac{\pi}{2}$.

θ	$r = 4 + 2\sin\theta$
$-\dfrac{\pi}{2}$	2
$-\dfrac{\pi}{3}$	$4 - \sqrt{3} \approx 2.3$
$-\dfrac{\pi}{6}$	3
0	4
$\dfrac{\pi}{6}$	5
$\dfrac{\pi}{3}$	$4 + \sqrt{3} \approx 5.7$
$\dfrac{\pi}{2}$	6

45. $r = 1 + 2\sin\theta$

The graph will be a limaçon with an inner loop.
Check for symmetry:

Polar axis: Replace θ by $-\theta$. The result is

$r = 1 + 2\sin(-\theta) = 1 - 2\sin\theta$. The test fails.

The line $\theta = \dfrac{\pi}{2}$: Replace θ by $\pi - \theta$.

$r = 1 + 2\sin(\pi - \theta)$
$\quad = 1 + 2\big[\sin(\pi)\cos\theta - \cos(\pi)\sin\theta\big]$
$\quad = 1 + 2(0 + \sin\theta)$
$\quad = 1 + 2\sin\theta$

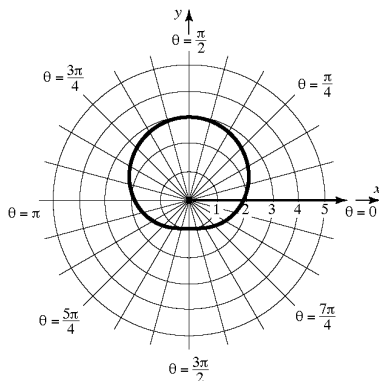The graph is symmetric with respect to the line

$\theta = \dfrac{\pi}{2}$.

The pole: Replace r by $-r$. $-r = 1 + 2\sin\theta$.
The test fails.

Due to symmetry with respect to the line $\theta = \dfrac{\pi}{2}$,

assign values to θ from $-\dfrac{\pi}{2}$ to $\dfrac{\pi}{2}$.

θ	$r = 1 + 2\sin\theta$
$-\dfrac{\pi}{2}$	-1
$-\dfrac{\pi}{3}$	$1 - \sqrt{3} \approx -0.7$
$-\dfrac{\pi}{6}$	0
0	1
$\dfrac{\pi}{6}$	2
$\dfrac{\pi}{3}$	$1 + \sqrt{3} \approx 2.7$
$\dfrac{\pi}{2}$	3

46. $r = 1 - 2\sin\theta$

The graph will be a limaçon with an inner loop.
Check for symmetry:

Polar axis: Replace θ by $-\theta$. The result is

$r = 1 - 2\sin(-\theta) = 1 + 2\sin\theta$. The test fails.

The line $\theta = \dfrac{\pi}{2}$: Replace θ by $\pi - \theta$.

$r = 1 - 2\sin(\pi - \theta)$
$\quad = 1 - 2\big[\sin(\pi)\cos\theta - \cos(\pi)\sin\theta\big]$
$\quad = 1 - 2(0 + \sin\theta)$
$\quad = 1 - 2\sin\theta$

The graph is symmetric with respect to the line

$\theta = \dfrac{\pi}{2}$.

The pole: Replace r by $-r$. $-r = 1 - 2\sin\theta$.
The test fails.

Due to symmetry with respect to the line $\theta = \dfrac{\pi}{2}$,

assign values to θ from $-\dfrac{\pi}{2}$ to $\dfrac{\pi}{2}$.

θ	$r = 1 - 2\sin\theta$
$-\dfrac{\pi}{2}$	3
$-\dfrac{\pi}{3}$	$1 + \sqrt{3} \approx 2.7$
$-\dfrac{\pi}{6}$	2
0	1
$\dfrac{\pi}{6}$	0
$\dfrac{\pi}{3}$	$1 - \sqrt{3} \approx -0.7$
$\dfrac{\pi}{2}$	-1

47. $r = 2 - 3\cos\theta$

The graph will be a limaçon with an inner loop. Check for symmetry:

Polar axis: Replace θ by $-\theta$. The result is

$r = 2 - 3\cos(-\theta) = 2 - 3\cos\theta$.

The graph is symmetric with respect to the polar axis.

The line $\theta = \dfrac{\pi}{2}$: Replace θ by $\pi - \theta$.

$r = 2 - 3\cos(\pi - \theta)$

$\quad = 2 - 3\left[\cos(\pi)\cos\theta + \sin(\pi)\sin\theta\right]$

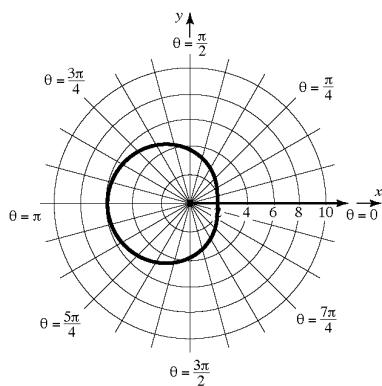$\quad = 2 - 3(-\cos\theta + 0)$

$\quad = 2 + 3\cos\theta$

The test fails.

The pole: Replace r by $-r$. $-r = 2 - 3\cos\theta$.

The test fails.

Due to symmetry with respect to the polar axis, assign values to θ from 0 to π.

θ	$r = 2 - 3\cos\theta$
0	-1
$\dfrac{\pi}{6}$	$2 - \dfrac{3\sqrt{3}}{2} \approx -0.6$
$\dfrac{\pi}{3}$	$\dfrac{1}{2}$
$\dfrac{\pi}{2}$	2
$\dfrac{2\pi}{3}$	$\dfrac{7}{2}$
$\dfrac{5\pi}{6}$	$2 + \dfrac{3\sqrt{3}}{2} \approx 4.6$
π	5
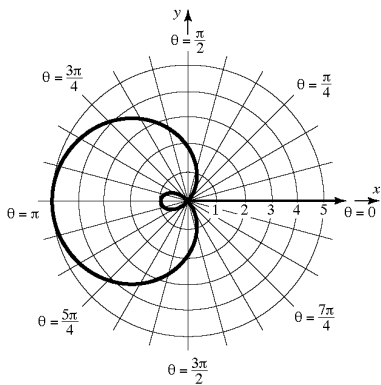
48. $r = 2 + 4\cos\theta$

The graph will be a limaçon with an inner loop. Check for symmetry:

Polar axis: Replace θ by $-\theta$. The result is

$r = 2 + 4\cos(-\theta) = 2 + 4\cos\theta$.

The graph is symmetric with respect to the polar axis.

The line $\theta = \dfrac{\pi}{2}$: Replace θ by $\pi - \theta$.

$r = 2 + 4\cos(\pi - \theta)$

$\quad = 2 + 4\left[\cos(\pi)\cos\theta + \sin(\pi)\sin\theta\right]$

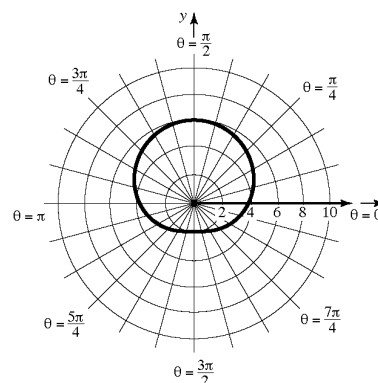$\quad = 2 + 4(-\cos\theta + 0)$

$\quad = 2 - 4\cos\theta$

The test fails.

The pole: Replace r by $-r$. $-r = 2 + 4\cos\theta$.

The test fails.

Due to symmetry with respect to the polar axis, assign values to θ from 0 to π.

θ	$r = 2 + 4\cos\theta$
0	6
$\dfrac{\pi}{6}$	$2 + 2\sqrt{3} \approx 5.5$
$\dfrac{\pi}{3}$	4
$\dfrac{\pi}{2}$	2
$\dfrac{2\pi}{3}$	0
$\dfrac{5\pi}{6}$	$2 - 2\sqrt{3} \approx -1.5$
π	-2
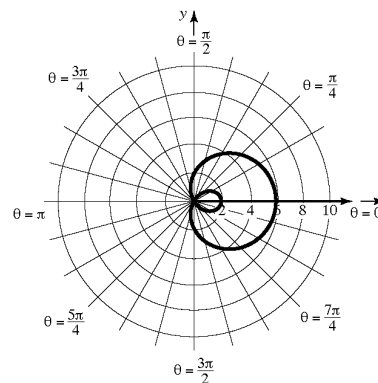
49. $r = 3\cos(2\theta)$

The graph will be a rose with four petals. Check for symmetry:

Polar axis: Replace θ by $-\theta$.

$r = 3\cos(2(-\theta)) = 3\cos(-2\theta) = 3\cos(2\theta)$. The graph is symmetric with respect to the polar axis.

The line $\theta = \dfrac{\pi}{2}$: Replace θ by $\pi - \theta$.

$\begin{aligned} r &= 3\cos\left[2(\pi - \theta)\right] \\ &= 3\cos(2\pi - 2\theta) \\ &= 3\left[\cos(2\pi)\cos(2\theta) + \sin(2\pi)\sin(2\theta)\right] \\ &= 3(\cos 2\theta + 0) \\ &= 3\cos(2\theta) \end{aligned}$

The graph is symmetric with respect to the line $\theta = \dfrac{\pi}{2}$.

The pole: Since the graph is symmetric with respect to both the polar axis and the line $\theta = \dfrac{\pi}{2}$, it is also symmetric with respect to the pole.

Due to symmetry, assign values to θ from 0 to $\dfrac{\pi}{2}$.

θ	$r = 3\cos(2\theta)$
	3
$\dfrac{\pi}{6}$	$\dfrac{3}{2}$
$\dfrac{\pi}{4}$	0
$\dfrac{\pi}{3}$	$-\dfrac{3}{2}$
$\dfrac{\pi}{2}$	-3

50. $r = 2\sin(3\theta)$

The graph will be a rose with three petals. Check for symmetry:

Polar axis: Replace θ by $-\theta$.

$r = 2\sin\left[3(-\theta)\right] = 2\sin(-3\theta) = -2\sin(3\theta)$. The test fails.

The line $\theta = \dfrac{\pi}{2}$: Replace θ by $\pi - \theta$.

$\begin{aligned} r &= 2\sin\left[3(\pi - \theta)\right] \\ &= 2\sin(3\pi - 3\theta) \\ &= 2\left[\sin(3\pi)\cos(3\theta) - \cos(3\pi)\sin(3\theta)\right] \\ &= 2\left[0 + \sin(3\theta)\right] \\ &= 2\sin(3\theta) \end{aligned}$
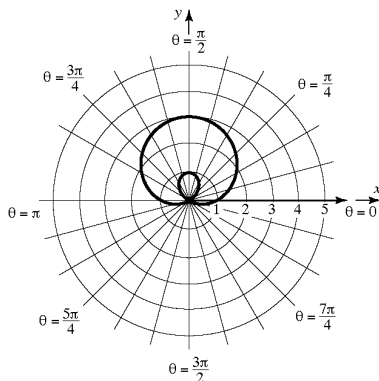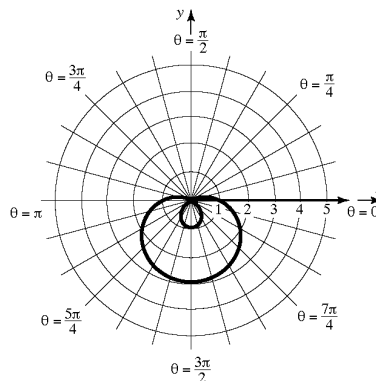
The graph is symmetric with respect to the line $\theta = \dfrac{\pi}{2}$.

The pole: Replace r by $-r$. $-r = 2\sin(3\theta)$. The test fails.

Due to symmetry with respect to the line $\theta = \dfrac{\pi}{2}$, assign values to θ from $-\dfrac{\pi}{2}$ to $\dfrac{\pi}{2}$.

θ	$r = 2\sin(3\theta)$	θ	$r = 2\sin(3\theta)$
$-\dfrac{\pi}{2}$	2	$\dfrac{\pi}{6}$	2
$-\dfrac{\pi}{3}$	0	$\dfrac{\pi}{4}$	$\sqrt{2} \approx 1.4$
$-\dfrac{\pi}{4}$	$-\sqrt{2} \approx -1.4$	$\dfrac{\pi}{3}$	0
$-\dfrac{\pi}{6}$	-2	$\dfrac{\pi}{2}$	-2
0	0		
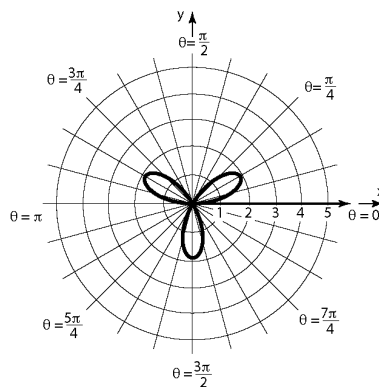
51. $r = 4\sin(5\theta)$

The graph will be a rose with five petals. Check for symmetry:

Polar axis: Replace θ by $-\theta$.

$r = 4\sin\left[5(-\theta)\right] = 4\sin(-5\theta) = -4\sin(5\theta)$.

The test fails.

The line $\theta = \dfrac{\pi}{2}$: Replace θ by $\pi - \theta$.

$r = 4\sin\left[5(\pi - \theta)\right] = 4\sin(5\pi - 5\theta)$

$\quad = 4\left[\sin(5\pi)\cos(5\theta) - \cos(5\pi)\sin(5\theta)\right]$

$\quad = 4\left[0 + \sin(5\theta)\right]$

$\quad = 4\sin(5\theta)$

The graph is symmetric with respect to the line $\theta = \dfrac{\pi}{2}$.

The pole: Replace r by $-r$. $-r = 4\sin(5\theta)$.

The test fails.

Due to symmetry with respect to the line $\theta = \dfrac{\pi}{2}$, assign values to θ from $-\dfrac{\pi}{2}$ to $\dfrac{\pi}{2}$.

θ	$r = 4\sin(5\theta)$		θ	$r = 4\sin(5\theta)$
$-\dfrac{\pi}{2}$	-4		$\dfrac{\pi}{6}$	2
$-\dfrac{\pi}{3}$	$2\sqrt{3} \approx 3.5$		$\dfrac{\pi}{4}$	$-2\sqrt{2} \approx -2.8$
$-\dfrac{\pi}{4}$	$2\sqrt{2} \approx 2.8$		$\dfrac{\pi}{3}$	$-2\sqrt{3} \approx -3.5$
$-\dfrac{\pi}{6}$	-2		$\dfrac{\pi}{2}$	4
0	0			
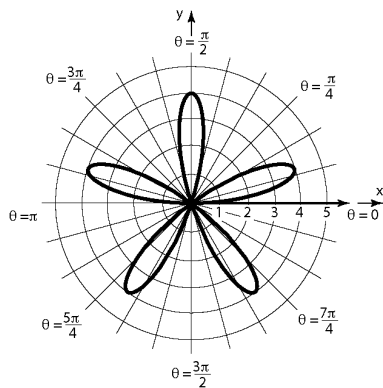
52. $r = 3\cos(4\theta)$

The graph will be a rose with eight petals. Check for symmetry:

Polar axis: Replace θ by $-\theta$.

$r = 3\cos(4(-\theta)) = 3\cos(-4\theta) = 3\cos(4\theta)$. The graph is symmetric with respect to the polar axis.

The line $\theta = \dfrac{\pi}{2}$: Replace θ by $\pi - \theta$.

$r = 3\cos\left[4(\pi - \theta)\right]$

$\quad = 3\cos(4\pi - 4\theta)$

$\quad = 3\left[\cos(4\pi)\cos(4\theta) + \sin(4\pi)\sin(4\theta)\right]$

$\quad = 3(\cos 4\theta + 0)$

$\quad = 3\cos(4\theta)$

The graph is symmetric with respect to the line $\theta = \dfrac{\pi}{2}$.

The pole: Since the graph is symmetric with respect to both the polar axis and the line $\theta = \dfrac{\pi}{2}$, it is also symmetric with respect to the pole.

Due to symmetry, assign values to θ from 0 to $\dfrac{\pi}{2}$.

θ	$r = 3\cos(4\theta)$
0	2
$\dfrac{\pi}{6}$	$-\dfrac{3}{2}$
$\dfrac{\pi}{4}$	-3
$\dfrac{\pi}{3}$	$-\dfrac{3}{2}$
$\dfrac{\pi}{2}$	3

53. $r^2 = 9\cos(2\theta)$

The graph will be a lemniscate. Check for symmetry:

Polar axis: Replace θ by $-\theta$.

$r^2 = 9\cos(2(-\theta)) = 9\cos(-2\theta) = 9\cos(2\theta)$. The graph is symmetric with respect to the polar axis.

The line $\theta = \dfrac{\pi}{2}$: Replace θ by $\pi - \theta$.

$r^2 = 9\cos[2(\pi - \theta)]$
$= 9\cos(2\pi - 2\theta)$
$= 9[\cos(2\pi)\cos 2\theta + \sin(2\pi)\sin 2\theta]$
$= 9(\cos 2\theta + 0)$
$= 9\cos(2\theta)$
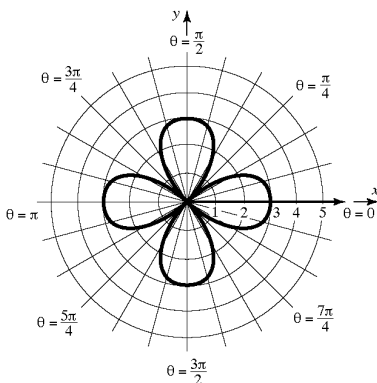
The graph is symmetric with respect to the line $\theta = \dfrac{\pi}{2}$.

The pole: Since the graph is symmetric with respect to both the polar axis and the line $\theta = \dfrac{\pi}{2}$, it is also symmetric with respect to the pole.

Due to symmetry, assign values to θ from 0 to $\dfrac{\pi}{2}$.

θ	$r = \pm\sqrt{9\cos(2\theta)}$
0	± 3
$\dfrac{\pi}{6}$	$\pm\dfrac{3\sqrt{2}}{2} \approx \pm 2.1$
$\dfrac{\pi}{4}$	0
$\dfrac{\pi}{3}$	undefined
$\dfrac{\pi}{2}$	undefined

54. $r^2 = \sin(2\theta)$

The graph will be a lemniscate. Check for symmetry:

Polar axis: Replace θ by $-\theta$.

$r^2 = \sin(2(-\theta)) = \sin(-2\theta) = -\sin(2\theta)$. The test fails.

The line $\theta = \dfrac{\pi}{2}$: Replace θ by $\pi - \theta$.

$r^2 = \sin[2(\pi - \theta)]$
$= \sin(2\pi - 2\theta)$
$= \sin(2\pi)\cos 2\theta - \cos(2\pi)\sin(2\theta)$
$= 0 - \sin(2\theta)$
$= -\sin(2\theta)$

The test fails.

The pole: Replace r by $-r$.

$(-r)^2 = \sin(2\theta)$
$r^2 = \sin(2\theta)$

The graph is symmetric with respect to the pole.

Due to symmetry, assign values to θ from 0 to π.

θ	$r = \pm\sqrt{\sin(2\theta)}$
0	0
$\dfrac{\pi}{6}$	$\pm\sqrt{\dfrac{\sqrt{3}}{2}}$
$\dfrac{\pi}{3}$	$\pm\sqrt{\dfrac{\sqrt{3}}{2}}$
$\dfrac{\pi}{2}$	0
$\dfrac{2\pi}{3}$	undefined
$\dfrac{5\pi}{6}$	undefined
π	0

55. $r = 2^\theta$

The graph will be a spiral. Check for symmetry:

Polar axis: Replace θ by $-\theta$. $r = 2^{-\theta}$. The test fails.

The line $\theta = \dfrac{\pi}{2}$: Replace θ by $\pi - \theta$. $r = 2^{\pi-\theta}$. The test fails.

The pole: Replace r by $-r$. $-r = 2^\theta$. The test fails.

θ	$r = 2^\theta$
$-\pi$	0.1
$-\dfrac{\pi}{2}$	0.3
$-\dfrac{\pi}{4}$	0.6
0	1
$\dfrac{\pi}{4}$	1.7
$\dfrac{\pi}{2}$	3.0
π	8.8
$\dfrac{3\pi}{2}$	26.2
2π	77.9

56. $r = 3^\theta$

The graph will be a spiral. Check for symmetry:

Polar axis: Replace θ by $-\theta$. $r = 3^{-\theta}$. The test fails.

The line $\theta = \dfrac{\pi}{2}$: Replace θ by $\pi - \theta$. $r = 3^{\pi-\theta}$. The test fails.

The pole: Replace r by $-r$. $-r = 3^\theta$. The test fails.

θ	$r = 3^\theta$
$-\pi$	0.03
$-\dfrac{\pi}{2}$	0.2
$-\dfrac{\pi}{4}$	0.4
0	1
$\dfrac{\pi}{4}$	2.4
$\dfrac{\pi}{2}$	5.6
π	31.5
$\dfrac{3\pi}{2}$	117.2
2π	995

57. $r = 1 - \cos\theta$

The graph will be a cardioid. Check for symmetry:

Polar axis: Replace θ by $-\theta$. The result is $r = 1 - \cos(-\theta) = 1 - \cos\theta$.

The graph is symmetric with respect to the polar axis.

The line $\theta = \dfrac{\pi}{2}$: Replace θ by $\pi - \theta$.

$r = 1 - \cos(\pi - \theta)$
$\quad = 1 - (\cos(\pi)\cos\theta + \sin(\pi)\sin\theta)$
$\quad = 1 - (-\cos\theta + 0)$
$\quad = 1 + \cos\theta$

The test fails.

The pole: Replace r by $-r$. $-r = 1 - \cos\theta$.

The test fails.

Due to symmetry, assign values to θ from 0 to π.

θ	$r = 1 - \cos\theta$
0	0
$\dfrac{\pi}{6}$	$1 - \dfrac{\sqrt{3}}{2} \approx 0.1$
$\dfrac{\pi}{3}$	$\dfrac{1}{2}$
$\dfrac{\pi}{2}$	1
$\dfrac{2\pi}{3}$	$\dfrac{3}{2}$
$\dfrac{5\pi}{6}$	$1 + \dfrac{\sqrt{3}}{2} \approx 1.9$
π	2

58. $r = 3 + \cos\theta$

The graph will be a limaçon without an inner loop. Check for symmetry:

Polar axis: Replace θ by $-\theta$. The result is $r = 3 + \cos(-\theta) = 3 + \cos\theta$. The graph is symmetric with respect to the polar axis.

The line $\theta = \dfrac{\pi}{2}$: Replace θ by $\pi - \theta$.

$r = 3 + \cos(\pi - \theta)$
$\quad = 3 + \left[\cos(\pi)\cos\theta + \sin(\pi)\sin\theta \right]$
$\quad = 3 + (-\cos\theta + 0)$
$\quad = 3 - \cos\theta$

The test fails.

The pole: Replace r by $-r$. $-r = 3 + \cos\theta$.

The test fails.

Due to symmetry, assign values to θ from 0 to π.

θ	$r = 3 + \cos\theta$
0	4
$\dfrac{\pi}{6}$	$3 + \dfrac{\sqrt{3}}{2} \approx 3.9$
$\dfrac{\pi}{3}$	$\dfrac{7}{2}$
$\dfrac{\pi}{2}$	3
$\dfrac{2\pi}{3}$	$\dfrac{5}{2}$
$\dfrac{5\pi}{6}$	$3 - \dfrac{\sqrt{3}}{2} \approx 2.1$
π	2

59. $r = 1 - 3\cos\theta$

The graph will be a limaçon with an inner loop.
Check for symmetry:

Polar axis: Replace θ by $-\theta$. The result is
$r = 1 - 3\cos(-\theta) = 1 - 3\cos\theta$. The graph is
symmetric with respect to the polar axis.

The line $\theta = \dfrac{\pi}{2}$: Replace θ by $\pi - \theta$.

$r = 1 - 3\cos(\pi - \theta)$
$\quad = 1 - 3\left[\cos(\pi)\cos\theta + \sin(\pi)\sin\theta\right]$
$\quad = 1 - 3(-\cos\theta + 0)$
$\quad = 1 + 3\cos\theta$

The test fails.

The pole: Replace r by $-r$. $-r = 1 - 3\cos\theta$.
The test fails.

Due to symmetry, assign values to
θ from 0 to π.

θ	$r = 1 - 3\cos\theta$
0	-2
$\dfrac{\pi}{6}$	$1 - \dfrac{3\sqrt{3}}{2} \approx -1.6$
$\dfrac{\pi}{3}$	$-\dfrac{1}{2}$
$\dfrac{\pi}{2}$	1
$\dfrac{2\pi}{3}$	$\dfrac{5}{2}$
$\dfrac{5\pi}{6}$	$1 + \dfrac{3\sqrt{3}}{2} \approx 3.6$
π	4

60. $r = 4\cos(3\theta)$

The graph will be a rose with three petals.
Check for symmetry:

Polar axis: Replace θ by $-\theta$.
$r = 4\cos(3(-\theta)) = 4\cos(-3\theta) = 4\cos(3\theta)$. The
graph is symmetric with respect to the polar axis.

The line $\theta = \dfrac{\pi}{2}$: Replace θ by $\pi - \theta$.

$r = 4\cos\left[3(\pi - \theta)\right]$
$\quad = 4\cos(3\pi - 3\theta)$
$\quad = 4\left[\cos(3\pi)\cos 3\theta + \sin(3\pi)\sin(3\theta)\right]$
$\quad = 4(-\cos 3\theta + 0)$
$\quad = -4\cos(3\theta)$

The test fails.

The pole: Replace r by $-r$. $-r = 4\cos(3\theta)$.
The test fails.

Due to symmetry, assign values to
θ from 0 to π.

θ	$r = 4\cos(3\theta)$
0	4
$\dfrac{\pi}{6}$	0
$\dfrac{\pi}{3}$	-4
$\dfrac{\pi}{2}$	0
$\dfrac{2\pi}{3}$	4
$\dfrac{5\pi}{6}$	0
π	-4
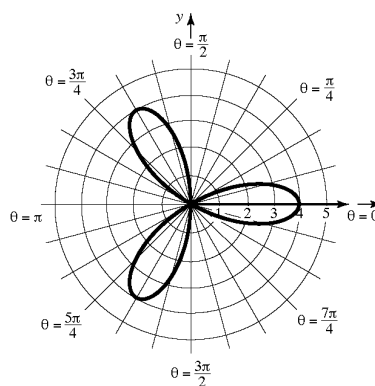
61. $r = 8\cos\theta$

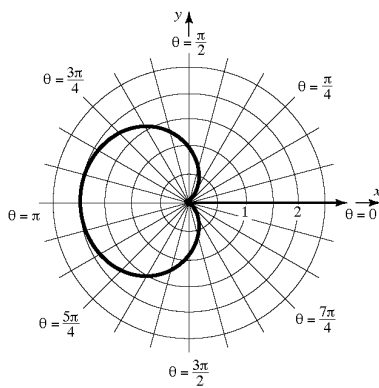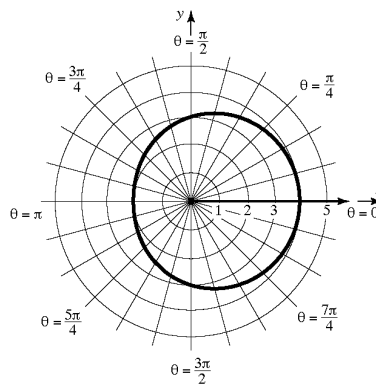The equation is of the form $r = \pm 2a\cos\theta, \ a > 0$.
It is a circle, passing through the pole, and center on the polar axis.

$r = 2\sec\theta$

$r = \dfrac{2}{\cos\theta}$

$r\cos\theta = 2$

The equation is of the form $r\cos\theta = a$. It is a vertical line, 2 units to the right of the pole.
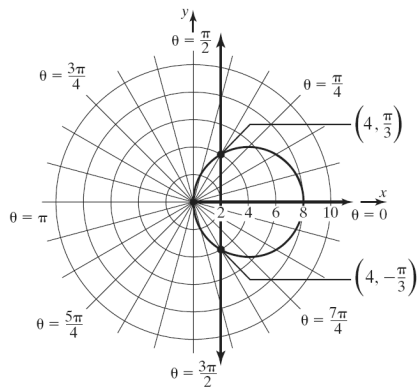
Use substitution to find the point(s) of intersection:

$8\cos\theta = 2\sec\theta$

$8\cos\theta = \dfrac{2}{\cos\theta}$

$\cos^2\theta = \dfrac{1}{4}$

$\cos\theta = \pm\dfrac{1}{2}$

$\theta = \dfrac{\pi}{3}, \dfrac{2\pi}{3}, \dfrac{4\pi}{3}, \dfrac{5\pi}{3}$ for $0 \le \theta < 2\pi$

If $\theta = \dfrac{\pi}{3}$, $r = 8\cos\dfrac{\pi}{3} = 8\left(\dfrac{1}{2}\right) = 4$.

If $\theta = \dfrac{2\pi}{3}$, $r = 8\cos\dfrac{2\pi}{3} = 8\left(-\dfrac{1}{2}\right) = -4$.

If $\theta = \dfrac{4\pi}{3}$, $r = 8\cos\dfrac{4\pi}{3} = 8\left(-\dfrac{1}{2}\right) = -4$.

If $\theta = \dfrac{5\pi}{3}$, $r = 8\cos\dfrac{5\pi}{3} = 8\left(\dfrac{1}{2}\right) = 4$.

The points of intersection are $\left(4, \dfrac{\pi}{3}\right)$,

$\left(-4, \dfrac{2\pi}{3}\right)$, $\left(-4, \dfrac{4\pi}{3}\right)$, and $\left(4, \dfrac{5\pi}{3}\right)$.

62. $r = 8\sin\theta$

The equation is of the form $r = \pm 2a\sin\theta, \ a > 0$.
It is a circle, passing through the pole, and center on the line $\theta = \dfrac{\pi}{2}$.

$r = 4\csc\theta$

$r = \dfrac{4}{\sin\theta}$

$r\sin\theta = 4$

The equation is of the form $r\sin\theta = b$. It is a horizontal line, 4 units above the pole.
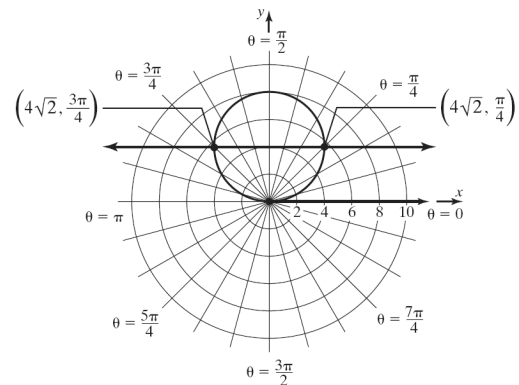
Use substitution to find the point(s) of intersection:

$8\sin\theta = 4\csc\theta$

$8\sin\theta = \dfrac{4}{\sin\theta}$

$\sin^2\theta = \dfrac{1}{2}$

$\sin\theta = \pm\dfrac{\sqrt{2}}{2}$

$\theta = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$ for $0 \le \theta < 2\pi$

If $\theta = \dfrac{\pi}{4}$, $r = 8\sin\dfrac{\pi}{4} = 8\left(\dfrac{\sqrt{2}}{2}\right) = 4\sqrt{2}$.

If $\theta = \dfrac{3\pi}{4}$, $r = 8\sin\dfrac{3\pi}{4} = 8\left(\dfrac{\sqrt{2}}{2}\right) = 4\sqrt{2}$.

If $\theta = \dfrac{5\pi}{4}$, $r = 8\sin\dfrac{5\pi}{4} = 8\left(-\dfrac{\sqrt{2}}{2}\right) = -4\sqrt{2}$.

If $\theta = \dfrac{7\pi}{4}$, $r = 8\sin\dfrac{7\pi}{4} = 8\left(-\dfrac{\sqrt{2}}{2}\right) = -4\sqrt{2}$.

The points of intersection are $\left(4\sqrt{2}, \dfrac{\pi}{4}\right)$,

$\left(4\sqrt{2}, \dfrac{3\pi}{4}\right)$, $\left(-4\sqrt{2}, \dfrac{5\pi}{4}\right)$, and $\left(-4\sqrt{2}, \dfrac{7\pi}{4}\right)$.

63. $r = \sin\theta$

The equation is of the form $r = \pm 2a\sin\theta, \ a > 0$.
It is a circle, passing through the pole, and center

on the line $\theta = \dfrac{\pi}{2}$.

$r = 1 + \cos\theta$

The graph will be a limaçon without an inner
loop. Check for symmetry:

Polar axis: Replace θ by $-\theta$. The result is
$r = 1 + \cos(-\theta) = 1 + \cos\theta$. The graph is
symmetric with respect to the polar axis.

The line $\theta = \dfrac{\pi}{2}$: Replace θ by $\pi - \theta$.

$r = 1 + \cos(\pi - \theta)$

$\quad = 1 + \left[\cos(\pi)\cos\theta + \sin(\pi)\sin\theta\right]$
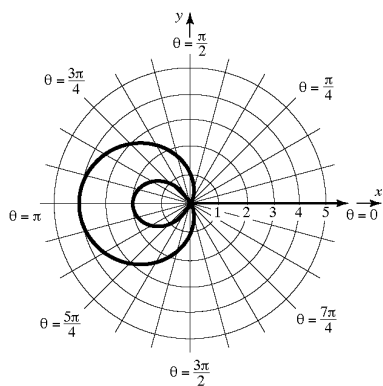
$\quad = 1 + (-\cos\theta + 0)$

$\quad = 1 - \cos\theta$

The test fails.

The pole: Replace r by $-r$. $-r = 1 + \cos\theta$.
The test fails.

Due to symmetry, assign values to
θ from 0 to π.

θ	$r = 1 + \cos\theta$
0	2
$\dfrac{\pi}{6}$	$1 + \dfrac{\sqrt{3}}{2} \approx 1.9$
$\dfrac{\pi}{3}$	$\dfrac{3}{2}$
$\dfrac{\pi}{2}$	1
$\dfrac{2\pi}{3}$	$\dfrac{1}{2}$
$\dfrac{5\pi}{6}$	$1 - \dfrac{\sqrt{3}}{2} \approx 0.1$
π	0

Use substitution to find the point(s) of
intersection:

$$\sin\theta = 1 + \cos\theta$$
$$\sin\theta - \cos\theta = 1$$
$$\left(\sin\theta - \cos\theta\right)^2 = \left(1\right)^2$$
$$\sin^2\theta - 2\sin\theta\cos\theta + \cos^2\theta = 1$$
$$1 - 2\sin\theta\cos\theta = 1$$
$$-2\sin\theta\cos\theta = 0$$
$$\sin\theta\cos\theta = 0$$

$\sin\theta = 0$ or $\cos\theta = 0$

$\theta = 0, \pi$ or $\theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$ for $0 \le \theta < 2\pi$

If $\theta = 0$, $r = \sin 0 = 0$.
If $\theta = \pi$, $r = \sin\pi = 0$.

If $\theta = \dfrac{\pi}{2}$, $r = \sin\dfrac{\pi}{2} = 1$.

If $\theta = \dfrac{3\pi}{2}$, $r = \sin\dfrac{3\pi}{2} = -1$.

The points of intersection are $(0, 0)$, $(0, \pi)$,

$\left(1, \dfrac{\pi}{2}\right)$, and $\left(-1, \dfrac{3\pi}{2}\right)$.

64. $r = 3$

The equation is of the form $r = a, \ a > 0$. It is a
circle, center at the pole and radius 3.

$r = 2 + 2\cos\theta$

The graph will be a limaçon without an inner
loop. Check for symmetry:

Polar axis: Replace θ by $-\theta$. The result is
$r = 2 + 2\cos(-\theta) = 2 + 2\cos\theta$. The graph is
symmetric with respect to the polar axis.

The line $\theta = \dfrac{\pi}{2}$: Replace θ by $\pi - \theta$.

$r = 2 + 2\cos(\pi - \theta)$

$\quad = 2 + 2\left[\cos(\pi)\cos\theta + \sin(\pi)\sin\theta\right]$

$\quad = 2 + 2(-\cos\theta + 0)$

$\quad = 2 - 2\cos\theta$

The test fails.

The pole: Replace r by $-r$. $-r = 2 + 2\cos\theta$.
The test fails.

Due to symmetry, assign values to
θ from 0 to π.

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θ	$r = 2 + 2\cos\theta$
0	4
$\dfrac{\pi}{6}$	$2 + \sqrt{3} \approx 3.7$
$\dfrac{\pi}{3}$	3
$\dfrac{\pi}{2}$	2
$\dfrac{2\pi}{3}$	1
$\dfrac{5\pi}{6}$	$2 - \sqrt{3} \approx 0.3$
π	0

Use substitution to find the point(s) of intersection:

$3 = 2 + 2\cos\theta$

$1 = 2\cos\theta$

$\dfrac{1}{2} = \cos\theta$

$\theta = \dfrac{\pi}{3}, \dfrac{5\pi}{3}$ for $0 \le \theta < 2\pi$

If $\theta = \dfrac{\pi}{3}$, $r = 2 + 2\cos\dfrac{\pi}{3} = 2 + 2\left(\dfrac{1}{2}\right) = 3$.

If $\theta = \dfrac{5\pi}{3}$, $r = 2 + 2\cos\dfrac{5\pi}{3} = 2 + 2\left(\dfrac{1}{2}\right) = 3$.

The points of intersection are $\left(3, \dfrac{\pi}{3}\right)$ and

$\left(3, \dfrac{5\pi}{3}\right)$.

65. $r = 1 + \sin\theta$

The graph will be a cardioid. Check for symmetry:

Polar axis: Replace θ by $-\theta$. The result is

$r = 1 + \sin(-\theta) = 1 - \sin\theta$. The test fails.

The line $\theta = \dfrac{\pi}{2}$: Replace θ by $\pi - \theta$.

$r = 1 + \sin(\pi - \theta)$

$\quad = 1 + \left[\sin(\pi)\cos\theta - \cos(\pi)\sin\theta\right]$

$\quad = 1 + (0 + \sin\theta)$

$\quad = 1 + \sin\theta$

The graph is symmetric with respect to the line

$\theta = \dfrac{\pi}{2}$.

The pole: Replace r by $-r$. $-r = 1 + \sin\theta$.

The test fails.

Due to symmetry with respect to the line

$\theta = \dfrac{\pi}{2}$, assign values to θ from $-\dfrac{\pi}{2}$ to $\dfrac{\pi}{2}$.

θ	$r = 1 + \sin\theta$
$-\dfrac{\pi}{2}$	0
$-\dfrac{\pi}{3}$	$1 - \dfrac{\sqrt{3}}{2} \approx 0.1$
$-\dfrac{\pi}{6}$	$\dfrac{1}{2}$
0	1
$\dfrac{\pi}{6}$	$\dfrac{3}{2}$
$\dfrac{\pi}{3}$	$1 + \dfrac{\sqrt{3}}{2} \approx 1.9$
$\dfrac{\pi}{2}$	2

$r = 1 + \cos\theta$

The graph will be a limaçon without an inner loop. Check for symmetry:

Polar axis: Replace θ by $-\theta$. The result is

$r = 1 + \cos(-\theta) = 1 + \cos\theta$. The graph is symmetric with respect to the polar axis.

The line $\theta = \dfrac{\pi}{2}$: Replace θ by $\pi - \theta$.

$r = 1 + \cos(\pi - \theta)$

$\quad = 1 + \left[\cos(\pi)\cos\theta + \sin(\pi)\sin\theta\right]$
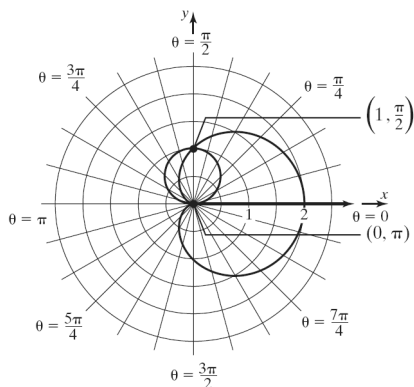
$\quad = 1 + (-\cos\theta + 0)$

$\quad = 1 - \cos\theta$

The test fails.

The pole: Replace r by $-r$. $-r = 1 + \cos\theta$.

The test fails.

Due to symmetry, assign values to θ from 0 to π.

θ	$r = 1 + \cos\theta$
0	2
$\dfrac{\pi}{6}$	$1 + \dfrac{\sqrt{3}}{2} \approx 1.9$
$\dfrac{\pi}{3}$	$\dfrac{3}{2}$
$\dfrac{\pi}{2}$	1
$\dfrac{2\pi}{3}$	$\dfrac{1}{2}$
$\dfrac{5\pi}{6}$	$1 - \dfrac{\sqrt{3}}{2} \approx 0.1$
π	0

Use substitution to find the point(s) of intersection:

$1 + \sin\theta = 1 + \cos\theta$

$\sin\theta = \cos\theta$

$\dfrac{\sin\theta}{\cos\theta} = 1$

$\tan\theta = 1$

$\theta = \dfrac{\pi}{4}, \dfrac{5\pi}{4}$ for $0 \le \theta < 2\pi$

If $\theta = \dfrac{\pi}{4}$, $r = 1 + \sin\dfrac{\pi}{4} = 1 + \dfrac{\sqrt{2}}{2} \approx 1.7$.

If $\theta = \dfrac{5\pi}{4}$, $r = 1 + \sin\dfrac{5\pi}{4} = 1 - \dfrac{\sqrt{2}}{2} \approx 0.3$.

The points of intersection are $\left(1 + \dfrac{\sqrt{2}}{2}, \dfrac{\pi}{4}\right)$ and $\left(1 - \dfrac{\sqrt{2}}{2}, \dfrac{5\pi}{4}\right)$.

66. $r = 1 + \cos\theta$

The graph will be a limaçon without an inner loop. Check for symmetry:

Polar axis: Replace θ by $-\theta$. The result is $r = 1 + \cos(-\theta) = 1 + \cos\theta$. The graph is symmetric with respect to the polar axis.

The line $\theta = \dfrac{\pi}{2}$: Replace θ by $\pi - \theta$.

$r = 1 + \cos(\pi - \theta)$

$= 1 + \left[\cos(\pi)\cos\theta + \sin(\pi)\sin\theta\right]$

$= 1 + (-\cos\theta + 0)$

$= 1 - \cos\theta$

The test fails.

The pole: Replace r by $-r$. $-r = 1 + \cos\theta$.

The test fails.

Due to symmetry, assign values to θ from 0 to π.

θ	$r = 1 + \cos\theta$
0	2
$\dfrac{\pi}{6}$	$1 + \dfrac{\sqrt{3}}{2} \approx 1.9$
$\dfrac{\pi}{3}$	$\dfrac{3}{2}$
$\dfrac{\pi}{2}$	1
$\dfrac{2\pi}{3}$	$\dfrac{1}{2}$
$\dfrac{5\pi}{6}$	$1 - \dfrac{\sqrt{3}}{2} \approx 0.1$
π	0

$r = 3\cos\theta$

The equation is of the form $r = \pm 2a\cos\theta$, $a > 0$. It is a circle, passing through the pole, and center on the polar axis.

Use substitution to find the point(s) of intersection:

$$1 + \cos\theta = 3\cos\theta$$
$$1 = 2\cos\theta$$
$$\cos\theta = \frac{1}{2}$$
$$\theta = \frac{\pi}{3}, \frac{5\pi}{3} \text{ for } 0 \le \theta < 2\pi$$

If $\theta = \frac{\pi}{3}$, $r = 1 + \cos\frac{\pi}{3} = 1 + \frac{1}{2} = \frac{3}{2}$.

If $\theta = \frac{5\pi}{3}$, $r = 1 + \cos\frac{5\pi}{3} = 1 + \frac{1}{2} = \frac{3}{2}$.

The points of intersection are $\left(\frac{3}{2}, \frac{\pi}{3}\right)$ and $\left(\frac{3}{2}, \frac{5\pi}{3}\right)$.

67. The graph is a cardioid whose equation is of the form $r = a + b\cos\theta$. The graph contains the point $(6, 0)$, so we have

$$6 = a + b\cos 0$$
$$6 = a + b(1)$$
$$6 = a + b$$

The graph contains the point $\left(3, \frac{\pi}{2}\right)$, so we have

$$3 = a + b\cos\frac{\pi}{2}$$
$$3 = a + b(0)$$
$$3 = a$$

Substituting $a = 3$ into the first equation yields:
$$6 = a + b$$
$$6 = 3 + b$$
$$3 = b$$
Therefore, the graph has equation $r = 3 + 3\cos\theta$.

68. The graph is a cardioid whose equation is of the form $r = a + b\cos\theta$. The graph contains the point $(6, \pi)$, so we have

$$6 = a + b\cos\pi$$
$$6 = a + b(-1)$$
$$6 = a - b$$

The graph contains the point $\left(3, \frac{\pi}{2}\right)$, so we have

$$3 = a + b\cos\frac{\pi}{2}$$
$$3 = a + b(0)$$
$$3 = a$$
Substituting $a = 3$ into the first equation yields:
$$6 = a - b$$
$$6 = 3 - b$$
$$b = -3$$
Therefore, the graph has equation $r = 3 - 3\cos\theta$.

69. The graph is a limaçon without inner loop whose equation is of the form $r = a + b\sin\theta$, where $0 < b < a$. The graph contains the point $(4, 0)$, so we have

$$4 = a + b\sin 0$$
$$4 = a + b(0)$$
$$4 = a$$

The graph contains the point $\left(5, \frac{\pi}{2}\right)$, so we have

$$5 = a + b\sin\frac{\pi}{2}$$
$$5 = a + b(1)$$
$$5 = a + b$$
Substituting $a = 4$ into the second equation yields:
$$5 = a + b$$
$$5 = 4 + b$$
$$1 = b$$
Therefore, the graph has equation $r = 4 + \sin\theta$.

70. The graph is a limaçon with inner loop whose equation is of the form $r = a + b\sin\theta$, where $0 < a < b$. The graph contains the point $(1, 0)$, so we have
$$1 = a + b\sin 0$$
$$1 = a + b(0)$$
$$1 = a$$

The graph contains the point $\left(5, \frac{\pi}{2}\right)$, so we have

$$5 = a + b\sin\frac{\pi}{2}$$
$$5 = a + b(1)$$
$$5 = a + b$$
Substituting $a = 1$ into the second equation yields:
$$5 = a + b$$
$$5 = 1 + b$$
$$4 = b$$
Therefore, the graph has equation $r = 1 + 4\sin\theta$.

71. $r = \dfrac{2}{1 - \cos\theta}$ Check for symmetry:

Polar axis: Replace θ by $-\theta$. The result is

$$r = \frac{2}{1 - \cos(-\theta)} = \frac{2}{1 - \cos\theta}.$$

The graph is symmetric with respect to the polar axis.

The line $\theta = \dfrac{\pi}{2}$: Replace θ by $\pi - \theta$.

$$r = \dfrac{2}{1 - \cos(\pi - \theta)}$$

$$= \dfrac{2}{1 - \left(\cos\pi\cos\theta + \sin\pi\sin\theta\right)}$$

$$= \dfrac{2}{1 - (-\cos\theta + 0)}$$

$$= \dfrac{2}{1 + \cos\theta}$$

The test fails.

The pole: Replace r by $-r$. $\quad -r = \dfrac{2}{1 - \cos\theta}$.

The test fails.

Due to symmetry, assign values to θ from 0 to π.

θ	$r = \dfrac{2}{1 - \cos\theta}$
0	undefined
$\dfrac{\pi}{6}$	$\dfrac{2}{1 - \sqrt{3}/2} \approx 14.9$
$\dfrac{\pi}{3}$	4
$\dfrac{\pi}{2}$	2
$\dfrac{2\pi}{3}$	$\dfrac{4}{3}$
$\dfrac{5\pi}{6}$	$\dfrac{2}{1 + \sqrt{3}/2} \approx 1.1$
π	1
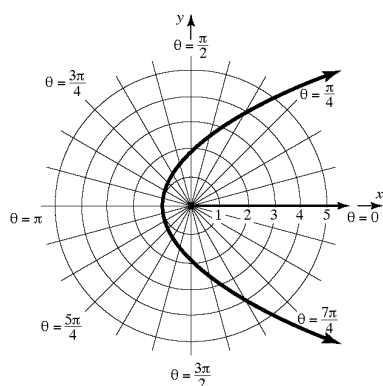
72. $r = \dfrac{2}{1 - 2\cos\theta}$ Check for symmetry:

Polar axis: Replace θ by $-\theta$. The result is

$$r = \dfrac{2}{1 - 2\cos(-\theta)} = \dfrac{2}{1 - 2\cos\theta}.$$ The graph is symmetric with respect to the polar axis.

The line $\theta = \dfrac{\pi}{2}$: Replace θ by $\pi - \theta$.

$$r = \dfrac{2}{1 - 2\cos(\pi - \theta)}$$

$$= \dfrac{2}{1 - 2\left(\cos\pi\cos\theta + \sin\pi\sin\theta\right)}$$

$$= \dfrac{2}{1 - 2(-\cos\theta + 0)} = \dfrac{2}{1 + 2\cos\theta}$$

The test fails.

The pole: Replace r by $-r$. $\quad -r = \dfrac{2}{1 - 2\cos\theta}$.

The test fails.

Due to symmetry, assign values to θ from 0 to π.

θ	$r = \dfrac{2}{1 - 2\cos\theta}$
0	-2
$\dfrac{\pi}{6}$	$\dfrac{2}{1 - \sqrt{3}} \approx -2.7$
$\dfrac{\pi}{3}$	undefined
$\dfrac{\pi}{2}$	2
$\dfrac{2\pi}{3}$	1
$\dfrac{5\pi}{6}$	$\dfrac{2}{1 + \sqrt{3}} \approx 0.7$
π	$\dfrac{2}{3}$
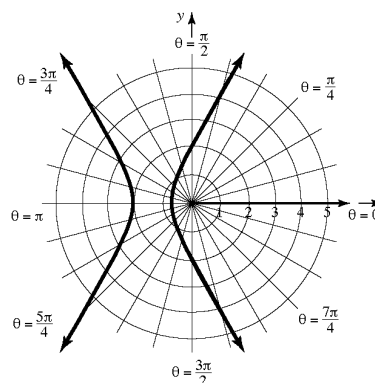
73. $r = \dfrac{1}{3-2\cos\theta}$ Check for symmetry:

Polar axis: Replace θ by $-\theta$. The result is

$r = \dfrac{1}{3-2\cos(-\theta)} = \dfrac{1}{3-2\cos\theta}$. The graph is

symmetric with respect to the polar axis.

The line $\theta = \dfrac{\pi}{2}$: Replace θ by $\pi-\theta$.

$r = \dfrac{1}{3-2\cos(\pi-\theta)}$

$= \dfrac{1}{3-2(\cos\pi\cos\theta + \sin\pi\sin\theta)}$

$= \dfrac{1}{3-2(-\cos\theta+0)} = \dfrac{1}{3+2\cos\theta}$

The test fails.

The pole: Replace r by $-r$. $-r = \dfrac{1}{3-2\cos\theta}$.

The test fails.

Due to symmetry, assign values to θ from 0 to π.

θ	$r = \dfrac{1}{3-2\cos\theta}$
0	1
$\dfrac{\pi}{6}$	$\dfrac{1}{3-\sqrt{3}} \approx 0.8$
$\dfrac{\pi}{3}$	$\dfrac{1}{2}$
$\dfrac{\pi}{2}$	$\dfrac{1}{3}$
$\dfrac{2\pi}{3}$	$\dfrac{1}{4}$
$\dfrac{5\pi}{6}$	$\dfrac{1}{3+\sqrt{3}} \approx 0.2$
π	$\dfrac{1}{5}$
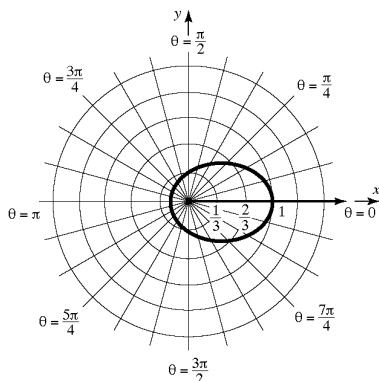
74. $r = \dfrac{1}{1-\cos\theta}$ Check for symmetry:

Polar axis: Replace θ by $-\theta$. The result is

$r = \dfrac{1}{1-\cos(-\theta)} = \dfrac{1}{1-\cos\theta}$. The graph is

symmetric with respect to the polar axis.

The line $\theta = \dfrac{\pi}{2}$: Replace θ by $\pi-\theta$.

$r = \dfrac{1}{1-\cos(\pi-\theta)}$

$= \dfrac{1}{1-(\cos\pi\cos\theta + \sin\pi\sin\theta)}$

$= \dfrac{1}{1-(-\cos\theta+0)}$

$= \dfrac{1}{1+\cos\theta}$

The test fails.

The pole: Replace r by $-r$. $-r = \dfrac{1}{1-\cos\theta}$.

The test fails.

Due to symmetry, assign values to θ from 0 to π.

θ	$r = \dfrac{1}{1-\cos\theta}$
0	undefined
$\dfrac{\pi}{6}$	$\dfrac{1}{1-\sqrt{3}/2} \approx 7.5$
$\dfrac{\pi}{3}$	2
$\dfrac{\pi}{2}$	2
$\dfrac{2\pi}{3}$	$\dfrac{2}{3}$
$\dfrac{5\pi}{6}$	$\dfrac{1}{1+\sqrt{3}/2} \approx 0.5$
π	$\dfrac{1}{2}$

75. $r = \theta,\ \theta \geq 0$

Check for symmetry:

Polar axis: Replace θ by $-\theta$. $r = -\theta$.
The test fails.

The line $\theta = \dfrac{\pi}{2}$: Replace θ by $\pi - \theta$. $r = \pi - \theta$.
The test fails.

The pole: Replace r by $-r$. $-r = \theta$. The test fails.

θ	$r = \theta$
0	0
$\dfrac{\pi}{6}$	$\dfrac{\pi}{6} \approx 0.5$
$\dfrac{\pi}{3}$	$\dfrac{\pi}{3} \approx 1.0$
$\dfrac{\pi}{2}$	$\dfrac{\pi}{2} \approx 1.6$
π	$\pi \approx 3.1$
$\dfrac{3\pi}{2}$	$\dfrac{3\pi}{2} \approx 4.7$
2π	$2\pi \approx 6.3$

76. $r = \dfrac{3}{\theta}$ Check for symmetry:

Polar axis: Replace θ by $-\theta$. $r = \dfrac{3}{-\theta}$. The test fails.

The line $\theta = \dfrac{\pi}{2}$: Replace θ by $\pi - \theta$.

$r = \dfrac{3}{\pi - \theta}$. The test fails.

The pole: Replace r by $-r$. $-r = \dfrac{3}{\theta}$. The test fails.

θ	$r = \dfrac{3}{\theta}$
0	undefined
$\dfrac{\pi}{6}$	$\dfrac{18}{\pi} \approx 5.7$
$\dfrac{\pi}{3}$	$\dfrac{9}{\pi} \approx 2.9$
$\dfrac{\pi}{2}$	$\dfrac{6}{\pi} \approx 1.9$
π	$\dfrac{3}{\pi} \approx 1.0$
$\dfrac{3\pi}{2}$	$\dfrac{2}{\pi} \approx 0.6$
2π	$\dfrac{3}{2\pi} \approx 0.5$

77. $r = \csc\theta - 2 = \dfrac{1}{\sin\theta} - 2,\ \ 0 < \theta < \pi$

Check for symmetry:

Polar axis: Replace θ by $-\theta$.

$r = \csc(-\theta) - 2 = -\csc\theta - 2$. The test fails.

The line $\theta = \dfrac{\pi}{2}$: Replace θ by $\pi - \theta$.

$r = \csc(\pi - \theta) - 2$

$\quad = \dfrac{1}{\sin(\pi - \theta)} - 2$

$\quad = \dfrac{1}{\sin\pi\cos\theta - \cos\pi\sin\theta} - 2$

$\quad = \dfrac{1}{0\cdot\cos\theta - 1\cdot\sin\theta} - 2$

$\quad = \dfrac{1}{\sin\theta} - 2$

$\quad = \csc\theta - 2$

The graph is symmetric with respect to the line

$\theta = \dfrac{\pi}{2}$.

The pole: Replace r by $-r$. $-r = \csc\theta - 2$.
The test fails.

Due to symmetry, assign values to θ from 0 to $\dfrac{\pi}{2}$.

θ	$r = \csc\theta - 2$
0	undefined
$\dfrac{\pi}{6}$	0
$\dfrac{\pi}{4}$	$\sqrt{2} - 2 \approx -0.6$
$\dfrac{\pi}{3}$	$\dfrac{2\sqrt{3}}{3} - 2 \approx -0.8$
$\dfrac{\pi}{2}$	-1

78. $r = \sin\theta\tan\theta$

Check for symmetry:

Polar axis: Replace θ by $-\theta$.

$r = \sin(-\theta)\tan(-\theta)$

$\quad = (-\sin\theta)(-\tan\theta)$

$\quad = \sin\theta\tan\theta$

The graph is symmetric with respect to the polar axis.

The line $\theta = \dfrac{\pi}{2}$: Replace θ by $\pi - \theta$.

$r = \sin(\pi - \theta)\tan(\pi - \theta)$

$\quad = (\sin\pi\cos\theta - \cos\pi\sin\theta)\left(\dfrac{\tan\pi - \tan\theta}{1 + \tan\pi\tan\theta}\right)$

$\quad = \sin\theta \cdot \dfrac{-\tan\theta}{1}$

$\quad = -\sin\theta\tan\theta$

The test fails.

The pole: Replace r by $-r$. $-r = \sin\theta\tan\theta$.
The test fails.

Due to symmetry, assign values to θ from 0 to π.

θ	$r = \sin\theta\tan\theta$
0	0
$\dfrac{\pi}{6}$	$\dfrac{1}{2}\cdot\dfrac{\sqrt{3}}{3} \approx 0.3$
$\dfrac{\pi}{3}$	$\dfrac{3}{2}$
$\dfrac{\pi}{2}$	undefined
$\dfrac{2\pi}{3}$	$-\dfrac{3}{2}$
$\dfrac{5\pi}{6}$	$\dfrac{1}{2}\cdot\left(-\dfrac{\sqrt{3}}{3}\right) \approx -0.3$
π	0

79. $r = \tan\theta, \quad -\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$

Check for symmetry:

Polar axis: Replace θ by $-\theta$.

$r = \tan(-\theta) = -\tan\theta$. The test fails.

The line $\theta = \dfrac{\pi}{2}$: Replace θ by $\pi - \theta$.

$r = \tan(\pi - \theta) = \dfrac{\tan(\pi) - \tan\theta}{1 + \tan(\pi)\tan\theta} = \dfrac{-\tan\theta}{1} = -\tan\theta$

The test fails.

The pole: Replace r by $-r$. $-r = \tan\theta$. The test fails.

θ	$r = \tan\theta$
$-\dfrac{\pi}{3}$	$-\sqrt{3} \approx -1.7$
$-\dfrac{\pi}{4}$	-1
$-\dfrac{\pi}{6}$	$-\dfrac{\sqrt{3}}{3} \approx -0.6$
0	0
$\dfrac{\pi}{6}$	$\dfrac{\sqrt{3}}{3} \approx 0.6$
$\dfrac{\pi}{4}$	1
$\dfrac{\pi}{3}$	$\sqrt{3} \approx 1.7$

80. $r = \cos\dfrac{\theta}{2}$

Check for symmetry:

Polar axis: Replace θ by $-\theta$.

$r = \cos\left(-\dfrac{\theta}{2}\right) = \cos\dfrac{\theta}{2}$. The graph is symmetric with respect to the polar axis.

The line $\theta = \dfrac{\pi}{2}$: Replace θ by $\pi - \theta$.

$r = \cos\left(\dfrac{\pi - \theta}{2}\right) = \cos\left(\dfrac{\pi}{2} - \dfrac{\theta}{2}\right)$

$\quad = \cos\dfrac{\pi}{2}\cdot\cos\dfrac{\theta}{2} + \sin\dfrac{\pi}{2}\cdot\sin\dfrac{\theta}{2}$

$\quad = \sin\dfrac{\theta}{2}$

The test fails.

The pole: Replace r by $-r$. $-r = \cos\dfrac{\theta}{2}$. The test fails.

Due to symmetry, assign values to θ from 0 to π.

θ	$r = \cos\dfrac{\theta}{2}$
0	1
$\dfrac{\pi}{6}$	0.97
$\dfrac{\pi}{3}$	$\dfrac{\sqrt{3}}{2} \approx 0.87$
$\dfrac{\pi}{2}$	$\dfrac{\sqrt{2}}{2} \approx 0.71$
$\dfrac{2\pi}{3}$	$\dfrac{1}{2}$
$\dfrac{5\pi}{6}$	0.26
π	0
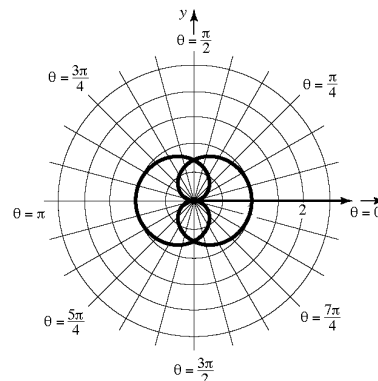
81. Convert the equation to rectangular form:
$$r\sin\theta = a$$
$$y = a$$
The graph of $r\sin\theta = a$ is a horizontal line a units above the pole if $a \geq 0$, and $|a|$ units below the pole if $a < 0$.

82. Convert the equation to rectangular form:
$$r\cos\theta = a$$
$$x = a$$
The graph of $r\cos\theta = a$ is a vertical line a units to the right of the pole if $a \geq 0$ and $|a|$ units to the left of the pole if $a < 0$.

83. Convert the equation to rectangular form:
$$r = 2a\sin\theta, a > 0$$
$$r^2 = 2ar\sin\theta$$
$$x^2 + y^2 = 2ay$$
$$x^2 + y^2 - 2ay = 0$$
$$x^2 + (y-a)^2 = a^2$$
Circle: radius a, center at rectangular coordinates $(0, a)$.

84. Convert the equation to rectangular form:
$$r = -2a\sin\theta, a > 0$$
$$r^2 = -2ar\sin\theta$$
$$x^2 + y^2 = -2ay$$
$$x^2 + y^2 + 2ay = 0$$
$$x^2 + (y+a)^2 = a^2$$
Circle: radius a, center at rectangular coordinates $(0, -a)$.

85. Convert the equation to rectangular form:
$$r = 2a\cos\theta, a > 0$$
$$r^2 = 2ar\cos\theta$$
$$x^2 + y^2 = 2ax$$
$$x^2 - 2ax + y^2 = 0$$
$$(x-a)^2 + y^2 = a^2$$
Circle: radius a, center at rectangular coordinates $(a, 0)$.

86. Convert the equation to rectangular form:
$$r = -2a\cos\theta, a > 0$$
$$r^2 = -2ar\cos\theta$$
$$x^2 + y^2 = -2ax$$
$$x^2 + 2ax + y^2 = 0$$
$$(x+a)^2 + y^2 = a^2$$
Circle: radius a, center at rectangular coordinates $(-a, 0)$.

87. a. $r^2 = \cos\theta$: $r^2 = \cos(\pi - \theta)$
$$r^2 = -\cos\theta$$
Not equivalent; test fails.
$$(-r)^2 = \cos(-\theta)$$
$$r^2 = \cos\theta$$
New test works.

b. $r^2 = \sin\theta$: $r^2 = \sin(\pi - \theta)$
$$r^2 = \sin\theta$$
Test works.
$$(-r)^2 = \sin(-\theta)$$
$$r^2 = -\sin\theta$$
Not equivalent; new test fails.

88. Answers will vary.

89. If an equation passes one or more of these tests, then it will definitely have a graph that is symmetric with respect to the polar axis, the line $\theta = \dfrac{\pi}{2}$, or the pole, depending on the test(s) passed. However, an equation may fail these tests and still have a graph that is symmetric with respect to the polar axis, the line $\theta = \dfrac{\pi}{2}$, or the pole.

90. Answers will vary.

Section 9.3

1. $-4 + 3i$

2. $\sin A \cos B + \cos A \sin B$

3. $\cos A \cos B - \sin A \sin B$

4. $\dfrac{\sqrt{3}}{2}, \ -\dfrac{1}{2}$

5. real; imaginary

6. magnitude, modulus, argument

7. $r_1 r_2 \,; \theta_1 + \theta_2 \,; \theta_1 + \theta_2$

8. $r^n \,; n\theta \,; n\theta$

9. three

10. True

11. $r = \sqrt{x^2 + y^2} = \sqrt{1^2 + 1^2} = \sqrt{2}$

    $\tan \theta = \dfrac{y}{x} = 1$

    $\theta = 45°$

    The polar form of $z = 1 + i$ is

    $z = r(\cos \theta + i \sin \theta) = \sqrt{2}(\cos 45° + i \sin 45°)$
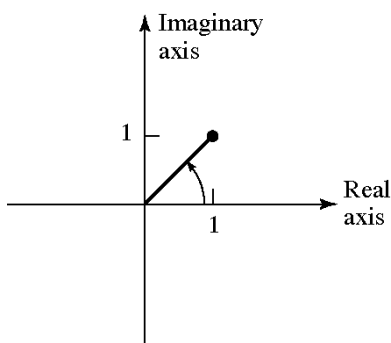
12. $r = \sqrt{x^2 + y^2} = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$

    $\tan \theta = \dfrac{y}{x} = -1$

    $\theta = 135°$

    The polar form of $z = -1 + i$ is

    $z = r(\cos \theta + i \sin \theta) = \sqrt{2}(\cos 135° + i \sin 135°)$
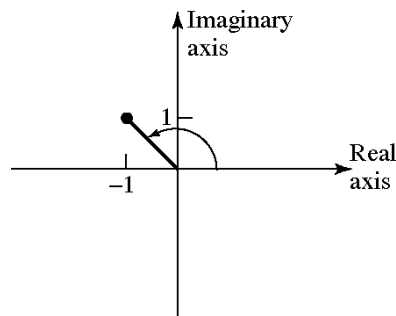
13. $r = \sqrt{x^2 + y^2} = \sqrt{\left(\sqrt{3}\right)^2 + (-1)^2} = \sqrt{4} = 2$

    $\tan \theta = \dfrac{y}{x} = \dfrac{-1}{\sqrt{3}} = -\dfrac{\sqrt{3}}{3}$

    $\theta = 330°$

    The polar form of $z = \sqrt{3} - i$ is

    $z = r(\cos \theta + i \sin \theta) = 2(\cos 330° + i \sin 330°)$
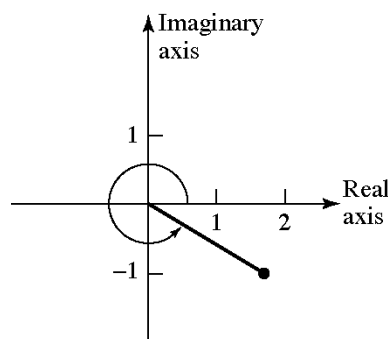
14. $r = \sqrt{x^2 + y^2} = \sqrt{1^2 + \left(-\sqrt{3}\right)^2} = \sqrt{4} = 2$

    $\tan \theta = \dfrac{y}{x} = \dfrac{-\sqrt{3}}{1} = -\sqrt{3}$

    $\theta = 300°$

    The polar form of $z = 1 - \sqrt{3}i$ is

    $z = r(\cos \theta + i \sin \theta) = 2(\cos 300° + i \sin 300°)$
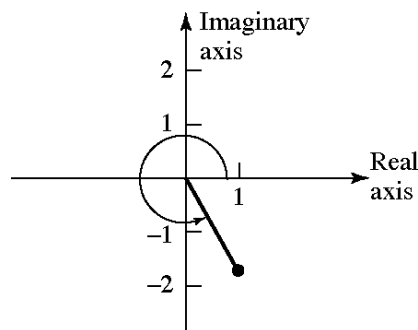
15. $r = \sqrt{x^2 + y^2} = \sqrt{0^2 + (-3)^2} = \sqrt{9} = 3$

$\tan\theta = \dfrac{y}{x} = \dfrac{-3}{0}$

$\theta = 270°$

The polar form of $z = -3i$ is

$z = r(\cos\theta + i\sin\theta) = 3(\cos 270° + i\sin 270°)$

16. $r = \sqrt{x^2 + y^2} = \sqrt{(-2)^2 + 0^2} = \sqrt{4} = 2$

$\tan\theta = \dfrac{y}{x} = \dfrac{0}{-2} = 0$

$\theta = 180°$

The polar form of $z = -2$ is

$z = r(\cos\theta + i\sin\theta) = 2(\cos 180° + i\sin 180°)$

17. $r = \sqrt{x^2 + y^2} = \sqrt{4^2 + (-4)^2} = \sqrt{32} = 4\sqrt{2}$

$\tan\theta = \dfrac{y}{x} = \dfrac{-4}{4} = -1$

$\theta = 315°$

The polar form of $z = 4 - 4i$ is

$z = r(\cos\theta + i\sin\theta) = 4\sqrt{2}(\cos 315° + i\sin 315°)$
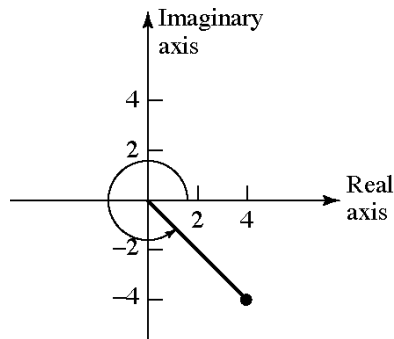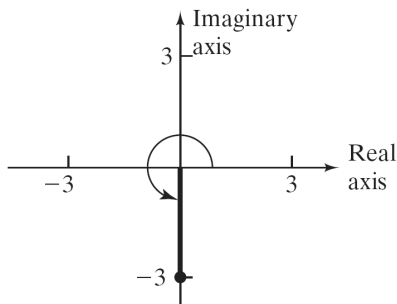
18. $r = \sqrt{x^2 + y^2} = \sqrt{\left(9\sqrt{3}\right)^2 + 9^2} = \sqrt{324} = 18$

$\tan\theta = \dfrac{y}{x} = \dfrac{9}{9\sqrt{3}} = \dfrac{\sqrt{3}}{3}$

$\theta = 30°$

The polar form of $z = 9\sqrt{3} + 9i$ is

$z = r(\cos\theta + i\sin\theta) = 18(\cos 30° + i\sin 30°)$.
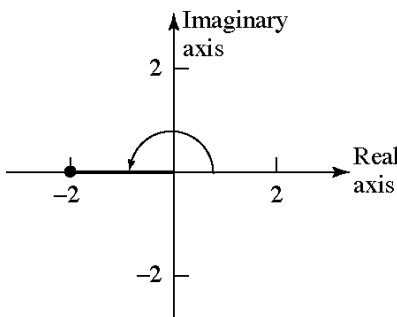
19. $r = \sqrt{x^2 + y^2} = \sqrt{3^2 + (-4)^2} = \sqrt{25} = 5$

$\tan\theta = \dfrac{y}{x} = \dfrac{-4}{3}$

$\theta \approx 306.9°$

The polar form of $z = 3 - 4i$ is

$z = r(\cos\theta + i\sin\theta) = 5(\cos 306.9° + i\sin 306.9°)$
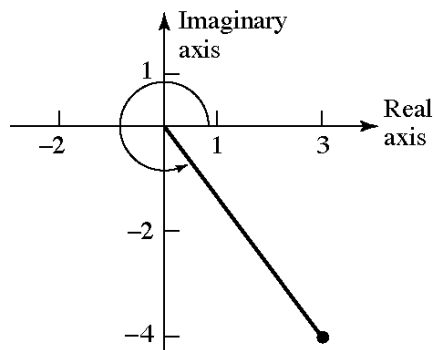
20. $r = \sqrt{x^2 + y^2} = \sqrt{2^2 + \left(\sqrt{3}\right)^2} = \sqrt{7}$

$\tan\theta = \dfrac{y}{x} = \dfrac{\sqrt{3}}{2}$

$\theta \approx 40.9°$

The polar form of $z = 2 + \sqrt{3}\,i$ is

$z = r(\cos\theta + i\sin\theta)$

$\quad = \sqrt{7}\left(\cos 40.9° + i\sin 40.9°\right)$
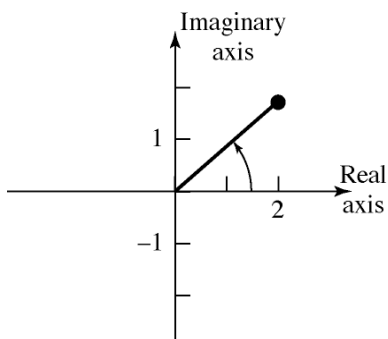
21. $r = \sqrt{x^2 + y^2} = \sqrt{(-2)^2 + 3^2} = \sqrt{13}$

$\tan\theta = \dfrac{y}{x} = \dfrac{3}{-2} = -\dfrac{3}{2}$

$\theta \approx 123.7°$

The polar form of $z = -2 + 3i$ is

$z = r(\cos\theta + i\sin\theta)$

$\quad = \sqrt{13}\left(\cos 123.7° + i\sin 123.7°\right)$
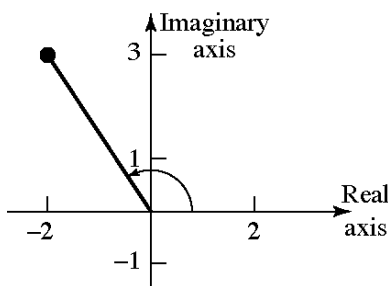
22. $r = \sqrt{x^2 + y^2} = \sqrt{\left(\sqrt{5}\right)^2 + (-1)^2} = \sqrt{6}$

$\tan\theta = \dfrac{y}{x} = \dfrac{-1}{\sqrt{5}} = -\dfrac{\sqrt{5}}{5}$

$\theta \approx 335.9°$

The polar form of $z = \sqrt{5} - i$ is

$z = r(\cos\theta + i\sin\theta)$

$\quad = \sqrt{6}\left(\cos 335.9° + i\sin 335.9°\right)$
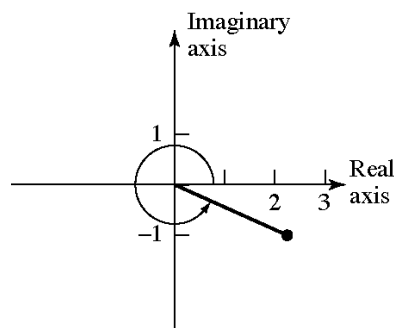
23. $2\left(\cos 120° + i\sin 120°\right) = 2\left(-\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i\right)$

$\qquad\qquad\qquad\qquad\quad = -1 + \sqrt{3}\,i$

24. $3\left(\cos 210° + i\sin 210°\right) = 3\left(-\dfrac{\sqrt{3}}{2} - \dfrac{1}{2}i\right)$

$\qquad\qquad\qquad\qquad\quad = -\dfrac{3\sqrt{3}}{2} - \dfrac{3}{2}i$

25. $4\left(\cos\dfrac{7\pi}{4} + i\sin\dfrac{7\pi}{4}\right) = 4\left(\dfrac{\sqrt{2}}{2} - \dfrac{\sqrt{2}}{2}i\right)$

$\qquad\qquad\qquad\qquad\quad = 2\sqrt{2} - 2\sqrt{2}\,i$

26. $2\left(\cos\dfrac{5\pi}{6} + i\sin\dfrac{5\pi}{6}\right) = 2\left(-\dfrac{\sqrt{3}}{2} + \dfrac{1}{2}i\right) = -\sqrt{3} + i$

27. $3\left(\cos\dfrac{3\pi}{2} + i\sin\dfrac{3\pi}{2}\right) = 3(0 - 1i) = -3i$

28. $4\left(\cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2}\right) = 4(0 + 1i) = 4i$

29. $0.2\left(\cos 100° + i\sin 100°\right) \approx 0.2(-0.1736 + 0.9848i)$

$\qquad\qquad\qquad\qquad\qquad\quad \approx -0.035 + 0.197i$

30. $0.4\left(\cos 200° + i\sin 200°\right) \approx 0.4(-0.9397 - 0.3420i)$

$\qquad\qquad\qquad\qquad\qquad\quad \approx -0.376 - 0.137i$

31. $2\left(\cos\dfrac{\pi}{18} + i\sin\dfrac{\pi}{18}\right) \approx 2(0.9848 + 0.1736i)$

$\qquad\qquad\qquad\qquad\qquad\quad \approx 1.970 + 0.347i$

32. $3\left(\cos\dfrac{\pi}{10} + i\sin\dfrac{\pi}{10}\right) \approx 3(0.9511 + 0.3090i)$

$\qquad\qquad\qquad\qquad\qquad\quad \approx 2.853 + 0.927i$

33. $z\,w = 2\left(\cos 40° + i\sin 40°\right)\cdot 4\left(\cos 20° + i\sin 20°\right)$

$\quad = 2\cdot 4\left[\cos(40° + 20°) + i\sin(40° + 20°)\right]$

$\quad = 8\left(\cos 60° + i\sin 60°\right)$

$\dfrac{z}{w} = \dfrac{2\left(\cos 40° + i\sin 40°\right)}{4\left(\cos 20° + i\sin 20°\right)}$

$\quad = \dfrac{2}{4}\left[\cos(40° - 20°) + i\sin(40° - 20°)\right]$

$\quad = \dfrac{1}{2}\left(\cos 20° + i\sin 20°\right)$

34. $z\,w = \left(\cos 120° + i\sin 120°\right)\cdot\left(\cos 100° + i\sin 100°\right)$

$\quad = \cos(120° + 100°) + i\sin(120° + 100°)$

$\quad = \cos 220° + i\sin 220°$

$\dfrac{z}{w} = \dfrac{\left(\cos 120° + i\sin 120°\right)}{\left(\cos 100° + i\sin 100°\right)}$

$\quad = \cos(120° - 100°) + i\sin(120° - 100°)$

$\quad = \cos 20° + i\sin 20°$

35. $z\,w = 3\left(\cos 130° + i\sin 130°\right)\cdot 4\left(\cos 270° + i\sin 270°\right)$

$\quad = 3\cdot 4\left[\cos(130° + 270°) + i\sin(130° + 270°)\right]$

$\quad = 12\left(\cos 400° + i\sin 400°\right)$

$\quad = 12\left[\cos(400° - 360°) + i\sin(400° - 360°)\right]$

$\quad = 12\left(\cos 40° + i\sin 40°\right)$

$\dfrac{z}{w} = \dfrac{3\left(\cos 130° + i\sin 130°\right)}{4\left(\cos 270° + i\sin 270°\right)}$

$\quad = \dfrac{3}{4}\left[\cos(130° - 270°) + i\sin(130° - 270°)\right]$

$\quad = \dfrac{3}{4}\left(\cos(-140°) + i\sin(-140°)\right)$

$\quad = \dfrac{3}{4}\left[\cos(360° - 140°) + i\sin(360° - 140°)\right]$

$\quad = \dfrac{3}{4}\left(\cos 220° + i\sin 220°\right)$

36. $z\,w = 2\left(\cos 80° + i\sin 80°\right)\cdot 6\left(\cos 200° + i\sin 200°\right)$

$\quad = 2\cdot 6\left[\cos(80° + 200°) + i\sin(80° + 200°)\right]$

$\quad = 12\left(\cos 280° + i\sin 280°\right)$

$\dfrac{z}{w} = \dfrac{2\left(\cos 80° + i\sin 80°\right)}{6\left(\cos 200° + i\sin 200°\right)}$

$\quad = \dfrac{2}{6}\left[\cos(80° - 200°) + i\sin(80° - 200°)\right]$

$\quad = \dfrac{1}{3}\left[\cos(-120°) + i\sin(-120°)\right]$

$\quad = \dfrac{1}{3}\left[\cos(360° - 120°) + i\sin(360° - 120°)\right]$

$\quad = \dfrac{1}{3}\left(\cos 240° + i\sin 240°\right)$

37. $z\,w = 2\left(\cos\dfrac{\pi}{8} + i\sin\dfrac{\pi}{8}\right)\cdot 2\left(\cos\dfrac{\pi}{10} + i\sin\dfrac{\pi}{10}\right)$

$\quad = 2\cdot 2\left[\cos\left(\dfrac{\pi}{8} + \dfrac{\pi}{10}\right) + i\sin\left(\dfrac{\pi}{8} + \dfrac{\pi}{10}\right)\right]$

$\quad = 4\left(\cos\dfrac{9\pi}{40} + i\sin\dfrac{9\pi}{40}\right)$

$\dfrac{z}{w} = \dfrac{2\left(\cos\dfrac{\pi}{8} + i\sin\dfrac{\pi}{8}\right)}{2\left(\cos\dfrac{\pi}{10} + i\sin\dfrac{\pi}{10}\right)}$

$\quad = \dfrac{2}{2}\left[\cos\left(\dfrac{\pi}{8} - \dfrac{\pi}{10}\right) + i\sin\left(\dfrac{\pi}{8} - \dfrac{\pi}{10}\right)\right]$

$\quad = \cos\dfrac{\pi}{40} + i\sin\dfrac{\pi}{40}$

38. $z\,w = 4\left(\cos\dfrac{3\pi}{8} + i\sin\dfrac{3\pi}{8}\right)\cdot 2\left(\cos\dfrac{9\pi}{16} + i\sin\dfrac{9\pi}{16}\right)$

$\quad = 4\cdot 2\left[\cos\left(\dfrac{3\pi}{8} + \dfrac{9\pi}{16}\right) + i\sin\left(\dfrac{3\pi}{8} + \dfrac{9\pi}{16}\right)\right]$

$\quad = 8\left(\cos\dfrac{15\pi}{16} + i\sin\dfrac{15\pi}{16}\right)$

$\dfrac{z}{w} = \dfrac{4\left(\cos\dfrac{3\pi}{8} + i\sin\dfrac{3\pi}{8}\right)}{2\left(\cos\dfrac{9\pi}{16} + i\sin\dfrac{9\pi}{16}\right)}$

$\quad = \dfrac{4}{2}\left[\cos\left(\dfrac{3\pi}{8} - \dfrac{9\pi}{16}\right) + i\sin\left(\dfrac{3\pi}{8} - \dfrac{9\pi}{16}\right)\right]$

$\quad = 2\left[\cos\left(-\dfrac{3\pi}{16}\right) + i\sin\left(-\dfrac{3\pi}{16}\right)\right]$

$\quad = 2\left[\cos\left(2\pi - \dfrac{3\pi}{16}\right) + i\sin\left(2\pi - \dfrac{3\pi}{16}\right)\right]$

$\quad = 2\left(\cos\dfrac{29\pi}{16} + i\sin\dfrac{29\pi}{16}\right)$

39. $z = 2 + 2i$

$r = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$

$\tan \theta = \dfrac{2}{2} = 1$

$\quad \theta = 45°$

$z = 2\sqrt{2}\left(\cos 45° + i\sin 45°\right)$

$w = \sqrt{3} - i$

$r = \sqrt{\left(\sqrt{3}\right)^2 + \left(-1\right)^2} = \sqrt{4} = 2$

$\tan \theta = \dfrac{-1}{\sqrt{3}} = -\dfrac{\sqrt{3}}{3}$

$\quad \theta = 330°$

$w = 2\left(\cos 330° + i\sin 330°\right)$

$zw = 2\sqrt{2}\left(\cos 45° + i\sin 45°\right) \cdot 2\left(\cos 330° + i\sin 330°\right)$

$\quad = 2\sqrt{2} \cdot 2\left[\cos\left(45° + 330°\right) + i\sin\left(45° + 330°\right)\right]$

$\quad = 4\sqrt{2}\left(\cos 375° + i\sin 375°\right)$

$\quad = 4\sqrt{2}\left[\cos\left(375° - 360°\right) + i\sin\left(375° - 360°\right)\right]$

$\quad = 4\sqrt{2}\left(\cos 15° + i\sin 15°\right)$

$\dfrac{z}{w} = \dfrac{2\sqrt{2}\left(\cos 45° + i\sin 45°\right)}{2\left(\cos 330° + i\sin 330°\right)}$

$\quad = \dfrac{2\sqrt{2}}{2}\left[\cos\left(45° - 330°\right) + i\sin\left(45° - 330°\right)\right]$

$\quad = \sqrt{2}\left[\cos\left(-285°\right) + i\sin\left(-285°\right)\right]$

$\quad = \sqrt{2}\left[\cos\left(360° - 285°\right) + i\sin\left(360° - 285°\right)\right]$

$\quad = \sqrt{2}\left(\cos 75° + i\sin 75°\right)$

40. $z = 1 - i$

$r = \sqrt{1^2 + \left(-1\right)^2} = \sqrt{2}$

$\tan \theta = \dfrac{-1}{1} = -1$

$\quad \theta = 315°$

$z = \sqrt{2}\left(\cos 315° + i\sin 315°\right)$

$w = 1 - \sqrt{3}\,i$

$r = \sqrt{1^2 + \left(-\sqrt{3}\right)^2} = \sqrt{4} = 2$

$\tan \theta = \dfrac{-\sqrt{3}}{1} = -\sqrt{3}$

$\quad \theta = 300°$

$w = 2\left(\cos 300° + i\sin 300°\right)$

$zw = \sqrt{2}\left(\cos 315° + i\sin 315°\right) \cdot 2\left(\cos 300° + i\sin 300°\right)$

$\quad = \sqrt{2} \cdot 2\left[\cos\left(315° + 300°\right) + i\sin\left(315° + 300°\right)\right]$

$\quad = 2\sqrt{2}\left(\cos 615° + i\sin 615°\right)$

$\quad = 2\sqrt{2}\left[\cos\left(615° - 360°\right) + i\sin\left(615° - 360°\right)\right]$

$\quad = 2\sqrt{2}\left(\cos 255° + i\sin 255°\right)$

$\dfrac{z}{w} = \dfrac{\sqrt{2}\left(\cos 315° + i\sin 315°\right)}{2\left(\cos 300° + i\sin 300°\right)}$

$\quad = \dfrac{\sqrt{2}}{2}\left[\cos\left(315° - 300°\right) + i\sin\left(315° - 300°\right)\right]$

$\quad = \dfrac{\sqrt{2}}{2}\left(\cos 15° + i\sin 15°\right)$

41. $\left[4\left(\cos 40° + i\sin 40°\right)\right]^3$

$\quad = 4^3\left[\cos\left(3 \cdot 40°\right) + i\sin\left(3 \cdot 40°\right)\right]$

$\quad = 64\left(\cos 120° + i\sin 120°\right)$

$\quad = 64\left(-\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}\,i\right)$

$\quad = -32 + 32\sqrt{3}\,i$

42. $\left[3\left(\cos 80° + i\sin 80°\right)\right]^3$

$\quad = 3^3\left[\cos(3 \cdot 80°) + i\sin(3 \cdot 80°)\right]$

$\quad = 27\left(\cos 240° + i\sin 240°\right)$

$\quad = 27\left(-\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}\,i\right)$

$\quad = -\dfrac{27}{2} - \dfrac{27\sqrt{3}}{2}\,i$

43. $\left[2\left(\cos \dfrac{\pi}{10} + i\sin \dfrac{\pi}{10}\right)\right]^5$

$\quad = 2^5\left[\cos\left(5 \cdot \dfrac{\pi}{10}\right) + i\sin\left(5 \cdot \dfrac{\pi}{10}\right)\right]$

$\quad = 32\left(\cos \dfrac{\pi}{2} + i\sin \dfrac{\pi}{2}\right)$

$\quad = 32\left(0 + 1\,i\right)$

$\quad = 32\,i$

44. $\left[\sqrt{2}\left(\cos\dfrac{5\pi}{16}+i\sin\dfrac{5\pi}{16}\right)\right]^{4}$

$=\left(\sqrt{2}\right)^{4}\left[\cos\left(4\cdot\dfrac{5\pi}{16}\right)+i\sin\left(4\cdot\dfrac{5\pi}{16}\right)\right]$

$=4\left(\cos\dfrac{5\pi}{4}+i\sin\dfrac{5\pi}{4}\right)$

$=4\left(-\dfrac{\sqrt{2}}{2}-\dfrac{\sqrt{2}}{2}i\right)$

$=-2\sqrt{2}-2\sqrt{2}\,i$

45. $\left[\sqrt{3}\left(\cos10°+i\sin10°\right)\right]^{6}$

$=\left(\sqrt{3}\right)^{6}\left[\cos(6\cdot10°)+i\sin(6\cdot10°)\right]$

$=27\left(\cos60°+i\sin60°\right)$

$=27\left(\dfrac{1}{2}+\dfrac{\sqrt{3}}{2}i\right)$

$=\dfrac{27}{2}+\dfrac{27\sqrt{3}}{2}i$

46. $\left[\dfrac{1}{2}\cdot\left(\cos72°+i\sin72°\right)\right]^{5}$

$=\left(\dfrac{1}{2}\right)^{5}\left[\cos(5\cdot72°)+i\sin(5\cdot72°)\right]$

$=\dfrac{1}{32}\cdot\left(\cos360°+i\sin360°\right)$

$=\dfrac{1}{32}\cdot\left(1+0\,i\right)$

$=\dfrac{1}{32}$

47. $\left[\sqrt{5}\left(\cos\dfrac{3\pi}{16}+i\sin\dfrac{3\pi}{16}\right)\right]^{4}$

$=\left(\sqrt{5}\right)^{4}\left[\cos\left(4\cdot\dfrac{3\pi}{16}\right)+i\sin\left(4\cdot\dfrac{3\pi}{16}\right)\right]$

$=25\left(\cos\dfrac{3\pi}{4}+i\sin\dfrac{3\pi}{4}\right)$

$=25\left(-\dfrac{\sqrt{2}}{2}+\dfrac{\sqrt{2}}{2}i\right)$

$=-\dfrac{25\sqrt{2}}{2}+\dfrac{25\sqrt{2}}{2}i$

48. $\left[\sqrt{3}\left(\cos\dfrac{5\pi}{18}+i\sin\dfrac{5\pi}{18}\right)\right]^{6}$

$=\left(\sqrt{3}\right)^{6}\left[\cos\left(6\cdot\dfrac{5\pi}{18}\right)+i\sin\left(6\cdot\dfrac{5\pi}{18}\right)\right]$

$=27\left(\cos\dfrac{5\pi}{3}+i\sin\dfrac{5\pi}{3}\right)$

$=27\left(\dfrac{1}{2}-\dfrac{\sqrt{3}}{2}i\right)$

$=\dfrac{27}{2}-\dfrac{27\sqrt{3}}{2}i$

49. $1-i$

$r=\sqrt{1^{2}+(-1)^{2}}=\sqrt{2}$

$\tan\theta=\dfrac{-1}{1}=-1$

$\theta=\dfrac{7\pi}{4}$

$1-i=\sqrt{2}\left(\cos\dfrac{7\pi}{4}+i\sin\dfrac{7\pi}{4}\right)$

$(1-i)^{5}=\left[\sqrt{2}\left(\cos\dfrac{7\pi}{4}+i\sin\dfrac{7\pi}{4}\right)\right]^{5}$

$=\left(\sqrt{2}\right)^{5}\left[\cos\left(5\cdot\dfrac{7\pi}{4}\right)+i\sin\left(5\cdot\dfrac{7\pi}{4}\right)\right]$

$=4\sqrt{2}\left(\cos\dfrac{35\pi}{4}+i\sin\dfrac{35\pi}{4}\right)$

$=4\sqrt{2}\left(-\dfrac{\sqrt{2}}{2}+\dfrac{\sqrt{2}}{2}i\right)$

$=-4+4\,i$

50. $\sqrt{3}-i$

$r=\sqrt{\left(\sqrt{3}\right)^{2}+(-1)^{2}}=\sqrt{4}=2$

$\tan\theta=\dfrac{-1}{\sqrt{3}}=-\dfrac{\sqrt{3}}{3}$

$\theta=330°$

$\sqrt{3}-i=2\left(\cos330°+i\sin330°\right)$

$\left(\sqrt{3}-i\right)^{6}=\left[2\left(\cos330°+i\sin330°\right)\right]^{6}$

$=2^{6}\left[\cos\left(6\cdot330°\right)+i\sin\left(6\cdot330°\right)\right]$

$=64\left(\cos1980°+i\sin1980°\right)$

$=64\left(-1+0\,i\right)$

$=-64$

51. $\sqrt{2}-i$

$$r = \sqrt{\left(\sqrt{2}\right)^2 + (-1)^2} = \sqrt{3}$$

$$\tan\theta = \frac{-1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$$

$$\theta \approx 324.736°$$

$$\sqrt{2} - i \approx \sqrt{3}\left(\cos 324.736° + i\sin 324.736°\right)$$

$$\left(\sqrt{2} - i\right)^6 \approx \left[\sqrt{3}\left(\cos 324.736° + i\sin 324.736°\right)\right]^6$$

$$= \left(\sqrt{3}\right)^6 \left[\cos\left(6 \cdot 324.736°\right) + i\sin\left(6 \cdot 324.736°\right)\right]$$

$$= 27\left(\cos 1948.416° + i\sin 1948.416°\right)$$

$$\approx 27\left(-0.8519 + 0.5237i\right)$$

$$\approx -23 + 14.142i$$

52. $1-\sqrt{5}i$

$$r = \sqrt{1^2 + \left(-\sqrt{5}\right)^2} = \sqrt{6}$$

$$\tan\theta = \frac{-\sqrt{5}}{1} = -\sqrt{5}$$

$$\theta \approx 294.095°$$

$$1 - \sqrt{5}\,i \approx \sqrt{6}\left(\cos 294.095° + i\sin 294.095°\right)$$

$$\left(1 - \sqrt{5}\,i\right)^8$$

$$\approx \left(\sqrt{6}\right)^8 \left[\cos\left(8 \cdot 294.095°\right) + i\sin\left(8 \cdot 294.095°\right)\right]$$

$$= 1296\left[\cos 2352.76° + i\sin 2352.76°\right]$$

$$\approx 1296\left(-0.9753 - 0.2208i\right)$$

$$\approx -1264 - 286.217i$$

53. $1+i$

$$r = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\tan\theta = \frac{1}{1} = 1$$

$$\theta = 45°$$

$$1 + i = \sqrt{2}\left(\cos 45° + i\sin 45°\right)$$

The three complex cube roots of

$1 + i = \sqrt{2}\left(\cos 45° + i\sin 45°\right)$ are:

$$z_k = \sqrt[3]{\sqrt{2}}\left[\cos\left(\frac{45°}{3} + \frac{360° k}{3}\right) + i\sin\left(\frac{45°}{3} + \frac{360° k}{3}\right)\right]$$

$$= \sqrt[6]{2}\left[\cos\left(15° + 120° k\right) + i\sin\left(15° + 120° k\right)\right]$$

$$z_0 = \sqrt[6]{2}\left[\cos\left(15° + 120° \cdot 0\right) + i\sin\left(15° + 120° \cdot 0\right)\right]$$

$$= \sqrt[6]{2}\left(\cos 15° + i\sin 15°\right)$$

$$z_1 = \sqrt[6]{2}\left[\cos\left(15° + 120° \cdot 1\right) + i\sin\left(15° + 120° \cdot 1\right)\right]$$

$$= \sqrt[6]{2}\left(\cos 135° + i\sin 135°\right)$$

$$z_2 = \sqrt[6]{2}\left[\cos\left(15° + 120° \cdot 2\right) + i\sin\left(15° + 120° \cdot 2\right)\right]$$

$$= \sqrt[6]{2}\left(\cos 255° + i\sin 255°\right)$$

54. $\sqrt{3}-i$

$$r = \sqrt{\left(\sqrt{3}\right)^2 + (-1)^2} = \sqrt{4} = 2$$

$$\tan\theta = \frac{-1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

$$\theta = 330°$$

$$\sqrt{3} - i = 2\left(\cos 330° + i\sin 330°\right)$$

The four complex fourth roots of

$\sqrt{3} - i = 2\left(\cos 330° + i\sin 330°\right)$ are:

$$z_k = \sqrt[4]{2}\left[\cos\left(\frac{330°}{4} + \frac{360° k}{4}\right) + i\sin\left(\frac{330°}{4} + \frac{360° k}{4}\right)\right]$$

$$= \sqrt[4]{2}\left[\cos\left(82.5° + 90° k\right) + i\sin\left(82.5° + 90° k\right)\right]$$

$$z_0 = \sqrt[4]{2}\left[\cos\left(82.5° + 90° \cdot 0\right) + i\sin\left(82.5° + 90° \cdot 0\right)\right]$$

$$= \sqrt[4]{2}\left[\cos\left(82.5°\right) + i\sin\left(82.5°\right)\right]$$

$$z_1 = \sqrt[4]{2}\left[\cos\left(82.5° + 90° \cdot 1\right) + i\sin\left(82.5° + 90° \cdot 1\right)\right]$$

$$= \sqrt[4]{2}\left[\cos\left(172.5°\right) + i\sin\left(172.5°\right)\right]$$

$$z_2 = \sqrt[4]{2}\left[\cos\left(82.5° + 90° \cdot 2\right) + i\sin\left(82.5° + 90° \cdot 2\right)\right]$$

$$= \sqrt[4]{2}\left[\cos\left(262.5°\right) + i\sin\left(262.5°\right)\right]$$

$$z_3 = \sqrt[4]{2}\left[\cos\left(82.5° + 90° \cdot 3\right) + i\sin\left(82.5° + 90° \cdot 3\right)\right]$$

$$= \sqrt[4]{2}\left[\cos\left(352.5°\right) + i\sin\left(352.5°\right)\right]$$

55. $4-4\sqrt{3}i$

$$r = \sqrt{4^2 + \left(-4\sqrt{3}\right)^2} = \sqrt{64} = 8$$

$$\tan\theta = \frac{-4\sqrt{3}}{4} = -\sqrt{3}$$

$$\theta = 300°$$

$$4 - 4\sqrt{3}\,i = 8\left(\cos 300° + i\sin 300°\right)$$

The four complex fourth roots of

$4 - 4\sqrt{3}\,i = 8\left(\cos 300° + i\sin 300°\right)$ are:

$z_k = \sqrt[4]{8}\left[\cos\left(\dfrac{300°}{4} + \dfrac{360° k}{4}\right) + i\sin\left(\dfrac{300°}{4} + \dfrac{360° k}{4}\right)\right]$

$\quad = \sqrt[4]{8}\left[\cos\left(75° + 90° k\right) + i\sin\left(75° + 90° k\right)\right]$

$z_0 = \sqrt[4]{8}\left[\cos\left(75° + 90° \cdot 0\right) + i\sin\left(75° + 90° \cdot 0\right)\right]$

$\quad = \sqrt[4]{8}\left(\cos 75° + i\sin 75°\right)$

$z_1 = \sqrt[4]{8}\left[\cos\left(75° + 90° \cdot 1\right) + i\sin\left(75° + 90° \cdot 1\right)\right]$

$\quad = \sqrt[4]{8}\left(\cos 165° + i\sin 165°\right)$

$z_2 = \sqrt[4]{8}\left[\cos\left(75° + 90° \cdot 2\right) + i\sin\left(75° + 90° \cdot 2\right)\right]$

$\quad = \sqrt[4]{8}\left(\cos 255° + i\sin 255°\right)$

$z_3 = \sqrt[4]{8}\left[\cos\left(75° + 90° \cdot 3\right) + i\sin\left(75° + 90° \cdot 3\right)\right]$

$\quad = \sqrt[4]{8}\left(\cos 345° + i\sin 345°\right)$

56. $-8 - 8i$

$r = \sqrt{(-8)^2 + (-8)^2} = 8\sqrt{2}$

$\tan\theta = \dfrac{-8}{-8} = 1$

$\theta = 225°$

$-8 - 8i = 8\sqrt{2}\left(\cos 225° + i\sin 225°\right)$

The three complex cube roots of

$-8 - 8i = 8\sqrt{2}\left(\cos 225° + i\sin 225°\right)$ are:

$z_k = \sqrt[3]{8\sqrt{2}}\left[\cos\left(\dfrac{225°}{3} + \dfrac{360° k}{3}\right) + i\sin\left(\dfrac{225°}{3} + \dfrac{360° k}{3}\right)\right]$

$\quad = 2\sqrt[6]{2}\left[\cos\left(75° + 120° k\right) + i\sin\left(75° + 120° k\right)\right]$

$z_0 = 2\sqrt[6]{2}\left[\cos\left(75° + 120° \cdot 0\right) + i\sin\left(75° + 120° \cdot 0\right)\right]$

$\quad = 2\sqrt[6]{2}\left(\cos 75° + i\sin 75°\right)$

$z_1 = 2\sqrt[6]{2}\left[\cos\left(75° + 120° \cdot 1\right) + i\sin\left(75° + 120° \cdot 1\right)\right]$

$\quad = 2\sqrt[6]{2}\left(\cos 195° + i\sin 195°\right)$

$z_2 = 2\sqrt[6]{2}\left[\cos\left(75° + 120° \cdot 2\right) + i\sin\left(75° + 120° \cdot 2\right)\right]$

$\quad = 2\sqrt[6]{2}\left(\cos 315° + i\sin 315°\right)$

57. $-16i$

$r = \sqrt{0^2 + (-16)^2} = \sqrt{256} = 16$

$\tan\theta = \dfrac{-16}{0}$

$\theta = 270°$

$-16i = 16\left(\cos 270° + i\sin 270°\right)$

The four complex fourth roots of

$-16i = 16\left(\cos 270° + i\sin 270°\right)$ are:

$z_k = \sqrt[4]{16}\left[\cos\left(\dfrac{270°}{4} + \dfrac{360° k}{4}\right) + i\sin\left(\dfrac{270°}{4} + \dfrac{360° k}{4}\right)\right]$

$\quad = 2\left[\cos\left(67.5° + 90° k\right) + i\sin\left(67.5° + 90° k\right)\right]$

$z_0 = 2\left[\cos\left(67.5° + 90° \cdot 0\right) + i\sin\left(67.5° + 90° \cdot 0\right)\right]$

$\quad = 2\left(\cos 67.5° + i\sin 67.5°\right)$

$z_1 = 2\left[\cos\left(67.5° + 90° \cdot 1\right) + i\sin\left(67.5° + 90° \cdot 1\right)\right]$

$\quad = 2\left(\cos 157.5° + i\sin 157.5°\right)$

$z_2 = 2\left[\cos\left(67.5° + 90° \cdot 2\right) + i\sin\left(67.5° + 90° \cdot 2\right)\right]$

$\quad = 2\left(\cos 247.5° + i\sin 247.5°\right)$

$z_3 = 2\left[\cos\left(67.5° + 90° \cdot 3\right) + i\sin\left(67.5° + 90° \cdot 3\right)\right]$

$\quad = 2\left(\cos 337.5° + i\sin 337.5°\right)$

58. -8

$r = \sqrt{(-8)^2 + 0^2} = 8$

$\tan\theta = \dfrac{0}{-8} = 0$

$\theta = 180°$

$-8 = 8\left(\cos 180° + i\sin 180°\right)$

The three complex cube roots of

$-8 = 8\left(\cos 180° + i\sin 180°\right)$ are:

$z_k = \sqrt[3]{8}\left[\cos\left(\dfrac{180°}{3} + \dfrac{360° k}{3}\right) + i\sin\left(\dfrac{180°}{3} + \dfrac{360° k}{3}\right)\right]$

$\quad = 2\left[\cos\left(60° + 120° k\right) + i\sin\left(60° + 120° k\right)\right]$

$z_0 = 2\left[\cos\left(60° + 120° \cdot 0\right) + i\sin\left(60° + 120° \cdot 0\right)\right]$

$\quad = 2\left(\cos 60° + i\sin 60°\right)$

$z_1 = 2\left[\cos\left(60° + 120° \cdot 1\right) + i\sin\left(60° + 120° \cdot 1\right)\right]$

$\quad = 2\left(\cos 180° + i\sin 180°\right)$

$z_2 = 2\left[\cos\left(60° + 120° \cdot 2\right) + i\sin\left(60° + 120° \cdot 2\right)\right]$

$\quad = 2\left(\cos 300° + i\sin 300°\right)$

59. i

$r = \sqrt{0^2 + 1^2} = \sqrt{1} = 1$

$\tan\theta = \dfrac{1}{0}$

$\theta = 90°$

$i = 1(\cos 90° + i\sin 90°)$

The five complex fifth roots of

$i = 1\left(\cos 90° + i\sin 90°\right)$ are:

$$z_k = \sqrt[5]{1}\left[\cos\left(\frac{90°}{5}+\frac{360°k}{5}\right)+i\sin\left(\frac{90°}{5}+\frac{360°k}{5}\right)\right]$$

$$= 1\left[\cos(18°+72°k)+i\sin(18°+72°k)\right]$$

$$z_0 = 1\left[\cos\left(18°+72°\cdot 0\right)+i\sin\left(18°+72°\cdot 0\right)\right]$$

$$= \cos 18°+i\sin 18°$$

$$z_1 = 1\left[\cos\left(18°+72°\cdot 1\right)+i\sin\left(18°+72°\cdot 1\right)\right]$$

$$= \cos 90°+i\sin 90°$$

$$z_2 = 1\left[\cos\left(18°+72°\cdot 2\right)+i\sin\left(18°+72°\cdot 2\right)\right]$$

$$= \cos 162°+i\sin 162°$$

$$z_3 = 1\left[\cos\left(18°+72°\cdot 3\right)+i\sin\left(18°+72°\cdot 3\right)\right]$$

$$= \cos 234°+i\sin 234°$$

$$z_4 = 1\left[\cos\left(18°+72°\cdot 4\right)+i\sin\left(18°+72°\cdot 4\right)\right]$$

$$= \cos 306°+i\sin 306°$$

60. $-i$

$$r = \sqrt{0^2+(-1)^2}=\sqrt{1}=1$$

$$\tan\theta = \frac{-1}{0} \;\Rightarrow\; \theta = 270°$$

$$-i = 1(\cos 270°+i\sin 270°)$$

The five complex fifth roots of
$-i = 1\left(\cos 270°+i\sin 270°\right)$ are:

$$z_k = \sqrt[5]{1}\left[\cos\left(\frac{270°}{5}+\frac{360°k}{5}\right)+i\sin\left(\frac{270°}{5}+\frac{360°k}{5}\right)\right]$$

$$= 1\left[\cos(54°+72°k)+i\sin(54°+72°k)\right]$$

$$z_0 = 1\left[\cos(54°+72°\cdot 0)+i\sin(54°+72°\cdot 0)\right]$$

$$= \cos 54°+i\sin 54°$$

$$z_1 = 1\left[\cos(54°+72°\cdot 1)+i\sin(54°+72°\cdot 1)\right]$$

$$= \cos 126°+i\sin 126°$$

$$z_2 = 1\left[\cos(54°+72°\cdot 2)+i\sin(54°+72°\cdot 2)\right]$$

$$= \cos 198°+i\sin 198°$$

$$z_3 = 1\left[\cos(54°+72°\cdot 3)+i\sin(54°+72°\cdot 3)\right]$$

$$= \cos 270°+i\sin 270°$$

$$z_4 = 1\left[\cos(54°+72°\cdot 4)+i\sin(54°+72°\cdot 4)\right]$$

$$= \cos 342°+i\sin 342°$$

61. $1 = 1+0i$

$$r = \sqrt{1^2+0^2}=\sqrt{1}=1$$

$$\tan\theta = \frac{0}{1}=0$$

$$\theta = 0°$$

$$1+0i = 1\left(\cos 0°+i\sin 0°\right)$$

The four complex fourth roots of unity are:

$$z_k = \sqrt[4]{1}\left[\cos\left(\frac{0°}{4}+\frac{360°k}{4}\right)+i\sin\left(\frac{0°}{4}+\frac{360°k}{4}\right)\right]$$

$$= 1\left[\cos\left(90°k\right)+i\sin\left(90°k\right)\right]$$

$$z_0 = \cos\left(90°\cdot 0\right)+i\sin\left(90°\cdot 0\right)$$

$$= \cos 0°+i\sin 0°=1+0i=1$$

$$z_1 = \cos\left(90°\cdot 1\right)+i\sin\left(90°\cdot 1\right)$$

$$= \cos 90°+i\sin 90°=0+1i=i$$

$$z_2 = \cos\left(90°\cdot 2\right)+i\sin\left(90°\cdot 2\right)$$

$$= \cos 180°+i\sin 180°=-1+0i=-1$$

$$z_3 = \cos\left(90°\cdot 3\right)+i\sin\left(90°\cdot 3\right)$$

$$= \cos 270°+i\sin 270°=0-1i=-i$$

The complex fourth roots of unity are:
$1, i, -1, -i$.
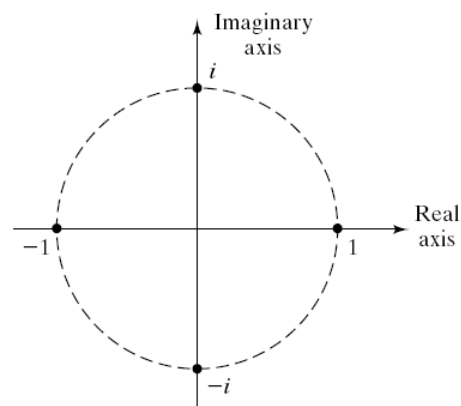
62. $1 = 1+0i$

$$r = \sqrt{1^2+0^2}=\sqrt{1}=1$$

$$\tan\theta = \frac{0}{1}=0$$

$$\theta = 0°$$

$$1+0i = 1\left(\cos 0°+i\sin 0°\right)$$

The six complex sixth roots of unity are:

$$z_k = \sqrt[6]{1}\left[\cos\left(\frac{0°}{6}+\frac{360°k}{6}\right)+i\sin\left(\frac{0°}{6}+\frac{360°k}{6}\right)\right]$$

$$= 1\left[\cos\left(60°k\right)+i\sin\left(60°k\right)\right]$$

$$z_0 = \cos(60° \cdot 0) + i\sin(60° \cdot 0)$$
$$= \cos 0° + i\sin 0° = 1 + 0i = 1$$
$$z_1 = \cos(60° \cdot 1) + i\sin(60° \cdot 1)$$
$$= \cos 60° + i\sin 60° = \frac{1}{2} + \frac{\sqrt{3}}{2}i$$
$$z_2 = \cos(60° \cdot 2) + i\sin(60° \cdot 2)$$
$$= \cos 120° + i\sin 120° = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$
$$z_3 = \cos(60° \cdot 3) + i\sin(60° \cdot 3)$$
$$= \cos 180° + i\sin 180° = -1 + 0i = -1$$
$$z_4 = \cos(60° \cdot 4) + i\sin(60° \cdot 4)$$
$$= \cos 240° + i\sin 240° = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$
$$z_5 = \cos(60° \cdot 5) + i\sin(60° \cdot 5)$$
$$= \cos 300° + i\sin 300° = \frac{1}{2} - \frac{\sqrt{3}}{2}i$$

The complex sixth roots of unity are:
$$1, \frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} + \frac{\sqrt{3}}{2}i, -1, -\frac{1}{2} - \frac{\sqrt{3}}{2}i, \frac{1}{2} - \frac{\sqrt{3}}{2}i \ .$$
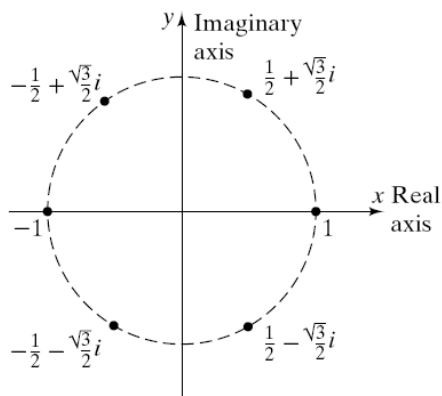
63. Let $w = r(\cos\theta + i\sin\theta)$ be a complex number. If $w \neq 0$, there are n distinct nth roots of w, given by the formula:

$$z_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)\right],$$

where $k = 0, 1, 2, \ldots, n-1$

$|z_k| = \sqrt[n]{r}$ for all k

64. Since $|z_k| = \sqrt[n]{r}$ for all k, each of the complex nth roots lies on a circle with center at the origin and radius $\sqrt[n]{|w|} = \sqrt[n]{r}$, where w is the original complex number.

65. Examining the formula for the distinct complex nth roots of the complex number $w = r(\cos\theta + i\sin\theta)$,

$$z_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)\right], \text{ where } k = 0, 1, 2, \ldots, n-1, \text{ we see that the } z_k \text{ are spaced apart by an}$$

angle of $\dfrac{2\pi}{n}$.

66. Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$. Then

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}$$

$$= \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} \cdot \frac{(\cos\theta_2 - i\sin\theta_2)}{(\cos\theta_2 - i\sin\theta_2)}$$

$$= \frac{r_1}{r_2} \cdot \frac{\cos\theta_1 \cdot \cos\theta_2 - i\cos\theta_1 \cdot \sin\theta_2 + i\sin\theta_1 \cdot \cos\theta_2 + \sin\theta_1 \cdot \sin\theta_2}{\cos^2\theta_2 + \sin^2\theta_2}$$

$$= \frac{r_1}{r_2} \cdot \frac{\cos\theta_1 \cdot \cos\theta_2 + \sin\theta_1 \cdot \sin\theta_2 + i(\sin\theta_1 \cdot \cos\theta_2 - \cos\theta_1 \cdot \sin\theta_2)}{1}$$

$$= \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right]$$

67. a.

$z = a_0$	a_1	a_2	a_3	a_4	a_5	a_6
$0.1 - 0.4i$	$-0.05 - 0.48i$	$-0.13 - 0.35i$	$-0.01 - 0.31i$	$0.01 - 0.35i$	$0.2 - 0.41i$	$0.07 - 0.42i$
$0.5 + 0.8i$	$0.11 + 1.6i$	$-2.05 + 1.15i$	$3.37 - 3.92i$	$-3.52 - 25.6i$	$-641.9 + 181.1i$	$379290 - 232509i$
$-0.9 + 0.7i$	$-0.58 - 0.56i$	$0.87 + 1.35i$	$-1.95 - 1.67i$	$0.13 + 7.21i$	$-52.88 + 2.56i$	$2788.5 - 269.6i$
$-1.1 + 0.1i$	$0.1 - 0.12i$	$-1.10 + 0.76i$	$0.11 - 0.068i$	$-1.09 + 0.085i$	$0.085 - 0.85i$	$-1.10 + 0.086i$
$0 - 1.3i$	$-1.69 - 1.3i$	$1.17 + 3.09i$	$-8.21 + 5.92i$	$32.46 - 98.47i$	$-8643.6 - 6393.7i$	$33833744 + 110529134.4i$
$1 + 1i$	$1 + 3i$	$-7 + 7i$	$1 - 97i$	$-9407 - 193i$	$88454401 + 3631103i$	$7.8 \times 10^{15} + 6.4 \times 10^{14} i$

b. z_1 and z_4 are in the Mandlebrot set. a_6 for the complex numbers not in the set have very large components.

c.

z	$|z|$	$|a_6|$
$0.1 - 0.4i$	0.4	0.4
$0.5 + 0.8i$	0.9	444884
$-0.9 + 0.7i$	1.1	2802
$-1.1 + 0.1i$	1.1	1.1
$0 - 1.3i$	1.3	115591573
$1 + 1i$	1.4	7.8×10^{15}

The numbers which are not in the Mandlebrot set satisfy this condition. The numbers which are in the Mandlebrot set satisfy the condition $|a_n| \le 2$.

Section 9.4

1. vector

2. **0**

3. unit

4. position

5. horizontal, vertical

6. resultant

7. True

8. False

9. $\mathbf{v} + \mathbf{w}$

10. $\mathbf{u} + \mathbf{v}$

11. $3\mathbf{v}$

12. $4\mathbf{w}$

NOT TO SCALE
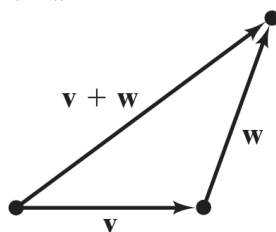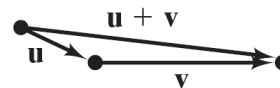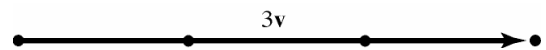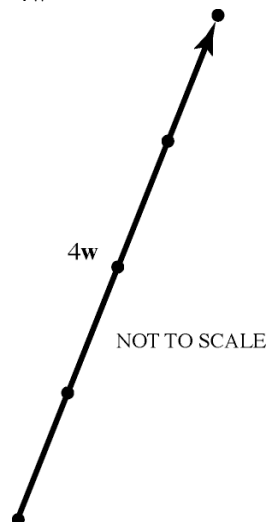
13. $\mathbf{v} - \mathbf{w}$
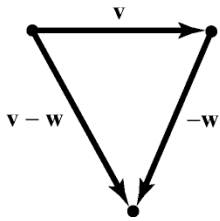
14. $\mathbf{u} - \mathbf{v}$
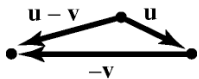
15. $3\mathbf{v} + \mathbf{u} - 2\mathbf{w}$
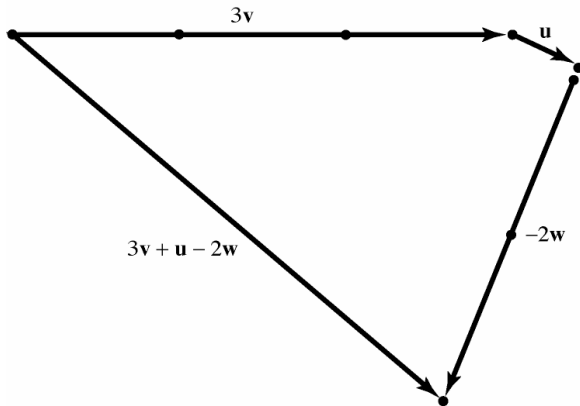
16. $2\mathbf{u} - 3\mathbf{v} + \mathbf{w}$
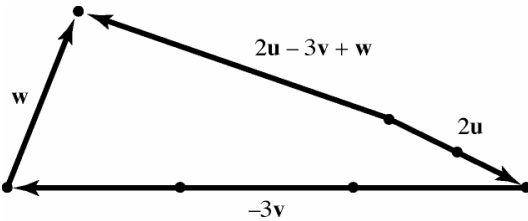
17. True

18. False $\mathbf{K} + \mathbf{G} = -\mathbf{F}$

19. False $\mathbf{C} = -\mathbf{F} + \mathbf{E} - \mathbf{D}$

20. True

21. False $\mathbf{D} - \mathbf{E} = \mathbf{H} + \mathbf{G}$

22. False $\mathbf{C} + \mathbf{H} = -\mathbf{G} - \mathbf{F}$

23. True

24. True

25. If $\|\mathbf{v}\| = 4$, then $\|3\mathbf{v}\| = |3| \cdot \|\mathbf{v}\| = 3 \cdot 4 = 12$.

26. If $\|\mathbf{v}\| = 2$, then $\|-4\mathbf{v}\| = |-4| \cdot \|\mathbf{v}\| = 4 \cdot 2 = 8$

27. $P = (0, 0)$, $Q = (3, 4)$
$\mathbf{v} = (3 - 0)\mathbf{i} + (4 - 0)\mathbf{j} = 3\mathbf{i} + 4\mathbf{j}$

28. $P = (0, 0)$, $Q = (-3, -5)$
$\mathbf{v} = (-3 - 0)\mathbf{i} + (-5 - 0)\mathbf{j} = -3\mathbf{i} - 5\mathbf{j}$

29. $P = (3, 2)$, $Q = (5, 6)$
$\mathbf{v} = (5 - 3)\mathbf{i} + (6 - 2)\mathbf{j} = 2\mathbf{i} + 4\mathbf{j}$

30. $P = (-3, 2)$, $Q = (6, 5)$
$\mathbf{v} = [6 - (-3)]\mathbf{i} + (5 - 2)\mathbf{j} = 9\mathbf{i} + 3\mathbf{j}$

31. $P = (-2, -1)$, $Q = (6, -2)$
$\mathbf{v} = [6 - (-2)]\mathbf{i} + [-2 - (-1)]\mathbf{j} = 8\mathbf{i} - \mathbf{j}$

32. $P = (-1, 4)$, $Q = (6, 2)$
$\mathbf{v} = [6 - (-1)]\mathbf{i} + (2 - 4)\mathbf{j} = 7\mathbf{i} - 2\mathbf{j}$

33. $P = (1, 0)$, $Q = (0, 1)$
$\mathbf{v} = (0 - 1)\mathbf{i} + (1 - 0)\mathbf{j} = -\mathbf{i} + \mathbf{j}$

34. $P = (1, 1)$, $Q = (2, 2)$
$\mathbf{v} = (2 - 1)\mathbf{i} + (2 - 1)\mathbf{j} = \mathbf{i} + \mathbf{j}$

35. For $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$, $\|\mathbf{v}\| = \sqrt{3^2 + (-4)^2} = \sqrt{25} = 5$.

36. For $\mathbf{v} = -5\mathbf{i} + 12\mathbf{j}$,
$\|\mathbf{v}\| = \sqrt{(-5)^2 + 12^2} = \sqrt{169} = 13$.

37. For $\mathbf{v} = \mathbf{i} - \mathbf{j}$, $\|\mathbf{v}\| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$.

38. For $\mathbf{v} = -\mathbf{i} - \mathbf{j}$, $\|\mathbf{v}\| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$.

39. For $\mathbf{v} = -2\mathbf{i} + 3\mathbf{j}$, $\|\mathbf{v}\| = \sqrt{(-2)^2 + 3^2} = \sqrt{13}$.

40. For $\mathbf{v} = 6\mathbf{i} + 2\mathbf{j}$, $\|\mathbf{v}\| = \sqrt{6^2 + 2^2} = \sqrt{40} = 2\sqrt{10}$.

41. $2\mathbf{v} + 3\mathbf{w} = 2(3\mathbf{i} - 5\mathbf{j}) + 3(-2\mathbf{i} + 3\mathbf{j})$
$= 6\mathbf{i} - 10\mathbf{j} - 6\mathbf{i} + 9\mathbf{j}$
$= -\mathbf{j}$

42. $3\mathbf{v} - 2\mathbf{w} = 3(3\mathbf{i} - 5\mathbf{j}) - 2(-2\mathbf{i} + 3\mathbf{j})$
$= 9\mathbf{i} - 15\mathbf{j} + 4\mathbf{i} - 6\mathbf{j}$
$= 13\mathbf{i} - 21\mathbf{j}$

43. $\| \mathbf{v} - \mathbf{w} \| = \| (3\mathbf{i} - 5\mathbf{j}) - (-2\mathbf{i} + 3\mathbf{j}) \|$
$= \| 5\mathbf{i} - 8\mathbf{j} \|$
$= \sqrt{5^2 + (-8)^2}$
$= \sqrt{89}$

44. $\| \mathbf{v} + \mathbf{w} \| = \| (3\mathbf{i} - 5\mathbf{j}) + (-2\mathbf{i} + 3\mathbf{j}) \|$
$= \| \mathbf{i} - 2\mathbf{j} \| = \sqrt{1^2 + (-2)^2} = \sqrt{5}$

45. $\| \mathbf{v} \| - \| \mathbf{w} \| = \| 3\mathbf{i} - 5\mathbf{j} \| - \| -2\mathbf{i} + 3\mathbf{j} \|$
$= \sqrt{3^2 + (-5)^2} - \sqrt{(-2)^2 + 3^2}$
$= \sqrt{34} - \sqrt{13}$

46. $\| \mathbf{v} \| + \| \mathbf{w} \| = \| 3\mathbf{i} - 5\mathbf{j} \| + \| -2\mathbf{i} + 3\mathbf{j} \|$
$= \sqrt{3^2 + (-5)^2} + \sqrt{(-2)^2 + 3^2}$
$= \sqrt{34} + \sqrt{13}$

47. $\mathbf{u} = \dfrac{\mathbf{v}}{\| \mathbf{v} \|} = \dfrac{5\mathbf{i}}{\| 5\mathbf{i} \|} = \dfrac{5\mathbf{i}}{\sqrt{25 + 0}} = \dfrac{5\mathbf{i}}{5} = \mathbf{i}$

48. $\mathbf{u} = \dfrac{\mathbf{v}}{\| \mathbf{v} \|} = \dfrac{-3\mathbf{j}}{\| -3\mathbf{j} \|} = \dfrac{-3\mathbf{j}}{\sqrt{0 + 9}} = \dfrac{-3\mathbf{j}}{3} = -\mathbf{j}$

49. $\mathbf{u} = \dfrac{\mathbf{v}}{\| \mathbf{v} \|} = \dfrac{3\mathbf{i} - 4\mathbf{j}}{\| 3\mathbf{i} - 4\mathbf{j} \|} = \dfrac{3\mathbf{i} - 4\mathbf{j}}{\sqrt{3^2 + (-4)^2}}$
$= \dfrac{3\mathbf{i} - 4\mathbf{j}}{\sqrt{25}}$
$= \dfrac{3\mathbf{i} - 4\mathbf{j}}{5}$
$= \dfrac{3}{5}\mathbf{i} - \dfrac{4}{5}\mathbf{j}$

50. $\mathbf{u} = \dfrac{\mathbf{v}}{\| \mathbf{v} \|} = \dfrac{-5\mathbf{i} + 12\mathbf{j}}{\| -5\mathbf{i} + 12\mathbf{j} \|} = \dfrac{-5\mathbf{i} + 12\mathbf{j}}{\sqrt{(-5)^2 + 12^2}}$
$= \dfrac{-5\mathbf{i} + 12\mathbf{j}}{\sqrt{169}}$
$= \dfrac{-5\mathbf{i} + 12\mathbf{j}}{13}$
$= -\dfrac{5}{13}\mathbf{i} + \dfrac{12}{13}\mathbf{j}$

51. $\mathbf{u} = \dfrac{\mathbf{v}}{\| \mathbf{v} \|} = \dfrac{\mathbf{i} - \mathbf{j}}{\| \mathbf{i} - \mathbf{j} \|} = \dfrac{\mathbf{i} - \mathbf{j}}{\sqrt{1^2 + (-1)^2}} = \dfrac{\mathbf{i} - \mathbf{j}}{\sqrt{2}}$
$= \dfrac{1}{\sqrt{2}}\mathbf{i} - \dfrac{1}{\sqrt{2}}\mathbf{j}$
$= \dfrac{\sqrt{2}}{2}\mathbf{i} - \dfrac{\sqrt{2}}{2}\mathbf{j}$

52. $\mathbf{u} = \dfrac{\mathbf{v}}{\| \mathbf{v} \|} = \dfrac{2\mathbf{i} - \mathbf{j}}{\| 2\mathbf{i} - \mathbf{j} \|} = \dfrac{2\mathbf{i} - \mathbf{j}}{\sqrt{2^2 + (-1)^2}} = \dfrac{2\mathbf{i} - \mathbf{j}}{\sqrt{5}}$
$= \dfrac{2}{\sqrt{5}}\mathbf{i} - \dfrac{1}{\sqrt{5}}\mathbf{j}$
$= \dfrac{2\sqrt{5}}{5}\mathbf{i} - \dfrac{\sqrt{5}}{5}\mathbf{j}$

53. Let $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$. We want $\| \mathbf{v} \| = 4$ and $a = 2b$.
$\| \mathbf{v} \| = \sqrt{a^2 + b^2} = \sqrt{(2b)^2 + b^2} = \sqrt{5b^2}$
$\sqrt{5b^2} = 4$
$5b^2 = 16$
$b^2 = \dfrac{16}{5}$
$b = \pm\sqrt{\dfrac{16}{5}} = \pm\dfrac{4}{\sqrt{5}} = \pm\dfrac{4\sqrt{5}}{5}$
$a = 2b = 2\left(\pm\dfrac{4\sqrt{5}}{5} \right) = \pm\dfrac{8\sqrt{5}}{5}$
$\mathbf{v} = \dfrac{8\sqrt{5}}{5}\mathbf{i} + \dfrac{4\sqrt{5}}{5}\mathbf{j}$ or $\mathbf{v} = -\dfrac{8\sqrt{5}}{5}\mathbf{i} - \dfrac{4\sqrt{5}}{5}\mathbf{j}$

54. Let $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$. We want $\| \mathbf{v} \| = 3$ and $a = b$.
$\| \mathbf{v} \| = \sqrt{a^2 + b^2} = \sqrt{b^2 + b^2} = \sqrt{2b^2}$
$\sqrt{2b^2} = 3$
$2b^2 = 9$
$b^2 = \dfrac{9}{2}$
$b = \pm\sqrt{\dfrac{9}{2}} = \pm\dfrac{3}{\sqrt{2}} = \pm\dfrac{3\sqrt{2}}{2}$
$a = b = \pm\dfrac{3\sqrt{2}}{2}$
$\mathbf{v} = \dfrac{3\sqrt{2}}{2}\mathbf{i} + \dfrac{3\sqrt{2}}{2}\mathbf{j}$ or $\mathbf{v} = -\dfrac{3\sqrt{2}}{2}\mathbf{i} - \dfrac{3\sqrt{2}}{2}\mathbf{j}$

55. $\mathbf{v} = 2\mathbf{i} - \mathbf{j}, \ \mathbf{w} = x\mathbf{i} + 3\mathbf{j}, \ \|\mathbf{v} + \mathbf{w}\| = 5$

$\|\mathbf{v} + \mathbf{w}\| = \|2\mathbf{i} - \mathbf{j} + x\mathbf{i} + 3\mathbf{j}\|$

$= \|(2 + x)\mathbf{i} + 2\mathbf{j}\|$

$= \sqrt{(2 + x)^2 + 2^2}$

$= \sqrt{x^2 + 4x + 4 + 4}$

$= \sqrt{x^2 + 4x + 8}$

Solve for x:

$\sqrt{x^2 + 4x + 8} = 5$

$x^2 + 4x + 8 = 25$

$x^2 + 4x - 17 = 0$

$x = \dfrac{-4 \pm \sqrt{16 - 4(1)(-17)}}{2(1)}$

$= \dfrac{-4 \pm \sqrt{84}}{2}$

$= \dfrac{-4 \pm 2\sqrt{21}}{2}$

$= -2 \pm \sqrt{21}$

The solution set is $\{-2 + \sqrt{21}, -2 - \sqrt{21}\}$.

56. $P = (-3, 1), Q = (x, 4),$

$\mathbf{v} = [x - (-3)]\mathbf{i} + (4 - 1)\mathbf{j} = (x + 3)\mathbf{i} + 3\mathbf{j}$

$\|\mathbf{v}\| = \sqrt{(x + 3)^2 + 3^2}$

$= \sqrt{x^2 + 6x + 9 + 9}$

$= \sqrt{x^2 + 6x + 18}$

Solve for x:

$\sqrt{x^2 + 6x + 18} = 5$

$x^2 + 6x + 18 = 25$

$x^2 + 6x - 7 = 0$

$(x + 7)(x - 1) = 0$

$x = -7 \ \text{ or } \ x = 1$

The solution set is $\{-7, 1\}$.

57. $\|\mathbf{v}\| = 5, \ \alpha = 60°$

$\mathbf{v} = \|\mathbf{v}\|(\cos\alpha\mathbf{i} + \sin\alpha\mathbf{j})$

$= 5(\cos(60°)\mathbf{i} + \sin(60°)\mathbf{j})$

$= 5\left(\dfrac{1}{2}\mathbf{i} + \dfrac{\sqrt{3}}{2}\mathbf{j}\right)$

$= \dfrac{5}{2}\mathbf{i} + \dfrac{5\sqrt{3}}{2}\mathbf{j}$

58. $\|\mathbf{v}\| = 8, \ \alpha = 45°$

$\mathbf{v} = \|\mathbf{v}\|(\cos\alpha\mathbf{i} + \sin\alpha\mathbf{j})$

$= 8\left[\cos(45°)\mathbf{i} + \sin(45°)\mathbf{j}\right]$

$= 8\left(\dfrac{\sqrt{2}}{2}\mathbf{i} + \dfrac{\sqrt{2}}{2}\mathbf{j}\right)$

$= 4\sqrt{2}\mathbf{i} + 4\sqrt{2}\mathbf{j}$

59. $\|\mathbf{v}\| = 14, \ \alpha = 120°$

$\mathbf{v} = \|\mathbf{v}\|(\cos\alpha\mathbf{i} + \sin\alpha\mathbf{j})$

$= 14(\cos(120°)\mathbf{i} + \sin(120°)\mathbf{j})$

$= 14\left(-\dfrac{1}{2}\mathbf{i} + \dfrac{\sqrt{3}}{2}\mathbf{j}\right)$

$= -7\mathbf{i} + 7\sqrt{3}\mathbf{j}$

60. $\|\mathbf{v}\| = 3, \ \alpha = 240°$

$\mathbf{v} = \|\mathbf{v}\|(\cos\alpha\mathbf{i} + \sin\alpha\mathbf{j})$

$= 3\left[\cos(240°)\mathbf{i} + \sin(240°)\mathbf{j}\right]$

$= 3\left(-\dfrac{1}{2}\mathbf{i} - \dfrac{\sqrt{3}}{2}\mathbf{j}\right)$

$= -\dfrac{3}{2}\mathbf{i} - \dfrac{3\sqrt{3}}{2}\mathbf{j}$

61. $\|\mathbf{v}\| = 25, \ \alpha = 330°$

$\mathbf{v} = \|\mathbf{v}\|(\cos\alpha\mathbf{i} + \sin\alpha\mathbf{j})$

$= 25\left[\cos(330°)\mathbf{i} + \sin(330°)\mathbf{j}\right]$

$= 25\left(\dfrac{\sqrt{3}}{2}\mathbf{i} - \dfrac{1}{2}\mathbf{j}\right)$

$= \dfrac{25\sqrt{3}}{2}\mathbf{i} - \dfrac{25}{2}\mathbf{j}$

62. $\|\mathbf{v}\| = 15, \ \alpha = 315°$

$\mathbf{v} = \|\mathbf{v}\|(\cos\alpha\mathbf{i} + \sin\alpha\mathbf{j})$

$= 15(\cos 315°\mathbf{i} + \sin 315°\mathbf{j})$

$= 15\left(\dfrac{\sqrt{2}}{2}\mathbf{i} - \dfrac{\sqrt{2}}{2}\mathbf{j}\right) = \dfrac{15\sqrt{2}}{2}\mathbf{i} - \dfrac{15\sqrt{2}}{2}\mathbf{j}$

63. $\mathbf{v} = 3\mathbf{i} + 3\mathbf{j} = 3(\mathbf{i} + \mathbf{j})$

$= \|\mathbf{v}\|(\cos\alpha\mathbf{i} + \sin\alpha\mathbf{j})$

$\tan\alpha = 1 \quad \alpha = \tan^{-1}(1) = 45°$

The angle is in quadrant I, thus, $\alpha = 45°$.

64. $\mathbf{v} = \mathbf{i} + \sqrt{3}\mathbf{j}$

$= \|\mathbf{v}\|(\cos\alpha\mathbf{i} + \sin\alpha\mathbf{j})$

$\tan\alpha = \sqrt{3} \quad \alpha = \tan^{-1}(\sqrt{3}) = 60°$

The angle is in quadrant I, thus, $\alpha = 60°$.

65. $\mathbf{v} = -3\sqrt{3}\mathbf{i} + 3\mathbf{j} = 3(-\sqrt{3}\mathbf{i} + \mathbf{j})$

$= \|\mathbf{v}\|(\cos\alpha\mathbf{i} + \sin\alpha\mathbf{j})$

$\tan\alpha = -\dfrac{1}{\sqrt{3}} \quad \alpha = \tan^{-1}\left(-\dfrac{1}{\sqrt{3}}\right) = -30°$

The angle is in quadrant II, thus, $\alpha = 150°$.

66. $\mathbf{v} = -5\mathbf{i} - 5\mathbf{j} = 5(-\mathbf{i} - \mathbf{j})$

$= \|\mathbf{v}\|(\cos\alpha\mathbf{i} + \sin\alpha\mathbf{j})$

$\tan\alpha = 1 \quad \alpha = \tan^{-1}(1) = 45°$

The angle is in quadrant III, thus, $\alpha = 225°$.

67. $\mathbf{v} = 4\mathbf{i} - 2\mathbf{j} = 2(2\mathbf{i} - \mathbf{j})$

$= \|\mathbf{v}\|(\cos\alpha\mathbf{i} + \sin\alpha\mathbf{j})$

$\tan\alpha = -\dfrac{1}{2} \quad \alpha = \tan^{-1}\left(-\dfrac{1}{2}\right) = -26.6°$

The angle is in quadrant IV, thus, $\alpha = 333.4°$.

68. $\mathbf{v} = 6\mathbf{i} - 4\mathbf{j} = 2(3\mathbf{i} - 2\mathbf{j})$

$= \|\mathbf{v}\|(\cos\alpha\mathbf{i} + \sin\alpha\mathbf{j})$

$\tan\alpha = -\dfrac{2}{3} \quad \alpha = \tan^{-1}\left(-\dfrac{2}{3}\right) = -33.7°$

The angle is in quadrant IV, thus, $\alpha = 326.3°$.

69. $\mathbf{v} = -\mathbf{i} - 5\mathbf{j}$

$= \|\mathbf{v}\|(\cos\alpha\mathbf{i} + \sin\alpha\mathbf{j})$

$\tan\alpha = 5 \quad \alpha = \tan^{-1}(5) = 78.7°$

The angle is in quadrant III, thus, $\alpha = 258.7°$.

70. $\mathbf{v} = -\mathbf{i} + 3\mathbf{j}$

$= \|\mathbf{v}\|(\cos\alpha\mathbf{i} + \sin\alpha\mathbf{j})$

$\tan\alpha = -3 \quad \alpha = \tan^{-1}(-3) = -71.6°$

The angle is in quadrant II, thus, $\alpha = 108.4°$.

71. a. Let $\mathbf{u} = \langle 3, -1 \rangle$. Then

$\mathbf{u'} = \mathbf{u} + \mathbf{v} = \langle 3, -1 \rangle + \langle -4, 5 \rangle = \langle -1, 4 \rangle$.

The new coordinate will be $(-1, 4)$.

b.
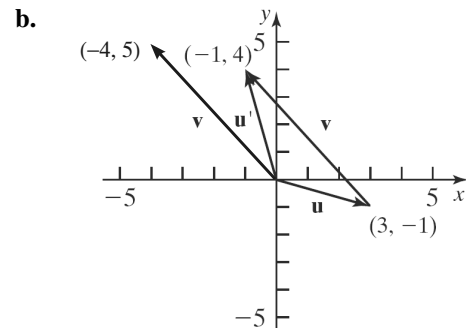
72. a. Let $\mathbf{a} = \langle -3, 0 \rangle$, $\mathbf{b} = \langle -1, -2 \rangle$, $\mathbf{c} = \langle 3, 1 \rangle$, and $\mathbf{d} = \langle 1, 3 \rangle$. Then

$\mathbf{a'} = \mathbf{a} + \mathbf{v} = \langle -3, 0 \rangle + \langle 3, -2 \rangle = \langle 0, -2 \rangle$,

$\mathbf{b'} = \mathbf{b} + \mathbf{v} = \langle -1, -2 \rangle + \langle 3, -2 \rangle = \langle 2, -4 \rangle$,

$\mathbf{c'} = \mathbf{c} + \mathbf{v} = \langle 3, 1 \rangle + \langle 3, -2 \rangle = \langle 6, -1 \rangle$, and

$\mathbf{d'} = \mathbf{d} + \mathbf{v} = \langle 1, 3 \rangle + \langle 3, -2 \rangle = \langle 4, 1 \rangle$.

The vertices of the new parallelogram $A'B'C'D'$ are $(0, -2)$, $(2, -4)$, $(6, -1)$, and $(4, 1)$.

b. $-\dfrac{1}{2}\mathbf{v} = -\dfrac{1}{2}\langle 3, -2 \rangle = \left\langle -\dfrac{3}{2}, 1 \right\rangle$. Then

$\mathbf{a'} = \mathbf{a} - \dfrac{1}{2}\mathbf{v} = \langle -3, 0 \rangle + \left\langle -\dfrac{3}{2}, 1 \right\rangle = \left\langle -\dfrac{9}{2}, 1 \right\rangle$,

$\mathbf{b'} = \mathbf{b} - \dfrac{1}{2}\mathbf{v} = \langle -1, -2 \rangle + \left\langle -\dfrac{3}{2}, 1 \right\rangle = \left\langle -\dfrac{5}{2}, -1 \right\rangle$,

$\mathbf{c'} = \mathbf{c} - \dfrac{1}{2}\mathbf{v} = \langle 3, 1 \rangle + \left\langle -\dfrac{3}{2}, 1 \right\rangle = \left\langle \dfrac{3}{2}, 2 \right\rangle$, and

$\mathbf{d'} = \mathbf{d} - \dfrac{1}{2}\mathbf{v} = \langle 1, 3 \rangle + \left\langle -\dfrac{3}{2}, 1 \right\rangle = \left\langle -\dfrac{1}{2}, 4 \right\rangle$.

The vertices of the new parallelogram $A'B'C'D'$ are $\left(-\dfrac{9}{2}, 1 \right)$, $\left(-\dfrac{5}{2}, -1 \right)$, $\left(\dfrac{3}{2}, 2 \right)$, and $\left(-\dfrac{1}{2}, 4 \right)$.

73. $\mathbf{F} = 40\left[\cos(30°)\mathbf{i} + \sin(30°)\mathbf{j} \right]$

$= 40\left(\dfrac{\sqrt{3}}{2}\mathbf{i} + \dfrac{1}{2}\mathbf{j} \right)$

$= 20\sqrt{3}\,\mathbf{i} + 20\mathbf{j}$

74. $\mathbf{F} = 100\left[\cos(20°)\mathbf{i} + \sin(20°)\mathbf{j} \right]$

75. $\mathbf{F}_1 = 40\left[\cos(30°)\mathbf{i} + \sin(30°)\mathbf{j}\right]$

$= 40\left(\dfrac{\sqrt{3}}{2}\mathbf{i} + \dfrac{1}{2}\mathbf{j}\right) = 20\sqrt{3}\,\mathbf{i} + 20\mathbf{j}$

$\mathbf{F}_2 = 60\left[\cos(-45°)\mathbf{i} + \sin(-45°)\mathbf{j}\right]$

$= 60\left(\dfrac{\sqrt{2}}{2}\mathbf{i} - \dfrac{\sqrt{2}}{2}\mathbf{j}\right) = 30\sqrt{2}\,\mathbf{i} - 30\sqrt{2}\,\mathbf{j}$

$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 = 20\sqrt{3}\,\mathbf{i} + 20\mathbf{j} + 30\sqrt{2}\,\mathbf{i} - 30\sqrt{2}\,\mathbf{j}$

$= \left(20\sqrt{3} + 30\sqrt{2}\right)\mathbf{i} + \left(20 - 30\sqrt{2}\right)\mathbf{j}$

76. $\mathbf{F}_1 = 30\left[\cos(45°)\mathbf{i} + \sin(45°)\mathbf{j}\right]$

$= 30\left(\dfrac{\sqrt{2}}{2}\mathbf{i} + \dfrac{\sqrt{2}}{2}\mathbf{j}\right) = 15\sqrt{2}\,\mathbf{i} + 15\sqrt{2}\,\mathbf{j}$

$\mathbf{F}_2 = 70\left[\cos(120°)\mathbf{i} + \sin(120°)\mathbf{j}\right]$

$= 70\left(-\dfrac{1}{2}\mathbf{i} + \dfrac{\sqrt{3}}{2}\mathbf{j}\right) = -35\mathbf{i} + 35\sqrt{3}\,\mathbf{j}$

$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 = 15\sqrt{2}\,\mathbf{i} + 15\sqrt{2}\,\mathbf{j} + (-35)\mathbf{i} + 35\sqrt{3}\,\mathbf{j}$

$= \left(15\sqrt{2} - 35\right)\mathbf{i} + \left(15\sqrt{2} + 35\sqrt{3}\right)\mathbf{j}$

77. Let $\mathbf{v_a}$ = the velocity of the plane in still air, $\mathbf{v_w}$ = the velocity of the wind, and $\mathbf{v_g}$ = the velocity of the plane relative to the ground.

a. $\mathbf{v}_a = 550\mathbf{j}$

$\mathbf{v}_w = 100(\cos 45°\mathbf{i} + \sin 45°\mathbf{j})$

$= 100\left(\dfrac{\sqrt{2}}{2}\mathbf{i} + \dfrac{\sqrt{2}}{2}\mathbf{j}\right)$

$= 50\sqrt{2}\,\mathbf{i} + 50\sqrt{2}\,\mathbf{j}$

b. $\mathbf{v}_g = \mathbf{v}_a + \mathbf{v}_w$

$= 550\mathbf{j} + 50\sqrt{2}\,\mathbf{i} + 50\sqrt{2}\,\mathbf{j}$

$= 50\sqrt{2}\,\mathbf{i} + \left(550 + 50\sqrt{2}\right)\mathbf{j}$

c. The speed of the plane relative to the ground is:

$\left\|\mathbf{v}_g\right\| = \sqrt{\left(50\sqrt{2}\right)^2 + \left(550 + 50\sqrt{2}\right)^2}$

$= \sqrt{390281.7459} \approx 624.7$

To find the direction, find the angle between $\mathbf{v_g}$ and the x-axis.

$\tan\alpha = \dfrac{550 + 50\sqrt{2}}{50\sqrt{2}}$

$\alpha = 83.5°$

The plane is traveling with a ground speed of 624.7 mph in an approximate direction of 6.5° degrees east of north (*N* 6.5° *E*).

78. Let $\mathbf{v_a}$ = the velocity of the plane in still air, $\mathbf{v_w}$ = the velocity of the wind, and $\mathbf{v_g}$ = the velocity of the plane relative to the ground.

a. $\mathbf{v}_a = -500\mathbf{i}$

$\mathbf{v}_w = 100(\cos 315°\mathbf{i} + \sin 315°\mathbf{j})$

$= 100\left(\dfrac{\sqrt{2}}{2}\mathbf{i} - \dfrac{\sqrt{2}}{2}\mathbf{j}\right)$

$= 50\sqrt{2}\,\mathbf{i} - 50\sqrt{2}\,\mathbf{j}$

b. $\mathbf{v}_g = \mathbf{v}_a + \mathbf{v}_w$

$= -500\mathbf{i} + 50\sqrt{2}\,\mathbf{i} - 50\sqrt{2}\,\mathbf{j}$

$= \left(50\sqrt{2} - 500\right)\mathbf{i} - 50\sqrt{2}\,\mathbf{j}$

c. The speed of the plane relative to the ground is:

$\left\|\mathbf{v}_g\right\| = \sqrt{\left(50\sqrt{2} - 500\right)^2 + \left(-50\sqrt{2}\right)^2}$

$= \sqrt{189289.3219} \approx 435.1$

To find the direction, find the angle between $\mathbf{v_g}$ and the x-axis and consider a convenient vector such as due south.

$\tan\alpha = \dfrac{-50\sqrt{2}}{50\sqrt{2} - 500}$

$\alpha = 9.4°$

The plane is traveling with a ground speed of 435.1 mph in an approximate direction of 80.6° degrees west of south (*S* 80.6° *W*).

79. Let $\mathbf{v_a}$ = the velocity of the plane in still air, $\mathbf{v_w}$ = the velocity of the wind, and $\mathbf{v_g}$ = the velocity of the plane relative to the ground.

$\mathbf{v_g} = \mathbf{v_a} + \mathbf{v_w}$

$$\mathbf{v_a} = 500\left[\cos\left(45°\right)\mathbf{i} + \sin\left(45°\right)\mathbf{j}\right]$$
$$= 500\left(\frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j}\right)$$
$$= 250\sqrt{2}\,\mathbf{i} + 250\sqrt{2}\,\mathbf{j}$$
$$\mathbf{v_w} = 60\left[\cos\left(120°\right)\mathbf{i} + \sin\left(120°\right)\mathbf{j}\right]$$
$$= 60\left(-\frac{1}{2}\mathbf{i} + \frac{\sqrt{3}}{2}\mathbf{j}\right)$$
$$= -30\mathbf{i} + 30\sqrt{3}\,\mathbf{j}$$
$$\mathbf{v_g} = \mathbf{v_a} + \mathbf{v_w}$$
$$= 250\sqrt{2}\,\mathbf{i} + 250\sqrt{2}\,\mathbf{j} - 30\mathbf{i} + 30\sqrt{3}\,\mathbf{j}$$
$$= \left(250\sqrt{2} - 30\right)\mathbf{i} + \left(250\sqrt{2} + 30\sqrt{3}\right)\mathbf{j}$$

The speed of the plane relative to the ground is:
$$\left\|\mathbf{v_g}\right\| = \sqrt{\left(-30 + 250\sqrt{2}\right)^2 + \left(250\sqrt{2} + 30\sqrt{3}\right)^2}$$
$$\approx \sqrt{269,129.1} \approx 518.8 \text{ km/hr}$$

To find the direction, find the angle between $\mathbf{v_g}$ and a convenient vector such as due east, $\mathbf{i}$.
$$\tan\theta = \frac{\mathbf{j}\ component}{\mathbf{i}\ component}$$
$$= \frac{\left(250\sqrt{2} + 30\sqrt{3}\right)}{\left(250\sqrt{2} - 30\right)}$$
$$\approx 1.2533$$
$$\theta \approx 51.5°$$
The plane is traveling with a ground speed of about 518.8 km/hr in a direction of 38.6° east of north $\left(\text{N38.6°E}\right)$.

80. Let $\mathbf{v_a}$ = the velocity of the plane in still air, $\mathbf{v_w}$ = the velocity of the wind, and $\mathbf{v_g}$ = the velocity of the plane relative to the ground.
$$\mathbf{v_g} = \mathbf{v_a} + \mathbf{v_w}$$
$$\mathbf{v_a} = 600\left[\cos\left(60°\right)\mathbf{i} - \sin\left(60°\right)\mathbf{j}\right]$$
$$= 600\left(\frac{1}{2}\mathbf{i} - \frac{\sqrt{3}}{2}\mathbf{j}\right) = 300\mathbf{i} - 300\sqrt{3}\,\mathbf{j}$$
$$\mathbf{v_w} = 40\left[\cos\left(45°\right)\mathbf{i} - \sin\left(45°\right)\mathbf{j}\right]$$
$$= 40\left(\frac{\sqrt{2}}{2}\mathbf{i} - \frac{\sqrt{2}}{2}\mathbf{j}\right)$$
$$= 20\sqrt{2}\,\mathbf{i} - 20\sqrt{2}\,\mathbf{j}$$

$$\mathbf{v_g} = \mathbf{v_a} + \mathbf{v_w}$$
$$= 300\mathbf{i} - 300\sqrt{3}\,\mathbf{j} + 20\sqrt{2}\,\mathbf{i} - 20\sqrt{2}\,\mathbf{j}$$
$$= \left(300 + 20\sqrt{2}\right)\mathbf{i} - \left(300\sqrt{3} + 20\sqrt{2}\right)\mathbf{j}$$
The speed of the plane relative to the ground is:
$$\left\|\mathbf{v_g}\right\| = \sqrt{\left(300 + 20\sqrt{2}\right)^2 + \left(-300\sqrt{3} - 20\sqrt{2}\right)^2}$$
$$\approx \sqrt{407,964} \approx 638.7 \text{ km/hr}$$

To find the direction, find the angle between $\mathbf{v_g}$ and a convenient vector such as due east, $\mathbf{i}$.
$$\tan\theta = \frac{\mathbf{j}\ component}{\mathbf{i}\ component}$$
$$= \frac{\left(-300\sqrt{3} - 20\sqrt{2}\right)}{\left(300 + 20\sqrt{2}\right)}$$
$$\approx -1.6689$$
$$\theta \approx -59.1°$$
The plane is traveling with a ground speed of about 638.7 km/hr in a direction of about 30.9° degrees east of south (S30.9°E).

81. Let $\mathbf{F_1}$ be the force of gravity and $\mathbf{F_2}$ be the force required to hold the weight on the ramp.
Then $\sin 10° = \dfrac{\left\|\mathbf{F_2}\right\|}{\left\|\mathbf{F_1}\right\|}$
$$\sin 10° = \frac{700}{\left\|\mathbf{F_1}\right\|}$$
$$\left\|\mathbf{F_1}\right\| = \frac{700}{\sin 10°}$$
$$= 4031$$

So the combined weight of the boat and its trailer is 4031 lbs.

82. Let $\mathbf{F_1}$ be the force of gravity and $\mathbf{F_2}$ be the force required to hold the weight on the ramp.
Then $\sin 15° = \dfrac{\left\|\mathbf{F_2}\right\|}{\left\|\mathbf{F_1}\right\|}$
$$\sin 15° = \frac{1200}{\left\|\mathbf{F_1}\right\|}$$
$$\left\|\mathbf{F_1}\right\| = \frac{1200}{\sin 15°}$$
$$= 4636$$

So the weight of the car is 4636 lbs.

83. Let the positive *x*-axis point downstream, so that the velocity of the current is $\mathbf{v_c} = 3\mathbf{i}$. Let $\mathbf{v_w} =$ the velocity of the boat in the water, and $\mathbf{v_g} =$ the velocity of the boat relative to the land. Then $\mathbf{v_g} = \mathbf{v_w} + \mathbf{v_c}$ and $\mathbf{v_g} = k$ since the boat is going directly across the river. The speed of the boat is $\|\mathbf{v_w}\| = 20$; we need to find the direction. Let $\mathbf{v_w} = a\mathbf{i} + b\mathbf{j}$, so

$$\|\mathbf{v_w}\| = \sqrt{a^2 + b^2} = 20$$
$$a^2 + b^2 = 400$$

Since $\mathbf{v_g} = \mathbf{v_w} + \mathbf{v_c}$,

$$k\,\mathbf{j} = a\mathbf{i} + b\mathbf{j} + 3\mathbf{i} = (a+3)\mathbf{i} + b\mathbf{j}$$
$$a + 3 = 0$$
$$a = -3$$
$$k = b$$
$$a^2 + b^2 = 400$$
$$9 + b^2 = 400$$
$$b^2 = 391$$
$$k = b = \sqrt{391} \approx 19.8$$
$$\mathbf{v_w} = -3\mathbf{i} + \sqrt{391}\,\mathbf{j} \text{ and } \mathbf{v_g} = \sqrt{391}\,\mathbf{j}$$

Find the angle between $\mathbf{v}_w$ and $\mathbf{j}$:

$$\cos\theta = \frac{\mathbf{v_w} \cdot \mathbf{j}}{\|\mathbf{v_w}\|\|\mathbf{j}\|}$$
$$= \frac{-3(0) + \sqrt{391}(1)}{20\sqrt{0^2 + 1^2}} = \frac{\sqrt{391}}{20} = 0.9887$$
$$\theta \approx 8.6°$$

The heading of the boat needs to be about $8.6°$ upstream. The velocity of the boat directly across the river is about 19.8 kilometers per hour. The time to cross the river is:

$$t = \frac{0.5}{19.8} \approx 0.025 \text{ hours or}$$
$$t = \frac{0.5}{19.8} \cdot 60 \approx 1.52 \text{ minutes}.$$

84. Let $\mathbf{v_a} = $ the velocity of the plane in still air, $\mathbf{v_w} = $ the velocity of the wind, and $\mathbf{v_g} = $ the velocity of the plane relative to the ground.

$$\mathbf{v_g} = \mathbf{v_a} + \mathbf{v_w}$$
$$\mathbf{v_a} = 250(\cos\alpha\,\mathbf{i} + \sin\alpha\,\mathbf{j})$$

$$\mathbf{v_w} = 40\left(\frac{\mathbf{i} - \mathbf{j}}{\|\mathbf{i} - \mathbf{j}\|}\right)$$
$$= 40\left(\frac{\mathbf{i} - \mathbf{j}}{\sqrt{1+1}}\right) = \frac{40}{\sqrt{2}}(\mathbf{i} - \mathbf{j}) = 20\sqrt{2}\,(\mathbf{i} - \mathbf{j})$$
$$\mathbf{v_g} = \mathbf{v_a} + \mathbf{v_w}$$
$$= 250\cos\alpha\,\mathbf{i} + 250\sin\alpha\,\mathbf{j} + 20\sqrt{2}\,\mathbf{i} - 20\sqrt{2}\,\mathbf{j}$$
$$= a\mathbf{i}$$

Examining the $\mathbf{j}$ components:

$$250\sin\alpha - 20\sqrt{2} = 0$$
$$250\sin\alpha = 20\sqrt{2}$$
$$\sin\alpha = \frac{20\sqrt{2}}{250} \approx 0.11314$$
$$\alpha \approx 6.5°$$

The heading of the plane should be about N83.5°E, that is, about $6.5°$ north of east.

Examining the $\mathbf{i}$ components:

$$250\cos(6.5°)\mathbf{i} + 20\sqrt{2}\,\mathbf{i} = a\mathbf{i}$$
$$276.7 \approx a$$

The speed of the plane relative to the ground is about 276.7 miles per hour.

85. Let $\mathbf{F_1}$ be the tension on the left cable and $\mathbf{F_2}$ be the tension on the right cable. Let $\mathbf{F_3}$ represent the force of the weight of the box.

$$\mathbf{F_1} = \|\mathbf{F_1}\|\left[\cos(155°)\mathbf{i} + \sin(155°)\mathbf{j}\right]$$
$$\approx \|\mathbf{F_1}\|(-0.9063\mathbf{i} + 0.4226\mathbf{j})$$
$$\mathbf{F_2} = \|\mathbf{F_2}\|\left[\cos(40°)\mathbf{i} + \sin(40°)\mathbf{j}\right]$$
$$\approx \|\mathbf{F_2}\|(0.7660\mathbf{i} + 0.6428\mathbf{j})$$
$$\mathbf{F_3} = -1000\mathbf{j}$$

For equilibrium, the sum of the force vectors must be zero.

$$\mathbf{F_1} + \mathbf{F_2} + \mathbf{F_3}$$
$$= -0.9063\|\mathbf{F_1}\|\mathbf{i} + 0.4226\|\mathbf{F_1}\|\mathbf{j}$$
$$\quad + 0.7660\|\mathbf{F_2}\|\mathbf{i} + 0.6428\|\mathbf{F_2}\|\mathbf{j} - 1000\mathbf{j}$$
$$= \left(-0.9063\|\mathbf{F_1}\| + 0.7660\|\mathbf{F_2}\|\right)\mathbf{i}$$
$$\quad + \left(0.4226\|\mathbf{F_1}\| + 0.6428\|\mathbf{F_2}\| - 1000\right)\mathbf{j}$$
$$= 0$$

Set the $\mathbf{i}$ and $\mathbf{j}$ components equal to zero and solve:

$$\begin{cases} -0.9063\|\mathbf{F_1}\| + 0.7660\|\mathbf{F_2}\| = 0 \\ 0.4226\|\mathbf{F_1}\| + 0.6428\|\mathbf{F_2}\| - 1000 = 0 \end{cases}$$

Solve the first equation for $\|\mathbf{F_2}\|$ and substitute the result into the second equation to solve the

system:

$\|\mathbf{F_2}\| = \dfrac{0.9063}{0.7660}\|\mathbf{F_1}\| \approx 1.1832\|\mathbf{F_1}\|$

$0.4226\|\mathbf{F_1}\| + 0.6428\left(1.1832\|\mathbf{F_1}\|\right) - 1000 = 0$

$1.1832\|\mathbf{F_1}\| = 1000$

$\|\mathbf{F_1}\| \approx 845.2$

$\|\mathbf{F_2}\| \approx 1.1832(845.2) \approx 1000$

The tension in the left cable is about 845.2 pounds and the tension in the right cable is about 1000 pounds.

86. Let $\mathbf{F_1}$ be the tension on the left cable and $\mathbf{F_2}$ be the tension on the right cable. Let $\mathbf{F_3}$ represent the force of the weight of the box.

$\mathbf{F_1} = \|\mathbf{F_1}\|\left[\cos\left(145°\right)\mathbf{i} + \sin\left(145°\right)\mathbf{j}\right]$

$\approx \|\mathbf{F_1}\|\left(-0.8192\mathbf{i} + 0.5736\mathbf{j}\right)$

$\mathbf{F_2} = \|\mathbf{F_2}\|\left[\cos\left(50°\right)\mathbf{i} + \sin\left(50°\right)\mathbf{j}\right]$

$\approx \|\mathbf{F_2}\|\left(0.6428\mathbf{i} + 0.7660\mathbf{j}\right)$

$\mathbf{F_3} = -800\mathbf{j}$

For equilibrium, the sum of the force vectors must be zero.

$\mathbf{F_1} + \mathbf{F_2} + \mathbf{F_3}$

$= -0.8192\|\mathbf{F_1}\|\mathbf{i} + 0.5736\|\mathbf{F_1}\|\mathbf{j}$

$\quad + 0.6428\|\mathbf{F_2}\|\mathbf{i} + 0.7660\|\mathbf{F_2}\|\mathbf{j} - 800\mathbf{j}$

$= \left(-0.8192\|\mathbf{F_1}\| + 0.6428\|\mathbf{F_2}\|\right)\mathbf{i}$

$\quad + \left(0.5736\|\mathbf{F_1}\| + 0.7660\|\mathbf{F_2}\| - 800\right)\mathbf{j}$

$= 0$

Set the $\mathbf{i}$ and $\mathbf{j}$ components equal to zero and solve:

$\begin{cases} -0.8192\|\mathbf{F_1}\| + 0.6428\|\mathbf{F_2}\| = 0 \\ 0.5736\|\mathbf{F_1}\| + 0.7660\|\mathbf{F_2}\| - 800 = 0 \end{cases}$

Solve the first equation for $\|\mathbf{F_2}\|$ and substitute the result into the second equation to solve the system:

$\|\mathbf{F_2}\| = \dfrac{0.8192}{0.6428}\|\mathbf{F_1}\| \approx 1.2744\|\mathbf{F_1}\|$

$0.5736\|\mathbf{F_1}\| + 0.7660\left(1.2744\|\mathbf{F_1}\|\right) - 800 = 0$

$1.5498\|\mathbf{F_1}\| = 800$

$\|\mathbf{F_1}\| \approx 516.2$

$\|\mathbf{F_2}\| \approx 1.2744(516.2) \approx 657.8$

The tension in the left cable is about 516.2 pounds and the tension in the right cable is about 657.8 pounds.

87. Let $\mathbf{F_1}$ be the tension on the left end of the rope and $\mathbf{F_2}$ be the tension on the right end of the rope. Let $\mathbf{F_3}$ represent the force of the weight of the tightrope walker.

$\mathbf{F_1} = \|\mathbf{F_1}\|\left[\cos\left(175.8°\right)\mathbf{i} + \sin\left(175.8°\right)\mathbf{j}\right]$

$\approx \|\mathbf{F_1}\|\left(-0.99731\mathbf{i} + 0.07324\mathbf{j}\right)$

$\mathbf{F_2} = \|\mathbf{F_2}\|\left[\cos\left(3.7°\right)\mathbf{i} + \sin\left(3.7°\right)\mathbf{j}\right]$

$\approx \|\mathbf{F_2}\|\left(0.99792\mathbf{i} + 0.06453\mathbf{j}\right)$

$\mathbf{F_3} = -150\mathbf{j}$

For equilibrium, the sum of the force vectors must be zero.

$\mathbf{F_1} + \mathbf{F_2} + \mathbf{F_3}$

$= -0.99731\|\mathbf{F_1}\|\mathbf{i} + 0.07324\|\mathbf{F_1}\|\mathbf{j}$

$\quad + 0.99792\|\mathbf{F_2}\|\mathbf{i} + 0.06453\|\mathbf{F_2}\|\mathbf{j} - 150\mathbf{j}$

$= \left(-0.99731\|\mathbf{F_1}\| + 0.99792\|\mathbf{F_2}\|\right)\mathbf{i}$

$\quad + \left(0.07324\|\mathbf{F_1}\| + 0.06453\|\mathbf{F_2}\| - 150\right)\mathbf{j}$

$= 0$

Set the $\mathbf{i}$ and $\mathbf{j}$ components equal to zero and solve:

$\begin{cases} -0.99731\|\mathbf{F_1}\| + 0.99792\|\mathbf{F_2}\| = 0 \\ 0.07324\|\mathbf{F_1}\| + 0.06453\|\mathbf{F_2}\| - 150 = 0 \end{cases}$

Solve the first equation for $\|\mathbf{F_2}\|$ and substitute the result into the second equation to solve the system:

$\|\mathbf{F_2}\| = \dfrac{0.99731}{0.99792}\|\mathbf{F_1}\| \approx 0.99939\|\mathbf{F_1}\|$

$0.07324\|\mathbf{F_1}\| + 0.06453\left(0.99939\|\mathbf{F_1}\|\right) - 150 = 0$

$0.13773\|\mathbf{F_1}\| = 150$

$\|\mathbf{F_1}\| = 1089.1$

$\|\mathbf{F_2}\| = 0.99939(1089.1) \approx 1088.4$

The tension in the left end of the rope is about 1089.1 pounds and the tension in the right end of the rope is about 1088.4 pounds.

88. Let $\mathbf{F_1}$ be the tension on the left end of the rope and $\mathbf{F_2}$ be the tension on the right end of the rope. Let $\mathbf{F_3}$ represent the force of the weight of the tightrope walker.

$\mathbf{F_1} = \|\mathbf{F_1}\|\left[\cos\left(176.2°\right)\mathbf{i} + \sin\left(176.2°\right)\mathbf{j}\right]$

$\approx \|\mathbf{F_1}\|\left(-0.99780\mathbf{i} + 0.06627\mathbf{j}\right)$

$\mathbf{F_2} = \|\mathbf{F_2}\|\left[\cos\left(2.6°\right)\mathbf{i} + \sin\left(2.6°\right)\mathbf{j}\right]$

$\approx \|\mathbf{F_2}\|\left(0.99897\mathbf{i} + 0.04536\mathbf{j}\right)$

$\mathbf{F_3} = -135\mathbf{j}$

For equilibrium, the sum of the force vectors must be zero.

$\mathbf{F_1} + \mathbf{F_2} + \mathbf{F_3}$
$= -0.99780\|\mathbf{F_1}\|\mathbf{i} + 0.06627\|\mathbf{F_1}\|\mathbf{j}$
$\qquad + 0.99897\|\mathbf{F_2}\|\mathbf{i} + 0.04536\|\mathbf{F_2}\|\mathbf{j} - 135\mathbf{j}$
$= \left(-0.99780\|\mathbf{F_1}\| + 0.99897\|\mathbf{F_2}\|\right)\mathbf{i}$
$\qquad + \left(0.06627\|\mathbf{F_1}\| + 0.04536\|\mathbf{F_2}\| - 135\right)\mathbf{j}$
$= 0$

Set the $\mathbf{i}$ and $\mathbf{j}$ components equal to zero and solve:

$\begin{cases} -0.99780\|\mathbf{F_1}\| + 0.99897\|\mathbf{F_2}\| = 0 \\ 0.06627\|\mathbf{F_1}\| + 0.04536\|\mathbf{F_2}\| - 135 = 0 \end{cases}$

Solve the first equation for $\|\mathbf{F_2}\|$ and substitute the result into the second equation to solve the system:

$\|\mathbf{F_2}\| = \dfrac{0.99780}{0.99897}\|\mathbf{F_1}\| \approx 0.99883\|\mathbf{F_1}\|$

$0.06627\|\mathbf{F_1}\| + 0.04536\left(0.99883\|\mathbf{F_1}\|\right) - 135 = 0$

$0.11158\|\mathbf{F_1}\| = 135$

$\|\mathbf{F_1}\| \approx 1209.9$

$\|\mathbf{F_2}\| = 0.99883(1209.9) = 1208.4$

The tension in the left end of the rope is about 1209.9 pounds and the tension in the right end of the rope is about 1208.4 pounds.

89. $\mathbf{F_1} = 3000\mathbf{i}$

$\mathbf{F_2} = 2000\left[\cos\left(45°\right)\mathbf{i} + \sin\left(45°\right)\mathbf{j}\right]$
$= 2000\left(\dfrac{\sqrt{2}}{2}\mathbf{i} + \dfrac{\sqrt{2}}{2}\mathbf{j}\right)$
$= 1000\sqrt{2}\,\mathbf{i} + 1000\sqrt{2}\mathbf{j}$

$\mathbf{F} = \mathbf{F_1} + \mathbf{F_2} = 3000\mathbf{i} + 1000\sqrt{2}\,\mathbf{i} + 1000\sqrt{2}\mathbf{j}$
$= \left(3000 + 1000\sqrt{2}\right)\mathbf{i} + 1000\sqrt{2}\mathbf{j}$

$\|\mathbf{F}\| = \sqrt{\left(3000 + 1000\sqrt{2}\right)^2 + \left(1000\sqrt{2}\right)^2}$
≈ 4635.2

The monster truck must pull with a force of approximately 4635.2 pounds in order to remain unmoved.

90. a. $\mathbf{F_1} = 7000\mathbf{i}$

$\mathbf{F_2} = 5500\left[\cos\left(40°\right)\mathbf{i} + \sin\left(40°\right)\mathbf{j}\right]$
$\approx 5500\left(0.766044\mathbf{i} + 0.642788\mathbf{j}\right)$
$\approx 4213.24\mathbf{i} + 3535.33\mathbf{j}$

$\mathbf{F} = \mathbf{F_1} + \mathbf{F_2} \approx 7000\mathbf{i} + 4213.24\mathbf{i} + 3535.33\mathbf{j}$
$= 11{,}213.24\mathbf{i} + 3535.33\mathbf{j}$

$\|\mathbf{F}\| = \sqrt{\left(11{,}213.24\right)^2 + \left(3535.33\right)^2}$
$\approx 11{,}757.4$

The farmer will not be successful in removing the stump. The two tractors will have a combined pull of only about 11,757.4 pounds, which is less than the 6 tons needed.

b. $\mathbf{F_1} = 7000\mathbf{i}$

$\mathbf{F_2} = 5500\left[\cos\left(25°\right)\mathbf{i} + \sin\left(25°\right)\mathbf{j}\right]$
$\approx 5500\left(0.906308\mathbf{i} + 0.422618\mathbf{j}\right)$
$\approx 4984.69\mathbf{i} + 2324.40\mathbf{j}$

$\mathbf{F} = \mathbf{F_1} + \mathbf{F_2} \approx 7000\mathbf{i} + 4984.69\mathbf{i} + 2324.40\mathbf{j}$
$= 11{,}984.69\mathbf{i} + 2324.40\mathbf{j}$

$\|\mathbf{F}\| = \sqrt{\left(11{,}984.69\right)^2 + \left(2324.40\right)^2}$
$\approx 12{,}208.0$

The farmer will be successful in removing the stump. The two tractors will have a combined pull of about 12,208 pounds, which is more than the 6 tons needed.

91. The given forces are:

$\mathbf{F_1} = -3\mathbf{i};\ \ \mathbf{F_2} = -\mathbf{i} + 4\mathbf{j};\ \ \mathbf{F_3} = 4\mathbf{i} - 2\mathbf{j};\ \ \mathbf{F_4} = -4\mathbf{j}$

A vector $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$ needs to be added for equilibrium. Find vector $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$:

$\mathbf{F_1} + \mathbf{F_2} + \mathbf{F_3} + \mathbf{F_4} + \mathbf{v} = 0$
$-3\mathbf{i} + (-\mathbf{i} + 4\mathbf{j}) + (4\mathbf{i} - 2\mathbf{j}) + (-4\mathbf{j}) + (a\mathbf{i} + b\mathbf{j}) = 0$
$0\mathbf{i} + -2\mathbf{j} + (a\mathbf{i} + b\mathbf{j}) = 0$
$a\mathbf{i} + (-2 + b)\mathbf{j} = 0$

$a = 0;\quad -2 + b = 0$
$\qquad\qquad b = 2$

Therefore, $\mathbf{v} = 2\mathbf{j}$.
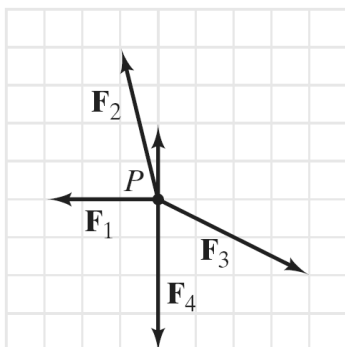
92 – 93. Interactive Exercises.

94 – 96. Answers will vary.

Section 9.5

1. $c^2 = a^2 + b^2 - 2ab \cos C$

2. dot product

3. orthogonal

4. parallel

5. True

6. False

7. $\mathbf{v} = \mathbf{i} - \mathbf{j}, \quad \mathbf{w} = \mathbf{i} + \mathbf{j}$

   a. $\mathbf{v} \cdot \mathbf{w} = 1(1) + (-1)(1) = 1 - 1 = 0$

   b. $\cos\theta = \dfrac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \dfrac{0}{\sqrt{1^2 + (-1)^2}\sqrt{1^2 + 1^2}} = 0$

     $\theta = 90°$

   c. The vectors are orthogonal.

8. $\mathbf{v} = \mathbf{i} + \mathbf{j}, \quad \mathbf{w} = -\mathbf{i} + \mathbf{j}$

   a. $\mathbf{v} \cdot \mathbf{w} = 1(-1) + 1(1) = -1 + 1 = 0$

   b. $\cos\theta = \dfrac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \dfrac{0}{\sqrt{1^2 + 1^2}\sqrt{(-1)^2 + 1^2}} = 0$

     $\theta = 90°$

   c. The vectors are orthogonal.

9. $\mathbf{v} = 2\mathbf{i} + \mathbf{j}, \quad \mathbf{w} = \mathbf{i} - 2\mathbf{j}$

   a. $\mathbf{v} \cdot \mathbf{w} = 2(1) + 1(-2) = 2 - 2 = 0$

   b. $\cos\theta = \dfrac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \dfrac{0}{\sqrt{2^2 + 1^2}\sqrt{1^2 + (-2)^2}}$

     $= \dfrac{0}{\sqrt{5}\sqrt{5}} = \dfrac{0}{5} = 0$

     $\theta = 90°$

   c. The vectors are orthogonal.

10. $\mathbf{v} = 2\mathbf{i} + 2\mathbf{j}, \quad \mathbf{w} = \mathbf{i} + 2\mathbf{j}$

   a. $\mathbf{v} \cdot \mathbf{w} = 2(1) + 2(2) = 2 + 4 = 6$

   b. $\cos\theta = \dfrac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \dfrac{6}{\sqrt{2^2 + 2^2}\sqrt{1^2 + 2^2}}$

     $= \dfrac{6}{2\sqrt{2}\sqrt{5}} = \dfrac{3}{\sqrt{10}} = \dfrac{3\sqrt{10}}{10}$

     $\theta \approx 18.4°$

   c. The vectors are neither parallel nor orthogonal.

11. $\mathbf{v} = \sqrt{3}\,\mathbf{i} - \mathbf{j}, \quad \mathbf{w} = \mathbf{i} + \mathbf{j}$

   a. $\mathbf{v} \cdot \mathbf{w} = \sqrt{3}\,(1) + (-1)(1) = \sqrt{3} - 1$

   b. $\cos\theta = \dfrac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \dfrac{\sqrt{3} - 1}{\sqrt{\left(\sqrt{3}\right)^2 + (-1)^2}\sqrt{1^2 + 1^2}}$

     $= \dfrac{\sqrt{3} - 1}{\sqrt{4}\sqrt{2}} = \dfrac{\sqrt{3} - 1}{2\sqrt{2}} = \dfrac{\sqrt{6} - \sqrt{2}}{4}$

     $\theta = 75°$

   c. The vectors are neither parallel nor orthogonal.

12. $\mathbf{v} = \mathbf{i} + \sqrt{3}\mathbf{j}, \quad \mathbf{w} = \mathbf{i} - \mathbf{j}$

   a. $\mathbf{v} \cdot \mathbf{w} = 1(1) + \sqrt{3}(-1) = 1 - \sqrt{3}$

   b. $\cos\theta = \dfrac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \dfrac{1 - \sqrt{3}}{\sqrt{1^2 + \left(\sqrt{3}\right)^2}\sqrt{1^2 + (-1)^2}}$

     $= \dfrac{1 - \sqrt{3}}{\sqrt{4}\sqrt{2}} = \dfrac{1 - \sqrt{3}}{2\sqrt{2}} = \dfrac{\sqrt{2} - \sqrt{6}}{4}$

     $\theta = 105°$

   c. The vectors are neither parallel nor orthogonal.

13. $\mathbf{v} = 3\mathbf{i} + 4\mathbf{j}$, $\mathbf{w} = -6\mathbf{i} - 8\mathbf{j}$

   a. $\mathbf{v} \cdot \mathbf{w} = 3(-6) + 4(-8) = -18 - 32 = -50$

   b. $\cos\theta = \dfrac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \dfrac{-50}{\sqrt{3^2 + 4^2}\sqrt{(-6)^2 + (-8)^2}}$

       $= \dfrac{-50}{\sqrt{25}\sqrt{100}} = \dfrac{-50}{50} = -1$

       $\theta = 180°$

   c. Note that $\mathbf{v} = -\dfrac{1}{2}\mathbf{w}$ and $\theta = 180°$, so the vectors are parallel.

14. $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$, $\mathbf{w} = 9\mathbf{i} - 12\mathbf{j}$

   a. $\mathbf{v} \cdot \mathbf{w} = 3(9) + (-4)(-12) = 27 + 48 = 75$

   b. $\cos\theta = \dfrac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}$

       $= \dfrac{75}{\sqrt{3^2 + (-4)^2}\sqrt{9^2 + (-12)^2}}$

       $= \dfrac{75}{\sqrt{25}\sqrt{225}} = \dfrac{75}{75} = 1$

       $\theta = 0°$

   c. Note that $\mathbf{v} = \dfrac{1}{3}\mathbf{w}$ and $\theta = 0°$, so the vectors are parallel.

15. $\mathbf{v} = 4\mathbf{i}$, $\mathbf{w} = \mathbf{j}$

   a. $\mathbf{v} \cdot \mathbf{w} = 4(0) + 0(1) = 0 + 0 = 0$

   b. $\cos\theta = \dfrac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \dfrac{0}{\sqrt{4^2 + 0^2}\sqrt{0^2 + 1^2}}$

       $= \dfrac{0}{4 \cdot 1} = \dfrac{0}{4} = 0$

       $\theta = 90°$

   c. The vectors are orthogonal.

16. $\mathbf{v} = \mathbf{i}$, $\mathbf{w} = -3\mathbf{j}$

   a. $\mathbf{v} \cdot \mathbf{w} = 1(0) + 0(-3) = 0 + 0 = 0$

   b. $\cos\theta = \dfrac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}$

       $= \dfrac{0}{\sqrt{1^2 + 0^2}\sqrt{0^2 + (-3)^2}} = \dfrac{0}{1 \cdot 3} = \dfrac{0}{3} = 0$

       $\theta = 90°$

   c. The vectors are orthogonal.

17. $\mathbf{v} = \mathbf{i} - a\mathbf{j}$, $\mathbf{w} = 2\mathbf{i} + 3\mathbf{j}$

   Two vectors are orthogonal if the dot product is zero. Solve for a:

       $\mathbf{v} \cdot \mathbf{w} = 0$

     $1(2) + (-a)(3) = 0$

       $2 - 3a = 0$

       $3a = 2$

       $a = \dfrac{2}{3}$

18. $\mathbf{v} = \mathbf{i} + \mathbf{j}$, $\mathbf{w} = \mathbf{i} + b\mathbf{j}$

   Two vectors are orthogonal if the dot product is zero. Solve for b:

       $\mathbf{v} \cdot \mathbf{w} = 0$

     $1(1) + 1(b) = 0$

       $1 + b = 0$

       $b = -1$

19. $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j}$, $\mathbf{w} = \mathbf{i} - \mathbf{j}$

   $\mathbf{v_1} = \dfrac{\mathbf{v} \cdot \mathbf{w}}{\left(\|\mathbf{w}\|\right)^2}\mathbf{w} = \dfrac{2(1) + (-3)(-1)}{\left(\sqrt{1^2 + (-1)^2}\right)^2}(\mathbf{i} - \mathbf{j})$

     $= \dfrac{5}{2}(\mathbf{i} - \mathbf{j}) = \dfrac{5}{2}\mathbf{i} - \dfrac{5}{2}\mathbf{j}$

   $\mathbf{v_2} = \mathbf{v} - \mathbf{v_1} = (2\mathbf{i} - 3\mathbf{j}) - \left(\dfrac{5}{2}\mathbf{i} - \dfrac{5}{2}\mathbf{j}\right) = -\dfrac{1}{2}\mathbf{i} - \dfrac{1}{2}\mathbf{j}$

20. $\mathbf{v} = -3\mathbf{i} + 2\mathbf{j}$, $\mathbf{w} = 2\mathbf{i} + \mathbf{j}$

   $\mathbf{v_1} = \dfrac{\mathbf{v} \cdot \mathbf{w}}{\left(\|\mathbf{w}\|\right)^2}\mathbf{w} = \dfrac{-3(2) + 2(1)}{\left(\sqrt{2^2 + 1^2}\right)^2}(2\mathbf{i} + \mathbf{j})$

     $= -\dfrac{4}{5}(2\mathbf{i} + \mathbf{j}) = -\dfrac{8}{5}\mathbf{i} - \dfrac{4}{5}\mathbf{j}$

   $\mathbf{v_2} = \mathbf{v} - \mathbf{v_1} = (-3\mathbf{i} + 2\mathbf{j}) - \left(-\dfrac{8}{5}\mathbf{i} - \dfrac{4}{5}\mathbf{j}\right)$

     $= -\dfrac{7}{5}\mathbf{i} + \dfrac{14}{5}\mathbf{j}$

21. $\mathbf{v} = \mathbf{i} - \mathbf{j}$, $\mathbf{w} = -\mathbf{i} - 2\mathbf{j}$

   $\mathbf{v_1} = \dfrac{\mathbf{v} \cdot \mathbf{w}}{\left(\|\mathbf{w}\|\right)^2}\mathbf{w} = \dfrac{1(-1) + (-1)(-2)}{\left(\sqrt{(-1)^2 + (-2)^2}\right)^2}(-\mathbf{i} - 2\mathbf{j})$

     $= -\dfrac{1}{5}(\mathbf{i} + 2\mathbf{j}) = -\dfrac{1}{5}\mathbf{i} - \dfrac{2}{5}\mathbf{j}$

   $\mathbf{v_2} = \mathbf{v} - \mathbf{v_1} = (\mathbf{i} - \mathbf{j}) - \left(-\dfrac{1}{5}\mathbf{i} - \dfrac{2}{5}\mathbf{j}\right) = \dfrac{6}{5}\mathbf{i} - \dfrac{3}{5}\mathbf{j}$

22. $v = 2i - j, \quad w = i - 2j$

$$v_1 = \frac{v \cdot w}{\left(\|w\|\right)^2} w = \frac{2(1)+(-1)(-2)}{\left(\sqrt{1^2+(-2)^2}\right)^2}(i-2j)$$

$$= \frac{4}{5}(i-2j) = \frac{4}{5}i - \frac{8}{5}j$$

$$v_2 = v - v_1 = (2i-j) - \left(\frac{4}{5}i - \frac{8}{5}j\right) = \frac{6}{5}i + \frac{3}{5}j$$

23. $v = 3i + j, \quad w = -2i - j$

$$v_1 = \frac{v \cdot w}{\left(\|w\|\right)^2} w = \frac{3(-2)+1(-1)}{\left(\sqrt{(-2)^2+(-1)^2}\right)^2}(-2i-j)$$

$$= -\frac{7}{5}(-2i-j) = \frac{14}{5}i + \frac{7}{5}j$$

$$v_2 = v - v_1 = (3i+j) - \left(\frac{14}{5}i + \frac{7}{5}j\right) = \frac{1}{5}i - \frac{2}{5}j$$

24. $v = i - 3j, \quad w = 4i - j$

$$v_1 = \frac{v \cdot w}{\left(\|w\|\right)^2} w = \frac{1(4)+(-3)(-1)}{\left(\sqrt{4^2+(-1)^2}\right)^2}(4i-j)$$

$$= \frac{7}{17}(4i-j) = \frac{28}{17}i - \frac{7}{17}j$$

$$v_2 = v - v_1 = (i-3j) - \left(\frac{28}{17}i - \frac{7}{17}j\right)$$

$$= -\frac{11}{17}i - \frac{44}{17}j$$

25. $F = 3\left[\cos(60°)i + \sin(60°)j\right]$

$$= 3\left(\frac{1}{2}i + \frac{\sqrt{3}}{2}j\right) = \frac{3}{2}i + \frac{3\sqrt{3}}{2}j$$

$$W = F \cdot \overrightarrow{AB} = \left(\frac{3}{2}i + \frac{3\sqrt{3}}{2}j\right) \cdot 6i$$

$$= \frac{3}{2}(6) + \frac{3\sqrt{3}}{2}(0) = 9 \text{ ft-lb}$$

26. $F = 20\left[\cos(30°)i + \sin(30°)j\right]$

$$= 20\left(\frac{\sqrt{3}}{2}i + \frac{1}{2}j\right) = 10\sqrt{3}i + 10j$$

$$W = F \cdot \overrightarrow{AB} = \left(10\sqrt{3}i + 10j\right) \cdot 100i$$

$$= 10\sqrt{3}(100) + 10(0)$$

$$= 1000\sqrt{3} \approx 1732 \text{ ft-lb}$$

27. a. $\|I\| = \sqrt{(-0.02)^2 + (-0.01)^2}$

$$= \sqrt{0.0005} \approx 0.022$$

The intensity of the sun's rays is about 0.022 watts per square centimeter.

$$\|A\| = \sqrt{(300)^2 + (400)^2} = \sqrt{250,000} = 500$$

The area of the solar panel is 500 square centimeter.

b. $W = |I \cdot A| = |(-0.02i - 0.01j) \cdot (300i + 400j)|$

$$= |(-0.02)(300) + (-0.01)(400)|$$

$$= |-6 + (-4)| = |-10| = 10$$

This means 10 watts of energy is collected.

c. To collect the maximum number of watts, **I** and **A** should be parallel with the solar panels facing the sun.

28. a. $\|R\| = \sqrt{(0.75)^2 + (-1.75)^2} = \sqrt{3.625} \approx 1.90$

About 1.90 inches of rain fell.

$$\|A\| = \sqrt{(0.3)^2 + (1)^2} = \sqrt{1.09} = 1.04$$

The area of the opening of the gauge is about 1.04 square inches.

b. $V = |R \cdot A| = |(0.75i - 1.75j) \cdot (0.3i + j)|$

$$= |(0.75)(0.3) + (-1.75)(1)|$$

$$= |0.225 + (-1.75)| = |-1.525| = 1.525$$

This means the gauge collected 1.525 cubic inches of rain.

c. To collect the maximum volume of rain, **R** and **A** should be parallel and oriented in opposite directions.

29. Split the force into the components going down the hill and perpendicular to the hill.
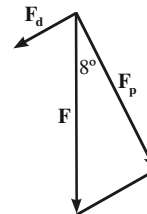
$$F_d = F \sin 8° = 5300 \sin 8° \approx 737.6$$

$$F_p = F \cos 8° = 5300 \cos 8° \approx 5248.4$$

The force required to keep the Sienna from rolling down the hill is about 737.6 pounds. The force perpendicular to the hill is approximately 5248.4 pounds.

30. Split the force into the components going down the hill and perpendicular to the hill.
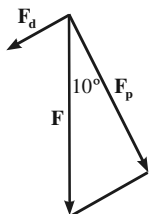
$\mathbf{F_d} = \mathbf{F}\sin 10° = 4500\sin 10° \approx 781.4$

$\mathbf{F_p} = \mathbf{F}\cos 10° = 4500\cos 10° \approx 4431.6$

The force required to keep the Bonneville from rolling down the hill is about 781.4 pounds. The force perpendicular to the hill is approximately 4431.6 pounds.

31. We must determine the component force going down the ramp.

$\mathbf{F_d} = \mathbf{F}\sin 20° = 250\sin 20° \approx 85.5$

Timmy must exert about 85.5 pounds of force to hold the piano in position.

32. We must determine the angle θ if the force of the boulder is $\mathbf{F} = 5000$ pounds and the component force going down the hill is $\mathbf{F_d} = 1000$ pounds.

$\mathbf{F_d} = \mathbf{F}\sin\theta$

$1000 = 5000\sin\theta$

$\sin\theta = \dfrac{1000}{5000} = 0.2$

$\theta = \sin^{-1}(0.2) \approx 11.5°$

The angle of inclination of the hill is about $11.5°$.

33. $W = \mathbf{F}\cdot\overrightarrow{AB}$ $W = 2$, $\overrightarrow{AB} = 4\mathbf{i}$

$\mathbf{F} = \cos\alpha\,\mathbf{i} - \sin\alpha\,\mathbf{j}$

$2 = (\cos\alpha\,\mathbf{i} - \sin\alpha\,\mathbf{j})\cdot 4\mathbf{i}$

$2 = 4\cos\alpha$

$\dfrac{1}{2} = \cos\alpha$

$\alpha = 60°$

34. Let $\mathbf{u} = a_1\mathbf{i} + b_1\mathbf{j}$, $\mathbf{v} = a_2\mathbf{i} + b_2\mathbf{j}$, and $\mathbf{w} = a_3\mathbf{i} + b_3\mathbf{j}$.

$\mathbf{u}\cdot(\mathbf{v} + \mathbf{w})$

$= (a_1\mathbf{i} + b_1\mathbf{j})(a_2\mathbf{i} + b_2\mathbf{j} + a_3\mathbf{i} + b_3\mathbf{j})$

$= (a_1\mathbf{i} + b_1\mathbf{j})(a_2\mathbf{i} + a_3\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{j})$

$= (a_1\mathbf{i} + b_1\mathbf{j})[(a_2 + a_3)\mathbf{i} + (b_2 + b_3)\mathbf{j}]$

$= a_1(a_2 + a_3) + b_1(b_2 + b_3)$

$= a_1a_2 + a_1a_3 + b_1b_2 + b_1b_3$

$= a_1a_2 + b_1b_2 + a_1a_3 + b_1b_3$

$= (a_1\mathbf{i} + b_1\mathbf{j})(a_2\mathbf{i} + b_2\mathbf{j}) + (a_1\mathbf{i} + b_1\mathbf{j})(a_3\mathbf{i} + b_3\mathbf{j})$

$= \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{w}$

35. Since $\mathbf{0} = 0\mathbf{i} + 0\mathbf{j}$ and $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$, we have that

$\mathbf{0}\cdot\mathbf{v} = 0\cdot a + 0\cdot b = 0$.

36. Let $\mathbf{v} = x\mathbf{i} + y\mathbf{j}$. Since $\mathbf{v}$ is a unit vector, we have that $\|\mathbf{v}\| = \sqrt{x^2 + y^2} = 1$, or $x^2 + y^2 = 1$. If α is the angle between $\mathbf{v}$ and $\mathbf{i}$, then

$\cos\alpha = \dfrac{\mathbf{v}\cdot\mathbf{i}}{\|\mathbf{v}\|\|\mathbf{i}\|} = \dfrac{(x\mathbf{i} + y\mathbf{j})\cdot\mathbf{i}}{1\cdot 1} = x$. Now,

$x^2 + y^2 = 1$

$\cos^2\alpha + y^2 = 1$

$\qquad y^2 = 1 - \cos^2\alpha$

$\qquad y^2 = \sin^2\alpha$

$\qquad y = \sin\alpha$

Thus, $\mathbf{v} = \cos\alpha\,\mathbf{i} + \sin\alpha\,\mathbf{j}$.

37. If $\mathbf{v} = a_1\mathbf{i} + b_1\mathbf{j} = \cos\alpha\,\mathbf{i} + \sin\alpha\,\mathbf{j}$ and $\mathbf{w} = a_2\mathbf{i} + b_2\mathbf{j} = \cos\beta\,\mathbf{i} + \sin\beta\,\mathbf{j}$, then

$\cos(\alpha - \beta) = \mathbf{v}\cdot\mathbf{w}$

$\qquad\qquad = a_1a_2 + b_1b_2$

$\qquad\qquad = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

38. Let $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$. The projection of $\mathbf{v}$ onto $\mathbf{i}$ is

$\mathbf{v_1} = $ the projection of $\mathbf{v}$ onto $\mathbf{i}$

$= \dfrac{\mathbf{v_1}\cdot\mathbf{i}}{(\|\mathbf{i}\|)^2}\mathbf{i} = \dfrac{(a\mathbf{i} + b\mathbf{i})\cdot\mathbf{i}}{\left(\sqrt{1^2 + 0^2}\right)^2}\mathbf{i} = \dfrac{a(1) + b(0)}{1^2}\mathbf{i} = a\mathbf{i}$

$\mathbf{v_2} = \mathbf{v} - \mathbf{v_1} = a\mathbf{i} + b\mathbf{j} - a\mathbf{i} = b\mathbf{j}$

Since $\mathbf{v}\cdot\mathbf{i} = (a\mathbf{i} + b\mathbf{j})\cdot\mathbf{i} = (a)(1) + (b)(0) = a$ and

$\mathbf{v}\cdot\mathbf{j} = (a\mathbf{i} + b\mathbf{j})\cdot\mathbf{j} = (a)(0) + (b)(1) = b$,

$\mathbf{v} = \mathbf{v_1} + \mathbf{v_2} = (\mathbf{v}\cdot\mathbf{i})\mathbf{i} + (\mathbf{v}\cdot\mathbf{j})\mathbf{j}$.

39. a. Let $\mathbf{u} = a_1\mathbf{i} + b_1\mathbf{j}$ and $\mathbf{v} = a_2\mathbf{i} + b_2\mathbf{j}$. Then,

$(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v})$

$= [(a_1 + a_2)\mathbf{i} + (b_1 + b_2)\mathbf{j}][(a_1 - a_2)\mathbf{i} + (b_1 - b_2)\mathbf{j}]$

$= (a_1 + a_2)(a_1 - a_2) + (b_1 + b_2)(b_1 - b_2)$

$= a_1^2 - a_2^2 + b_1^2 - b_2^2$

$= (a_1^2 + b_1^2) - (a_2^2 + b_2^2)$

$= \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2$

Since the vectors have the same magnitudes and $(\mathbf{u} + \mathbf{v})(\mathbf{u} - \mathbf{v}) = 0$, the vectors are orthogonal.

b. Because the vectors $\mathbf{u}$ and $\mathbf{v}$ are radii of the circle, we know they have the same magnitude. Since $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$ are sides of the angle inscribed in the semicircle and since we know that they are orthogonal by part (a), this angle must be a right angle.

40. $(\mathbf{v} - \alpha\mathbf{w}) \cdot \mathbf{w} = \mathbf{v} \cdot \mathbf{w} - \alpha\mathbf{w} \cdot \mathbf{w}$

$= \mathbf{v} \cdot \mathbf{w} - \alpha(\|\mathbf{w}\|)^2$

$= \mathbf{v} \cdot \mathbf{w} - \dfrac{\mathbf{v} \cdot \mathbf{w}}{(\|\mathbf{w}\|)^2}(\|\mathbf{w}\|)^2$

$= 0$

Therefore, the vectors are orthogonal.

41. $(\|\mathbf{w}\|\mathbf{v} + \|\mathbf{v}\|\mathbf{w}) \cdot (\|\mathbf{w}\|\mathbf{v} - \|\mathbf{v}\|\mathbf{w})$

$= (\|\mathbf{w}\|)^2 \mathbf{v} \cdot \mathbf{v} - \|\mathbf{w}\|\|\mathbf{v}\|\mathbf{v} \cdot \mathbf{w}$

$\qquad + \|\mathbf{w}\|\|\mathbf{v}\|\mathbf{v} \cdot \mathbf{w} - (\|\mathbf{v}\|)^2 \mathbf{w} \cdot \mathbf{w}$

$= (\|\mathbf{w}\|)^2 \mathbf{v} \cdot \mathbf{v} - (\|\mathbf{v}\|)^2 \mathbf{w} \cdot \mathbf{w}$

$= (\|\mathbf{w}\|)^2 (\|\mathbf{v}\|)^2 - (\|\mathbf{v}\|)^2 (\|\mathbf{w}\|)^2$

$= 0$

Therefore, the vectors are orthogonal.

42. If $\mathbf{F}$ is orthogonal to $\overrightarrow{AB}$, then $\mathbf{F} \cdot \overrightarrow{AB} = 0$. So, $W = \mathbf{F} \cdot \overrightarrow{AB} = 0$.

43. $(\|\mathbf{u} + \mathbf{v}\|)^2 - (\|\mathbf{u} - \mathbf{v}\|)^2$

$= (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) - (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v})$

$= (\mathbf{u} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v})$

$\qquad - (\mathbf{u} \cdot \mathbf{u} - \mathbf{u} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v})$

$= 2(\mathbf{u} \cdot \mathbf{v}) + 2(\mathbf{u} \cdot \mathbf{v})$

$= 4(\mathbf{u} \cdot \mathbf{v})$

44. Answers will vary.

Section 9.6

1. $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

2. xy-plane

3. components

4. 1

5. False

6. True

7. $y = 0$ is the set of all points of the form $(x, 0, z)$, the set of all points in the xz-plane.

8. $x = 0$ the set of all points of the form $(0, y, z)$, the set of all points in the yz-plane.

9. $z = 2$ is the set of all points of the form $(x, y, 2)$, the plain two units above the xy-plane.

10. $y = 3$ is the set of all points of the form $(x, 3, z)$, the plane three units to the right of the xz-plane.

11. $x = -4$ is the set of all points of the form $(-4, y, z)$, the plain four units to the left of the yz-plane.

12. $z = -3$ is the set of all points of the form $(x, y, -3)$, the plane three units below the xy-plane.

13. $x = 1$ and $y = 2$ is the set of all points of the form $(1, 2, z)$, a line parallel to the z-axis.

14. $x = 3$ and $z = 1$ is the set of all points of the form $(3, y, 1)$, a line parallel to the y-axis.

15. $d = \sqrt{(4-0)^2 + (1-0)^2 + (2-0)^2}$

$= \sqrt{16 + 1 + 4}$

$= \sqrt{21}$

16. $d = \sqrt{(1-0)^2 + (-2-0)^2 + (3-0)^2}$

$= \sqrt{1 + 4 + 9}$

$= \sqrt{14}$

17. $d = \sqrt{(0-(-1))^2 + (-2-2)^2 + (1-(-3))^2}$

    $= \sqrt{1+16+16}$

    $= \sqrt{33}$

18. $d = \sqrt{(4-(-2))^2 + (0-2)^2 + (-3-3)^2}$

    $= \sqrt{36+4+36}$

    $= \sqrt{76}$

    $= 2\sqrt{19}$

19. $d = \sqrt{(3-4)^2 + (2-(-2))^2 + (1-(-2))^2}$

    $= \sqrt{1+16+9}$

    $= \sqrt{26}$

20. $d = \sqrt{(4-2)^2 + (1-(-3))^2 + (-1-(-3))^2}$

    $= \sqrt{4+16+4}$

    $= \sqrt{24}$

    $= 2\sqrt{6}$

21. The bottom of the box is formed by the vertices (0, 0, 0), (2, 0, 0), (0, 1, 0), and (2, 1, 0). The top of the box is formed by the vertices (0, 0, 3), (2, 0, 3), (0, 1, 3), and (2, 1, 3).

22. The bottom of the box is formed by the vertices (0, 0, 0), (4, 0, 0), (4, 2, 0), and (0, 2, 0). The top of the box is formed by the vertices (0, 0, 2), (4, 0, 2), (0, 2, 2), and (4, 2, 2).

23. The bottom of the box is formed by the vertices (1, 2, 3), (3, 2, 3), (3, 4, 3), and (1, 4, 3). The top of the box is formed by the vertices (3, 4, 5), (1, 2, 5), (3, 2, 5), and (1, 4, 5).

24. The bottom of the box is formed by the vertices (5, 6, 1), (3, 6, 1), (5, 8, 1), and (3, 8, 1). The top of the box is formed by the vertices (3, 8, 2), (5, 6, 2), (3, 6, 2), and (5, 8, 2).

25. The bottom of the box is formed by the vertices (–1, 0, 2), (4, 0, 2), (–1, 2, 2), and (4, 2, 2). The top of the box is formed by the vertices (4, 2, 5), (–1, 0, 5), (4, 0, 5), and (–1, 2, 5).

26. The bottom of the box is formed by the vertices (–2, –3, 0), (–6, –3, 0), (–2, 7, 0), and (–6, 7, 0). The top of the box is formed by the vertices (–6, 7, 1), (–2, –3, 1), (–2, 7, 1), and (–6, –3, 1).

27. $\mathbf{v} = (3-0)\mathbf{i} + (4-0)\mathbf{j} + (-1-0)\mathbf{k}$

    $= 3\mathbf{i} + 4\mathbf{j} - \mathbf{k}$

28. $\mathbf{v} = (-3-0)\mathbf{i} + (-5-0)\mathbf{j} + (4-0)\mathbf{k}$

    $= -3\mathbf{i} - 5\mathbf{j} + 4\mathbf{k}$

29. $\mathbf{v} = (5-3)\mathbf{i} + (6-2)\mathbf{j} + (0-(-1))\mathbf{k}$

    $= 2\mathbf{i} + 4\mathbf{j} + \mathbf{k}$

30. $\mathbf{v} = (6-(-3))\mathbf{i} + (5-2)\mathbf{j} + (-1-0)\mathbf{k}$

    $= 9\mathbf{i} + 3\mathbf{j} - \mathbf{k}$

31. $\mathbf{v} = (6-(-2))\mathbf{i} + (-2-(-1))\mathbf{j} + (4-4)\mathbf{k}$

    $= 8\mathbf{i} - \mathbf{j}$

32. $\mathbf{v} = (6-(-1))\mathbf{i} + (2-4)\mathbf{j} + (2-(-2))\mathbf{k}$

    $= 7\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$

33. $\|\mathbf{v}\| = \sqrt{3^2 + (-6)^2 + (-2)^2}$

    $= \sqrt{9+36+4}$

    $= \sqrt{49}$

    $= 7$

34. $\|\mathbf{v}\| = \sqrt{(-6)^2 + 12^2 + 4^2}$

    $= \sqrt{36+144+16}$

    $= \sqrt{196}$

    $= 14$

35. $\|\mathbf{v}\| = \sqrt{1^2 + (-1)^2 + 1^2}$

    $= \sqrt{1+1+1}$

    $= \sqrt{3}$

36. $\|\mathbf{v}\| = \sqrt{(-1)^2 + (-1)^2 + 1^2}$

    $= \sqrt{1+1+1}$

    $= \sqrt{3}$

37. $\|\mathbf{v}\| = \sqrt{(-2)^2 + 3^2 + (-3)^2}$

    $= \sqrt{4+9+9}$

    $= \sqrt{22}$

38. $\|\mathbf{v}\| = \sqrt{6^2 + 2^2 + (-2)^2}$

    $= \sqrt{36+4+4}$

    $= \sqrt{44}$

    $= 2\sqrt{11}$

39. $2\mathbf{v}+3\mathbf{w}=2(3\mathbf{i}-5\mathbf{j}+2\mathbf{k})+3(-2\mathbf{i}+3\mathbf{j}-2\mathbf{k})$
$=6\mathbf{i}-10\mathbf{j}+4\mathbf{k}-6\mathbf{i}+9\mathbf{j}-6\mathbf{k}$
$=-\mathbf{j}-2\mathbf{k}$

40. $3\mathbf{v}-2\mathbf{w}=3(3\mathbf{i}-5\mathbf{j}+2\mathbf{k})-2(-2\mathbf{i}+3\mathbf{j}-2\mathbf{k})$
$=9\mathbf{i}-15\mathbf{j}+6\mathbf{k}+4\mathbf{i}-6\mathbf{j}+4\mathbf{k}$
$=13\mathbf{i}-21\mathbf{j}+10\mathbf{k}$

41. $\|\mathbf{v}-\mathbf{w}\|=\|(3\mathbf{i}-5\mathbf{j}+2\mathbf{k})-(-2\mathbf{i}+3\mathbf{j}-2\mathbf{k})\|$
$=\|3\mathbf{i}-5\mathbf{j}+2\mathbf{k}+2\mathbf{i}-3\mathbf{j}+2\mathbf{k}\|$
$=\|5\mathbf{i}-8\mathbf{j}+4\mathbf{k}\|$
$=\sqrt{5^2+(-8)^2+4^2}$
$=\sqrt{25+64+16}$
$=\sqrt{105}$

42. $\|\mathbf{v}+\mathbf{w}\|=\|(3\mathbf{i}-5\mathbf{j}+2\mathbf{k})+(-2\mathbf{i}+3\mathbf{j}-2\mathbf{k})\|$
$=\|\mathbf{i}-2\mathbf{j}+0\mathbf{k}\|$
$=\sqrt{1^2+(-2)^2+0^2}$
$=\sqrt{1+4+0}$
$=\sqrt{5}$

43. $\|\mathbf{v}\|-\|\mathbf{w}\|$
$=\|3\mathbf{i}-5\mathbf{j}+2\mathbf{k}\|-\|-2\mathbf{i}+3\mathbf{j}-2\mathbf{k}\|$
$=\sqrt{3^2+(-5)^2+2^2}-\sqrt{(-2)^2+3^2+(-2)^2}$
$=\sqrt{38}-\sqrt{17}$

44. $\|\mathbf{v}\|+\|\mathbf{w}\|$
$=\|3\mathbf{i}-5\mathbf{j}+2\mathbf{k}\|+\|-2\mathbf{i}+3\mathbf{j}-2\mathbf{k}\|$
$=\sqrt{3^2+(-5)^2+2^2}+\sqrt{(-2)^2+3^2+(-2)^2}$
$=\sqrt{38}+\sqrt{17}$

45. $\mathbf{u}=\dfrac{\mathbf{v}}{\|\mathbf{v}\|}=\dfrac{5\mathbf{i}}{\sqrt{5^2+0^2+0^2}}=\dfrac{5\mathbf{i}}{5}=\mathbf{i}$

46. $\mathbf{u}=\dfrac{\mathbf{v}}{\|\mathbf{v}\|}=\dfrac{-3\mathbf{j}}{\sqrt{0^2+(-3)^2+0^2}}=\dfrac{-3\mathbf{j}}{3}=-\mathbf{j}$

47. $\mathbf{u}=\dfrac{\mathbf{v}}{\|\mathbf{v}\|}=\dfrac{3\mathbf{i}-6\mathbf{j}-2\mathbf{k}}{\sqrt{3^2+(-6)^2+(-2)^2}}$
$=\dfrac{3\mathbf{i}-6\mathbf{j}-2\mathbf{k}}{7}$
$=\dfrac{3}{7}\mathbf{i}-\dfrac{6}{7}\mathbf{j}-\dfrac{2}{7}\mathbf{k}$

48. $\mathbf{u}=\dfrac{\mathbf{v}}{\|\mathbf{v}\|}=\dfrac{-6\mathbf{i}+12\mathbf{j}+4\mathbf{k}}{\sqrt{(-6)^2+12^2+4^2}}$
$=\dfrac{-6\mathbf{i}+12\mathbf{j}+4\mathbf{k}}{14}$
$=-\dfrac{3}{7}\mathbf{i}+\dfrac{6}{7}\mathbf{j}+\dfrac{2}{7}\mathbf{k}$

49. $\mathbf{u}=\dfrac{\mathbf{v}}{\|\mathbf{v}\|}=\dfrac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{1^2+1^2+1^2}}$
$=\dfrac{\mathbf{i}+\mathbf{j}+\mathbf{k}}{\sqrt{3}}$
$=\dfrac{1}{\sqrt{3}}\mathbf{i}+\dfrac{1}{\sqrt{3}}\mathbf{j}+\dfrac{1}{\sqrt{3}}\mathbf{k}$
$=\dfrac{\sqrt{3}}{3}\mathbf{i}+\dfrac{\sqrt{3}}{3}\mathbf{j}+\dfrac{\sqrt{3}}{3}\mathbf{k}$

50. $\mathbf{u}=\dfrac{\mathbf{v}}{\|\mathbf{v}\|}=\dfrac{2\mathbf{i}-\mathbf{j}+\mathbf{k}}{\sqrt{2^2+(-1)^2+1^2}}$
$=\dfrac{2\mathbf{i}-\mathbf{j}+\mathbf{k}}{\sqrt{6}}$
$=\dfrac{2}{\sqrt{6}}\mathbf{i}-\dfrac{1}{\sqrt{6}}\mathbf{j}+\dfrac{1}{\sqrt{6}}\mathbf{k}$
$=\dfrac{\sqrt{6}}{3}\mathbf{i}-\dfrac{\sqrt{6}}{6}\mathbf{j}+\dfrac{\sqrt{6}}{6}\mathbf{k}$

51. $\mathbf{v}\bullet\mathbf{w}=(\mathbf{i}-\mathbf{j})\bullet(\mathbf{i}+\mathbf{j}+\mathbf{k})$
$=1\cdot1+(-1)(1)+0\cdot1$
$=1-1+0$
$=0$

$\cos\theta=\dfrac{\mathbf{v}\bullet\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}$
$=\dfrac{0}{\sqrt{1^2+(-1)^2+0^2}\sqrt{1^2+1^2+1^2}}$
$=\dfrac{0}{\sqrt{2}\sqrt{3}}$
$=0$
$\theta=\dfrac{\pi}{2}$ radians $=90°$

52. $\mathbf{v}\bullet\mathbf{w}=(\mathbf{i}+\mathbf{j})\bullet(-\mathbf{i}+\mathbf{j}-\mathbf{k})$
$=1\cdot(-1)+1(1)+0\cdot(-1)$
$=-1+1+0$
$=0$

$$\cos\theta = \frac{\mathbf{v}\bullet\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}$$

$$= \frac{0}{\sqrt{1^2+1^2+0^2}\sqrt{(-1)^2+1^2+(-1)^2}}$$

$$= \frac{0}{\sqrt{2}\sqrt{3}}$$

$$= 0$$

$$\theta = \frac{\pi}{2}\text{ radians} = 90°$$

53. $\mathbf{v}\bullet\mathbf{w} = (2\mathbf{i}+\mathbf{j}-3\mathbf{k})\bullet(\mathbf{i}+2\mathbf{j}+2\mathbf{k})$

$$= 2\cdot1+1(2)+(-3)(2)$$

$$= 2+2-6$$

$$= -2$$

$$\cos\theta = \frac{\mathbf{v}\bullet\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}$$

$$= \frac{-2}{\sqrt{2^2+1^2+(-3)^2}\sqrt{1^2+2^2+2^2}}$$

$$= \frac{-2}{\sqrt{14}\sqrt{9}}$$

$$= \frac{-2}{3\sqrt{14}} \approx -0.1782$$

$$\theta \approx 1.75 \text{ radians} \approx 100.3°$$

54. $\mathbf{v}\bullet\mathbf{w} = (2\mathbf{i}+2\mathbf{j}-\mathbf{k})\bullet(\mathbf{i}+2\mathbf{j}+3\mathbf{k})$

$$= 2\cdot1+2(2)+(-1)(3)$$

$$= 2+4-3$$

$$= 3$$

$$\cos\theta = \frac{\mathbf{v}\bullet\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}$$

$$= \frac{3}{\sqrt{2^2+2^2+(-1)^2}\sqrt{1^2+2^2+3^2}}$$

$$= \frac{3}{\sqrt{9}\sqrt{14}}$$

$$= \frac{3}{3\sqrt{14}} \approx 0.2673$$

$$\theta \approx 1.30 \text{ radians} \approx 74.5°$$

55. $\mathbf{v}\bullet\mathbf{w} = (3\mathbf{i}-\mathbf{j}+2\mathbf{k})\bullet(\mathbf{i}+\mathbf{j}-\mathbf{k})$

$$= 3\cdot1+(-1)(1)+2(-1)$$

$$= 3-1-2$$

$$= 0$$

$$\cos\theta = \frac{\mathbf{v}\bullet\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}$$

$$= \frac{0}{\sqrt{3^2+(-1)^2+2^2}\sqrt{1^2+1^2+(-1)^2}}$$

$$= \frac{0}{\sqrt{14}\sqrt{3}}$$

$$= 0$$

$$\theta = \frac{\pi}{2}\text{ radians} = 90°$$

56. $\mathbf{v}\bullet\mathbf{w} = (\mathbf{i}+3\mathbf{j}+2\mathbf{k})\bullet(\mathbf{i}-\mathbf{j}+\mathbf{k})$

$$= 1\cdot1+3(-1)+2(1)$$

$$= 1-3+2$$

$$= 0$$

$$\cos\theta = \frac{\mathbf{v}\bullet\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}$$

$$= \frac{0}{\sqrt{1^2+3^2+2^2}\sqrt{1^2+(-1)^2+1^2}}$$

$$= \frac{0}{\sqrt{14}\sqrt{3}}$$

$$= 0$$

$$\theta = \frac{\pi}{2}\text{ radians} = 90°$$

57. $\mathbf{v}\bullet\mathbf{w} = (3\mathbf{i}+4\mathbf{j}+\mathbf{k})\bullet(6\mathbf{i}+8\mathbf{j}+2\mathbf{k})$

$$= 3\cdot6+4\cdot8+1\cdot2$$

$$= 18+32+2$$

$$= 52$$

$$\cos\theta = \frac{\mathbf{v}\bullet\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}$$

$$= \frac{52}{\sqrt{3^2+4^2+1^2}\sqrt{6^2+8^2+2^2}}$$

$$= \frac{52}{\sqrt{26}\sqrt{104}}$$

$$= \frac{52}{52}$$

$$= 1$$

$$\theta = 0 \text{ radians} = 0°$$

58. $\mathbf{v}\bullet\mathbf{w} = (3\mathbf{i}-4\mathbf{j}+\mathbf{k})\bullet(6\mathbf{i}-8\mathbf{j}+2\mathbf{k})$

$$= 3\cdot6+(-4)\cdot(-8)+1\cdot2$$

$$= 18+32+2$$

$$= 52$$

$$\cos\theta = \frac{\mathbf{v}\bullet\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}$$

$$= \frac{52}{\sqrt{3^2+(-4)^2+1^2}\sqrt{6^2+(-8)^2+2^2}}$$

$$= \frac{52}{\sqrt{26}\sqrt{104}}$$

$$= \frac{52}{52}$$

$$= 1$$

$$\theta = 0 \text{ radians} = 0°$$

59. $\cos\alpha = \dfrac{a}{\|\mathbf{v}\|} = \dfrac{3}{\sqrt{3^2+(-6)^2+(-2)^2}} = \dfrac{3}{\sqrt{49}} = \dfrac{3}{7}$

$\alpha \approx 64.6°$

$\cos\beta = \dfrac{b}{\|\mathbf{v}\|} = \dfrac{-6}{\sqrt{3^2+(-6)^2+(-2)^2}} = \dfrac{-6}{\sqrt{49}} = -\dfrac{6}{7}$

$\beta \approx 149.0°$

$\cos\gamma = \dfrac{c}{\|\mathbf{v}\|} = \dfrac{-2}{\sqrt{3^2+(-6)^2+(-2)^2}} = \dfrac{-2}{\sqrt{49}} = -\dfrac{2}{7}$

$\gamma \approx 106.6°$

$\mathbf{v} = 7\left(\cos 64.6°\,\mathbf{i} + \cos 149.0°\,\mathbf{j} + \cos 106.6°\,\mathbf{k}\right)$

60. $\cos\alpha = \dfrac{a}{\|\mathbf{v}\|} = \dfrac{-6}{\sqrt{(-6)^2+12^2+4^2}} = \dfrac{-6}{\sqrt{196}} = -\dfrac{3}{7}$

$\alpha \approx 115.4°$

$\cos\beta = \dfrac{b}{\|\mathbf{v}\|} = \dfrac{12}{\sqrt{(-6)^2+12^2+4^2}} = \dfrac{12}{\sqrt{196}} = \dfrac{6}{7}$

$\beta \approx 31.0°$

$\cos\gamma = \dfrac{c}{\|\mathbf{v}\|} = \dfrac{4}{\sqrt{(-6)^2+12^2+4^2}} = \dfrac{4}{\sqrt{196}} = \dfrac{2}{7}$

$\gamma \approx 73.4°$

$\mathbf{v} = 14\left(\cos 115.4°\,\mathbf{i} + \cos 31.0°\,\mathbf{j} + \cos 73.4°\,\mathbf{k}\right)$

61. $\cos\alpha = \dfrac{a}{\|\mathbf{v}\|} = \dfrac{1}{\sqrt{1^2+1^2+1^2}} = \dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$

$\alpha \approx 54.7°$

$\cos\beta = \dfrac{b}{\|\mathbf{v}\|} = \dfrac{1}{\sqrt{1^2+1^2+1^2}} = \dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$

$\beta \approx 54.7°$

$\cos\gamma = \dfrac{c}{\|\mathbf{v}\|} = \dfrac{1}{\sqrt{1^2+1^2+1^2}} = \dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$

$\gamma \approx 54.7°$

$\mathbf{v} = \sqrt{3}\left(\cos 54.7°\,\mathbf{i} + \cos 54.7°\,\mathbf{j} + \cos 54.7°\,\mathbf{k}\right)$

62. $\cos\alpha = \dfrac{a}{\|\mathbf{v}\|} = \dfrac{1}{\sqrt{1^2+(-1)^2+(-1)^2}} = \dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$

$\alpha \approx 54.7°$

$\cos\beta = \dfrac{b}{\|\mathbf{v}\|} = \dfrac{-1}{\sqrt{1^2+(-1)^2+(-1)^2}} = \dfrac{-1}{\sqrt{3}} = -\dfrac{\sqrt{3}}{3}$

$\beta \approx 125.3°$

$\cos\gamma = \dfrac{c}{\|\mathbf{v}\|} = \dfrac{-1}{\sqrt{1^2+(-1)^2+(-1)^2}} = \dfrac{-1}{\sqrt{3}} = -\dfrac{\sqrt{3}}{3}$

$\gamma \approx 125.3°$

$\mathbf{v} = \sqrt{3}\left(\cos 54.7°\,\mathbf{i} + \cos 125.3°\,\mathbf{j} + \cos 125.3°\,\mathbf{k}\right)$

63. $\cos\alpha = \dfrac{a}{\|\mathbf{v}\|} = \dfrac{1}{\sqrt{1^2+1^2+0^2}} = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$

$\alpha = 45°$

$\cos\beta = \dfrac{b}{\|\mathbf{v}\|} = \dfrac{1}{\sqrt{1^2+1^2+0^2}} = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$

$\beta = 45°$

$\cos\gamma = \dfrac{c}{\|\mathbf{v}\|} = \dfrac{0}{\sqrt{1^2+1^2+0^2}} = \dfrac{0}{\sqrt{2}} = 0$

$\gamma = 90°$

$\mathbf{v} = \sqrt{2}\left(\cos 45°\,\mathbf{i} + \cos 45°\,\mathbf{j} + \cos 90°\,\mathbf{k}\right)$

64. $\cos\alpha = \dfrac{a}{\|\mathbf{v}\|} = \dfrac{0}{\sqrt{0^2+1^2+1^2}} = \dfrac{0}{\sqrt{2}} = 0$

$\alpha = 90°$

$\cos\beta = \dfrac{b}{\|\mathbf{v}\|} = \dfrac{1}{\sqrt{0^2+1^2+1^2}} = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$

$\beta = 45°$

$\cos\gamma = \dfrac{c}{\|\mathbf{v}\|} = \dfrac{1}{\sqrt{0^2+1^2+1^2}} = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$

$\gamma = 45°$

$\mathbf{v} = \sqrt{2}\left(\cos 90°\,\mathbf{i} + \cos 45°\,\mathbf{j} + \cos 45°\,\mathbf{k}\right)$

65. $\cos\alpha = \dfrac{a}{\|\mathbf{v}\|} = \dfrac{3}{\sqrt{3^2+(-5)^2+2^2}} = \dfrac{3}{\sqrt{38}}$

$\alpha \approx 60.9°$

$\cos\beta = \dfrac{b}{\|\mathbf{v}\|} = \dfrac{-5}{\sqrt{3^2+(-5)^2+2^2}} = -\dfrac{5}{\sqrt{38}}$

$\beta \approx 144.2°$

$$\cos \gamma = \frac{c}{\|\mathbf{v}\|} = \frac{2}{\sqrt{3^2 + (-5)^2 + 2^2}} = \frac{2}{\sqrt{38}}$$

$$\gamma \approx 71.1°$$

$$\mathbf{v} = \sqrt{38}\left(\cos 60.9°\mathbf{i} + \cos 144.2°\mathbf{j} + \cos 71.1°\mathbf{k}\right)$$

66. $\cos \alpha = \dfrac{a}{\|\mathbf{v}\|} = \dfrac{2}{\sqrt{2^2 + 3^2 + (-4)^2}} = \dfrac{2}{\sqrt{29}}$

$$\alpha \approx 68.2°$$

$$\cos \beta = \frac{b}{\|\mathbf{v}\|} = \frac{3}{\sqrt{2^2 + 3^2 + (-4)^2}} = \frac{3}{\sqrt{29}}$$

$$\beta \approx 56.1°$$

$$\cos \gamma = \frac{c}{\|\mathbf{v}\|} = \frac{-4}{\sqrt{2^2 + 3^2 + (-4)^2}} = -\frac{4}{\sqrt{29}}$$

$$\gamma \approx 138.0°$$

$$\mathbf{v} = \sqrt{29}\left(\cos 68.2°\mathbf{i} + \cos 56.1°\mathbf{j} + \cos 138.0°\mathbf{k}\right)$$

67. a. $\mathbf{d} = \langle 2,3,4 \rangle + \langle 1,-1,3 \rangle + \langle 4,-1,-2 \rangle$

$$= \langle 2+1+4, 3-1-1, 4+3-2 \rangle$$

$$= \langle 7,1,5 \rangle$$

b. $\sqrt{(7-0)^2 + (1-0)^2 + (5-0)^2}$

$$= \sqrt{(7)^2 + (1)^2 + (5)^2}$$

$$= \sqrt{49+1+25}$$

$$= \sqrt{75} = 5\sqrt{3} \approx 8.66$$

The distance between the hand and the origin is approximately 8.66 feet.

68. If the point $P = (x, y, z)$ is on the sphere with center $C = (x_0, y_0, z_0)$ and radius r, then the distance between P and C is

$$d(P_0, P) = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2} = r.$$

Therefore, the equation of the sphere is

$$(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = r^2.$$

69. $(x-3)^2 + (y-1)^2 + (z-1)^2 = 1^2$

$$(x-3)^2 + (y-1)^2 + (z-1)^2 = 1$$

70. $(x-1)^2 + (y-2)^2 + (z-2)^2 = 2^2$

$$(x-1)^2 + (y-2)^2 + (z-2)^2 = 4$$

71. $\quad x^2 + y^2 + z^2 + 2x - 2y = 2$

$$(x^2 + 2x) + (y^2 - 2y) + z^2 = 2$$

$$(x^2 + 2x + 1) + (y^2 - 2y + 1) + z^2 = 2 + 1 + 1$$

$$(x+1)^2 + (y-1)^2 + (z-0)^2 = 4$$

Center: (–1, 1, 0); Radius: 2

72. $\quad x^2 + y^2 + z^2 + 2x - 2z = -1$

$$(x^2 + 2x) + y^2 + (z^2 - 2z) = -1$$

$$(x^2 + 2x + 1) + y^2 + (z^2 - 2z + 1) = -1 + 1 + 1$$

$$(x+1)^2 + (y-0)^2 + (z-1)^2 = 1$$

Center: (–1, 0, 1); Radius: 1

73. $\quad x^2 + y^2 + z^2 - 4x + 4y + 2z = 0$

$$(x^2 - 4x) + (y^2 + 4y) + (z^2 + 2z) = 0$$

$$(x^2 - 4x + 4) + (y^2 + 4y + 4)$$
$$+ (z^2 + 2z + 1) = 4 + 4 + 1$$

$$(x-2)^2 + (y+2)^2 + (z+1)^2 = 9$$

Center: (2, –2, –1); Radius: 3

74. $\quad x^2 + y^2 + z^2 - 4x = 0$

$$(x^2 - 4x) + y^2 + z^2 = 0$$

$$(x^2 - 4x + 4) + y^2 + z^2 = 4$$

$$(x-2)^2 + (y-0)^2 + (z-0)^2 = 4$$

Center: (2, 0, 0); Radius: 2

75. $\quad 2x^2 + 2y^2 + 2z^2 - 8x + 4z = -1$

$$(x^2 - 4x) + y^2 + (z^2 + 2z) = -\frac{1}{2}$$

$$(x^2 - 4x + 4) + y^2 + (z^2 + 2z + 1) = -\frac{1}{2} + 4 + 1$$

$$(x-2)^2 + (y-0)^2 + (z+1)^2 = \frac{9}{2}$$

Center: (2, 0, –1); Radius: $\dfrac{3\sqrt{2}}{2}$

76. $\quad 3x^2 + 3y^2 + 3z^2 + 6x - 6y = 3$

$$(x^2 + 2x) + (y^2 - 2y) + z^2 = 1$$

$$(x^2 + 2x + 1) + (y^2 - 2y + 1) + z^2 = 1 + 1 + 1$$

$$(x+1)^2 + (y-1)^2 + (z-0)^2 = 3$$

Center: (–1, 1, 0); Radius: $\sqrt{3}$

77. Write the force as a vector:

$$\cos\alpha = \frac{2}{\sqrt{2^2 + 1^2 + 2^2}} = \frac{2}{\sqrt{9}} = \frac{2}{3}$$

$$\cos\beta = \frac{1}{3}$$

$$\cos\gamma = \frac{2}{3}$$

$$\mathbf{F} = 3\left(\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}\right)$$

$$W = 3\left(\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}\right) \bullet 2\mathbf{j} = 3\left(\frac{1}{3}\cdot 2\right)$$

$$= 2 \text{ newton-meters (joules)}$$

78. Write the force as a vector:

$$\cos\alpha = \frac{2}{\sqrt{2^2 + 2^2 + 1^2}} = \frac{2}{\sqrt{9}} = \frac{2}{3}$$

$$\cos\beta = \frac{2}{3}$$

$$\cos\gamma = \frac{1}{3}$$

$$\mathbf{F} = 1\left(\frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}\right)$$

$$W = 1\left(\frac{2}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} + \frac{1}{3}\mathbf{k}\right) \bullet (1\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$$

$$= 1\left(\frac{2}{3}\cdot 1 + \frac{2}{3}\cdot 2 + \frac{1}{3}\cdot 2\right)$$

$$= \frac{8}{3} \text{ newton-meters (joules)}$$

79. $W = \mathbf{F} \bullet \mathbf{AB}$

$$= (2\mathbf{i} - \mathbf{j} - \mathbf{k}) \bullet (3\mathbf{i} + 2\mathbf{j} - 5\mathbf{k})$$

$$= 2\cdot 3 + (-1)(2) + (-1)(-5)$$

$$= 9 \text{ newton-meters (joules)}$$

Section 9.7

1. True

2. True

3. True

4. False; It is orthogonal to both.

5. False

6. True

7. $\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = 3\cdot 2 - 1\cdot 4 = 6 - 4 = 2$

8. $\begin{vmatrix} -2 & 5 \\ 2 & -3 \end{vmatrix} = -2(-3) - 2\cdot 5 = 6 - 10 = -4$

9. $\begin{vmatrix} 6 & 5 \\ -2 & -1 \end{vmatrix} = 6(-1) - (-2)(5) = -6 + 10 = 4$

10. $\begin{vmatrix} -4 & 0 \\ 5 & 3 \end{vmatrix} = -4\cdot 3 - 5\cdot 0 = -12 - 0 = -12$

11. $\begin{vmatrix} A & B & C \\ 2 & 1 & 4 \\ 1 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 4 \\ 3 & 1 \end{vmatrix} A - \begin{vmatrix} 2 & 4 \\ 1 & 1 \end{vmatrix} B + \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} C$

$$= (1 - 12)A - (2 - 4)B + (6 - 1)C$$

$$= -11A + 2B + 5C$$

12. $\begin{vmatrix} A & B & C \\ 0 & 2 & 4 \\ 3 & 1 & 3 \end{vmatrix} = \begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix} A - \begin{vmatrix} 0 & 4 \\ 3 & 3 \end{vmatrix} B + \begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix} C$

$$= (6 - 4)A - (0 - 12)B + (0 - 6)C$$

$$= 2A + 12B - 6C$$

13. $\begin{vmatrix} A & B & C \\ -1 & 3 & 5 \\ 5 & 0 & -2 \end{vmatrix}$

$$= \begin{vmatrix} 3 & 5 \\ 0 & -2 \end{vmatrix} A - \begin{vmatrix} -1 & 5 \\ 5 & -2 \end{vmatrix} B + \begin{vmatrix} -1 & 3 \\ 5 & 0 \end{vmatrix} C$$

$$= (-6 - 0)A - (2 - 25)B + (0 - 15)C$$

$$= -6A + 23B - 15C$$

14. $\begin{vmatrix} A & B & C \\ 1 & -2 & -3 \\ 0 & 2 & -2 \end{vmatrix}$

$$= \begin{vmatrix} -2 & -3 \\ 2 & -2 \end{vmatrix} A - \begin{vmatrix} 1 & -3 \\ 0 & -2 \end{vmatrix} B + \begin{vmatrix} 1 & -2 \\ 0 & 2 \end{vmatrix} C$$

$$= (4 + 6)A - (-2 - 0)B + (2 - 0)C$$

$$= 10A + 2B + 2C$$

15. a. $\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & 1 \\ 3 & -2 & -1 \end{vmatrix}$

$$= \begin{vmatrix} -3 & 1 \\ -2 & -1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & -3 \\ 3 & -2 \end{vmatrix} \mathbf{k}$$

$$= 5\mathbf{i} + 5\mathbf{j} + 5\mathbf{k}$$

b. $\mathbf{w} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & -1 \\ 2 & -3 & 1 \end{vmatrix}$

$= \begin{vmatrix} -2 & -1 \\ -3 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 3 & -1 \\ 2 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 3 & -2 \\ 2 & -3 \end{vmatrix} \mathbf{k}$

$= -5\mathbf{i} - 5\mathbf{j} - 5\mathbf{k}$

c. $\mathbf{w} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & -1 \\ 3 & -2 & -1 \end{vmatrix}$

$= \begin{vmatrix} -2 & -1 \\ -2 & -1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 3 & -1 \\ 3 & -1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 3 & -2 \\ 3 & -2 \end{vmatrix} \mathbf{k}$

$= 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$

d. $\mathbf{v} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & 1 \\ 2 & -3 & 1 \end{vmatrix}$

$= \begin{vmatrix} -3 & 1 \\ -3 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & -3 \\ 2 & -3 \end{vmatrix} \mathbf{k}$

$= 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$

16. a. $\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 3 & 2 \\ 3 & -2 & -1 \end{vmatrix}$

$= \begin{vmatrix} 3 & 2 \\ -2 & -1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -1 & 2 \\ 3 & -1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -1 & 3 \\ 3 & -2 \end{vmatrix} \mathbf{k}$

$= \mathbf{i} + 5\mathbf{j} - 7\mathbf{k}$

b. $\mathbf{w} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & -1 \\ -1 & 3 & 2 \end{vmatrix}$

$= \begin{vmatrix} -2 & -1 \\ 3 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 3 & -1 \\ -1 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 3 & -2 \\ -1 & 3 \end{vmatrix} \mathbf{k}$

$= -1\mathbf{i} - 5\mathbf{j} + 7\mathbf{k}$

$= -\mathbf{i} - 5\mathbf{j} + 7\mathbf{k}$

c. $\mathbf{w} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & -1 \\ 3 & -2 & -1 \end{vmatrix}$

$= \begin{vmatrix} -2 & -1 \\ -2 & -1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 3 & -1 \\ 3 & -1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 3 & -2 \\ 3 & -2 \end{vmatrix} \mathbf{k}$

$= 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$

d. $\mathbf{v} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 3 & 2 \\ -1 & 3 & 2 \end{vmatrix}$

$= \begin{vmatrix} 3 & 2 \\ 3 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -1 & 2 \\ -1 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -1 & 3 \\ -1 & 3 \end{vmatrix} \mathbf{k}$

$= 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$

17. a. $\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{vmatrix}$

$= \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} \mathbf{k}$

$= 1\mathbf{i} - 1\mathbf{j} - 1\mathbf{k}$

$= \mathbf{i} - \mathbf{j} - \mathbf{k}$

b. $\mathbf{w} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix}$

$= \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} \mathbf{k}$

$= -1\mathbf{i} + 1\mathbf{j} + 1\mathbf{k}$

$= -\mathbf{i} + \mathbf{j} + \mathbf{k}$

c. $\mathbf{w} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{vmatrix}$

$= \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} \mathbf{k}$

$= 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$

d. $\mathbf{v} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{vmatrix}$

$= \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} \mathbf{k}$

$= 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$

18. a. $\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -4 & 2 \\ 3 & 2 & 1 \end{vmatrix}$

$= \begin{vmatrix} -4 & 2 \\ 2 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & -4 \\ 3 & 2 \end{vmatrix} \mathbf{k}$

$= -8\mathbf{i} + 5\mathbf{j} + 14\mathbf{k}$

b. $\mathbf{w} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & 1 \\ 1 & -4 & 2 \end{vmatrix}$

$= \begin{vmatrix} 2 & 1 \\ -4 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 3 & 2 \\ 1 & -4 \end{vmatrix} \mathbf{k}$

$= 8\mathbf{i} - 5\mathbf{j} - 14\mathbf{k}$

c. $\mathbf{w} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & 1 \\ 3 & 2 & 1 \end{vmatrix}$

$= \begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 3 & 1 \\ 3 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 3 & 2 \\ 3 & 2 \end{vmatrix} \mathbf{k}$

$= 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$

d. $\mathbf{v} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -4 & 2 \\ 1 & -4 & 2 \end{vmatrix}$

$= \begin{vmatrix} -4 & 2 \\ -4 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & -4 \\ 1 & -4 \end{vmatrix} \mathbf{k}$

$= 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$

19. a. $\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 2 \\ 0 & 1 & -1 \end{vmatrix}$

$= \begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 2 \\ 0 & -1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} \mathbf{k}$

$= -1\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$

$= -\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$

b. $\mathbf{w} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & -1 \\ 2 & -1 & 2 \end{vmatrix}$

$= \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 0 & -1 \\ 2 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 0 & 1 \\ 2 & -1 \end{vmatrix} \mathbf{k}$

$= 1\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$

$= \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$

c. $\mathbf{w} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{vmatrix}$

$= \begin{vmatrix} 1 & -1 \\ 1 & -1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 0 & -1 \\ 0 & -1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix} \mathbf{k}$

$= 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$

d. $\mathbf{v} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 2 \\ 2 & -1 & 2 \end{vmatrix}$

$= \begin{vmatrix} -1 & 2 \\ -1 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 2 \\ 2 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & -1 \\ 2 & -1 \end{vmatrix} \mathbf{k}$

$= 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$

20. a. $\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & 3 \\ 1 & 0 & -1 \end{vmatrix}$

$= \begin{vmatrix} 1 & 3 \\ 0 & -1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 3 & 3 \\ 1 & -1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 3 & 1 \\ 1 & 0 \end{vmatrix} \mathbf{k}$

$= -\mathbf{i} + 6\mathbf{j} - \mathbf{k}$

b. $\mathbf{w} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -1 \\ 3 & 1 & 3 \end{vmatrix}$

$= \begin{vmatrix} 0 & -1 \\ 1 & 3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & -1 \\ 3 & 3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix} \mathbf{k}$

$= \mathbf{i} - 6\mathbf{j} + \mathbf{k}$

c. $\mathbf{w} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{vmatrix}$

$= \begin{vmatrix} 0 & -1 \\ 0 & -1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & -1 \\ 1 & -1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} \mathbf{k}$

$= 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$

d. $\mathbf{v} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & 3 \\ 3 & 1 & 3 \end{vmatrix}$

$= \begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 3 & 3 \\ 3 & 3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 3 & 1 \\ 3 & 1 \end{vmatrix} \mathbf{k}$

$= 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$

21. a. $\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & -1 \\ 4 & 0 & -3 \end{vmatrix}$

$= \begin{vmatrix} -1 & -1 \\ 0 & -3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & -1 \\ 4 & -3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & -1 \\ 4 & 0 \end{vmatrix} \mathbf{k}$

$= 3\mathbf{i} - 1\mathbf{j} + 4\mathbf{k}$

$= 3\mathbf{i} - \mathbf{j} + 4\mathbf{k}$

b. $\mathbf{w} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 0 & -3 \\ 1 & -1 & -1 \end{vmatrix}$

$= \begin{vmatrix} 0 & -3 \\ -1 & -1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 4 & -3 \\ 1 & -1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 4 & 0 \\ 1 & -1 \end{vmatrix} \mathbf{k}$

$= -3\mathbf{i} + 1\mathbf{j} - 4\mathbf{k}$

$= -3\mathbf{i} + \mathbf{j} - 4\mathbf{k}$

c. $\mathbf{w} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 0 & -3 \\ 4 & 0 & -3 \end{vmatrix}$

$= \begin{vmatrix} 0 & -3 \\ 0 & -3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 4 & -3 \\ 4 & -3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 4 & 0 \\ 4 & 0 \end{vmatrix} \mathbf{k}$

$= 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$

d. $\mathbf{v} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{vmatrix}$

$= \begin{vmatrix} -1 & -1 \\ -1 & -1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & -1 \\ 1 & -1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & -1 \\ 1 & -1 \end{vmatrix} \mathbf{k}$

$= 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$

22. a. $\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & 0 \\ 0 & 3 & -2 \end{vmatrix}$

$= \begin{vmatrix} -3 & 0 \\ 3 & -2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 0 \\ 0 & -2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & -3 \\ 0 & 3 \end{vmatrix} \mathbf{k}$

$= 6\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$

b. $\mathbf{w} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 3 & -2 \\ 2 & -3 & 0 \end{vmatrix}$

$= \begin{vmatrix} 3 & -2 \\ -3 & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 0 & -2 \\ 2 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 0 & 3 \\ 2 & -3 \end{vmatrix} \mathbf{k}$

$= -6\mathbf{i} - 4\mathbf{j} - 6\mathbf{k}$

c. $\mathbf{w} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 3 & -2 \\ 0 & 3 & -2 \end{vmatrix}$

$= \begin{vmatrix} 3 & -2 \\ 3 & -2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 0 & -2 \\ 0 & -2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 0 & 3 \\ 0 & 3 \end{vmatrix} \mathbf{k}$

$= 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$

d. $\mathbf{v} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & 0 \\ 2 & -3 & 0 \end{vmatrix}$

$= \begin{vmatrix} -3 & 0 \\ -3 & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 0 \\ 2 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & -3 \\ 2 & -3 \end{vmatrix} \mathbf{k}$

$= 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$

23. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & 1 \\ -3 & 3 & 2 \end{vmatrix}$

$= \begin{vmatrix} -3 & 1 \\ 3 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 1 \\ -3 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & -3 \\ -3 & 3 \end{vmatrix} \mathbf{k}$

$= -9\mathbf{i} - 7\mathbf{j} - 3\mathbf{k}$

24. $\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 3 & 2 \\ 1 & 1 & 3 \end{vmatrix}$

$= \begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -3 & 2 \\ 1 & 3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -3 & 3 \\ 1 & 1 \end{vmatrix} \mathbf{k}$

$= 7\mathbf{i} + 11\mathbf{j} - 6\mathbf{k}$

25. $\mathbf{v} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 3 & 2 \\ 2 & -3 & 1 \end{vmatrix}$

$= \begin{vmatrix} 3 & 2 \\ -3 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -3 & 2 \\ 2 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -3 & 3 \\ 2 & -3 \end{vmatrix} \mathbf{k}$

$= 9\mathbf{i} + 7\mathbf{j} + 3\mathbf{k}$

26. $\mathbf{w} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 3 \\ -3 & 3 & 2 \end{vmatrix}$

$= \begin{vmatrix} 1 & 3 \\ 3 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 3 \\ -3 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 1 \\ -3 & 3 \end{vmatrix} \mathbf{k}$

$= -7\mathbf{i} - 11\mathbf{j} + 6\mathbf{k}$

27. $\mathbf{v} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 3 & 2 \\ -3 & 3 & 2 \end{vmatrix}$

$= \begin{vmatrix} 3 & 2 \\ 3 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -3 & 2 \\ -3 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -3 & 3 \\ -3 & 3 \end{vmatrix} \mathbf{k}$

$= 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$

28. $\mathbf{w} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 3 \\ 1 & 1 & 3 \end{vmatrix}$

$= \begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} \mathbf{k}$

$= 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$

29. $(3\mathbf{u}) \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & -9 & 3 \\ -3 & 3 & 2 \end{vmatrix}$

$= \begin{vmatrix} -9 & 3 \\ 3 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 6 & 3 \\ -3 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 6 & -9 \\ -3 & 3 \end{vmatrix} \mathbf{k}$

$= -27\mathbf{i} - 21\mathbf{j} - 9\mathbf{k}$

30. $\mathbf{v} \times (4\mathbf{w}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 3 & 2 \\ 4 & 4 & 12 \end{vmatrix}$

$= \begin{vmatrix} 3 & 2 \\ 4 & 12 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -3 & 2 \\ 4 & 12 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -3 & 3 \\ 4 & 4 \end{vmatrix} \mathbf{k}$

$= 28\mathbf{i} + 44\mathbf{j} - 24\mathbf{k}$

31. $\mathbf{u} \times (2\mathbf{v}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & 1 \\ -6 & 6 & 4 \end{vmatrix}$

$= \begin{vmatrix} -3 & 1 \\ 6 & 4 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 1 \\ -6 & 4 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & -3 \\ -6 & 6 \end{vmatrix} \mathbf{k}$

$= -18\mathbf{i} - 14\mathbf{j} - 6\mathbf{k}$

32. $(-3\mathbf{v}) \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 9 & -9 & -6 \\ 1 & 1 & 3 \end{vmatrix}$

$= \begin{vmatrix} -9 & -6 \\ 1 & 3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 9 & -6 \\ 1 & 3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 9 & -9 \\ 1 & 1 \end{vmatrix} \mathbf{k}$

$= -21\mathbf{i} - 33\mathbf{j} + 18\mathbf{k}$

33. $\mathbf{u} \bullet (\mathbf{u} \times \mathbf{v})$

$= \mathbf{u} \bullet \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & 1 \\ -3 & 3 & 2 \end{vmatrix}$

$= \mathbf{u} \bullet \left(\begin{vmatrix} -3 & 1 \\ 3 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 1 \\ -3 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & -3 \\ -3 & 3 \end{vmatrix} \mathbf{k} \right)$

$= (2\mathbf{i} - 3\mathbf{j} + \mathbf{k}) \bullet (-9\mathbf{i} - 7\mathbf{j} - 3\mathbf{k})$

$= 2(-9) + (-3)(-7) + 1(-3)$

$= -18 + 21 - 3$

$= 0$

34. $\mathbf{v} \bullet (\mathbf{v} \times \mathbf{w}) = \mathbf{v} \bullet \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 3 & 2 \\ 1 & 1 & 3 \end{vmatrix}$

$= \mathbf{v} \bullet \left(\begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -3 & 2 \\ 1 & 3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -3 & 3 \\ 1 & 1 \end{vmatrix} \mathbf{k} \right)$

$= (-3\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}) \bullet (7\mathbf{i} + 11\mathbf{j} - 6\mathbf{k})$

$= -3 \cdot 7 + 3(11) + 2(-6)$

$= -21 + 33 - 12$

$= 0$

35. $\mathbf{u} \bullet (\mathbf{v} \times \mathbf{w}) = \mathbf{u} \bullet \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 3 & 2 \\ 1 & 1 & 3 \end{vmatrix}$

$= \mathbf{u} \bullet \left(\begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -3 & 2 \\ 1 & 3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -3 & 3 \\ 1 & 1 \end{vmatrix} \mathbf{k} \right)$

$= (2\mathbf{i} - 3\mathbf{j} + \mathbf{k}) \bullet (7\mathbf{i} + 11\mathbf{j} - 6\mathbf{k})$

$= 2 \cdot 7 + (-3)(11) + 1(-6)$

$= 14 - 33 - 6$

$= -25$

36. $(\mathbf{u} \times \mathbf{v}) \bullet \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & 1 \\ -3 & 3 & 2 \end{vmatrix} \bullet \mathbf{w}$

$= \left(\begin{vmatrix} -3 & 1 \\ 3 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 1 \\ -3 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & -3 \\ -3 & 3 \end{vmatrix} \mathbf{k} \right) \bullet \mathbf{w}$

$= (-9\mathbf{i} - 7\mathbf{j} - 3\mathbf{k}) \bullet (\mathbf{i} + \mathbf{j} + 3\mathbf{k})$

$= -9 \cdot 1 + (-7)(1) + (-3)(3)$

$= -9 - 7 - 9$

$= -25$

37. $\mathbf{v} \bullet (\mathbf{u} \times \mathbf{w}) = \mathbf{v} \bullet \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & 1 \\ 1 & 1 & 3 \end{vmatrix}$

$= \mathbf{v} \bullet \left(\begin{vmatrix} -3 & 1 \\ 1 & 3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix} \mathbf{k} \right)$

$= (-3\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}) \bullet (-10\mathbf{i} - 5\mathbf{j} + 5\mathbf{k})$

$= -3(-10) + 3(-5) + 2 \cdot 5$

$= 30 - 15 + 10$

$= 25$

38. $(\mathbf{v} \times \mathbf{u}) \bullet \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 3 & 2 \\ 2 & -3 & 1 \end{vmatrix} \bullet \mathbf{w}$

$= \left(\begin{vmatrix} 3 & 2 \\ -3 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -3 & 2 \\ 2 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -3 & 3 \\ 2 & -3 \end{vmatrix} \mathbf{k} \right) \bullet \mathbf{w}$

$= (9\mathbf{i} + 7\mathbf{j} + 3\mathbf{k}) \bullet (\mathbf{i} + \mathbf{j} + 3\mathbf{k})$

$= 9 \cdot 1 + 7 \cdot 1 + 3 \cdot 3$

$= 9 + 7 + 9$

$= 25$

39. $\mathbf{u} \times (\mathbf{v} \times \mathbf{v}) = \mathbf{u} \times \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 3 & 2 \\ -3 & 3 & 2 \end{vmatrix}$

$= \mathbf{u} \times \left(\begin{vmatrix} 3 & 2 \\ 3 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -3 & 2 \\ -3 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -3 & 3 \\ -3 & 3 \end{vmatrix} \mathbf{k} \right)$

$= (2\mathbf{i} - 3\mathbf{j} + \mathbf{k}) \times (0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k})$

$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & 1 \\ 0 & 0 & 0 \end{vmatrix}$

$= \begin{vmatrix} -3 & 1 \\ 0 & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 1 \\ 0 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & -3 \\ 0 & 0 \end{vmatrix} \mathbf{k}$

$= 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$

40. $(\mathbf{w} \times \mathbf{w}) \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 3 \\ 1 & 1 & 3 \end{vmatrix} \times \mathbf{v}$

$= \left(\begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} \mathbf{k} \right) \times \mathbf{v}$

$= (0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}) \times \mathbf{v}$

$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 0 \\ -3 & 3 & 2 \end{vmatrix}$

$= \begin{vmatrix} 0 & 0 \\ 3 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 0 & 0 \\ -3 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 0 & 0 \\ -3 & 3 \end{vmatrix} \mathbf{k}$

$= 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}$

41. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & 1 \\ -3 & 3 & 2 \end{vmatrix}$

$= \begin{vmatrix} -3 & 1 \\ 3 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 1 \\ -3 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & -3 \\ -3 & 3 \end{vmatrix} \mathbf{k}$

$= -9\mathbf{i} - 7\mathbf{j} - 3\mathbf{k}$

Actually, any vector of the form $c(-9\mathbf{i} - 7\mathbf{j} - 3\mathbf{k})$, where c is a nonzero scalar, is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

42. $\mathbf{u} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & 1 \\ 1 & 1 & 3 \end{vmatrix}$

$= \begin{vmatrix} -3 & 1 \\ 1 & 3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix} \mathbf{k}$

$= -10\mathbf{i} - 5\mathbf{j} + 5\mathbf{k}$

Actually, any vector of the form $c(-10\mathbf{i} - 5\mathbf{j} + 5\mathbf{k})$, where c is a nonzero scalar, is orthogonal to both $\mathbf{u}$ and $\mathbf{w}$.

43. A vector that is orthogonal to both $\mathbf{u}$ and $\mathbf{i} + \mathbf{j}$ is $\mathbf{u} \times (\mathbf{i} + \mathbf{j})$.

$\mathbf{u} \times (\mathbf{i} + \mathbf{j}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & 1 \\ 1 & 1 & 0 \end{vmatrix}$

$= \begin{vmatrix} -3 & 1 \\ 1 & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix} \mathbf{k}$

$= -1\mathbf{i} + 1\mathbf{j} + 5\mathbf{k}$

Actually, any vector of the form $c(-\mathbf{i} + \mathbf{j} + 5\mathbf{k})$, where c is a nonzero scalar, is orthogonal to both $\mathbf{u}$ and $\mathbf{i} + \mathbf{j}$

44. A vector that is orthogonal to both **u** and **j** + **k** is **u** × (**j** + **k**).

$$\mathbf{u} \times (\mathbf{j}+\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & 1 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} -3 & 1 \\ 1 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & -3 \\ 0 & 1 \end{vmatrix}\mathbf{k}$$

$$= -4\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$$

Actually, any vector of the form $c(-4\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$, where c is a nonzero scalar, is orthogonal to both **u** and **j** + **k**.

45. $\mathbf{u} = P_1P_2 = 1\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$
$\mathbf{v} = P_1P_3 = -2\mathbf{i} + 3\mathbf{j} + 0\mathbf{k}$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ -2 & 3 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 3 \\ 3 & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 3 \\ -2 & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 2 \\ -2 & 3 \end{vmatrix}\mathbf{k}$$

$$= -9\mathbf{i} - 6\mathbf{j} + 7\mathbf{k}$$

Area $= \|\mathbf{u} \times \mathbf{v}\| = \sqrt{(-9)^2 + (-6)^2 + 7^2} = \sqrt{166}$

46. $\mathbf{u} = P_1P_2 = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$
$\mathbf{v} = P_1P_3 = -2\mathbf{i} + 4\mathbf{j} + \mathbf{k}$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 1 \\ -2 & 4 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 1 \\ 4 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 1 \\ -2 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & 3 \\ -2 & 4 \end{vmatrix}\mathbf{k}$$

$$= -\mathbf{i} - 4\mathbf{j} + 14\mathbf{k}$$

Area $= \|\mathbf{u} \times \mathbf{v}\| = \sqrt{(-1)^2 + (-4)^2 + 14^2} = \sqrt{213}$

47. $\mathbf{u} = P_1P_2 = -3\mathbf{i} + 1\mathbf{j} + 4\mathbf{k}$
$\mathbf{v} = P_1P_3 = -1\mathbf{i} - 4\mathbf{j} + 3\mathbf{k}$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 1 & 4 \\ -1 & -4 & 3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 4 \\ -4 & 3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -3 & 4 \\ -1 & 3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -3 & 1 \\ -1 & -4 \end{vmatrix}\mathbf{k}$$

$$= 19\mathbf{i} + 5\mathbf{j} + 13\mathbf{k}$$

Area $= \|\mathbf{u} \times \mathbf{v}\| = \sqrt{19^2 + 5^2 + 13^2} = \sqrt{555}$

48. $\mathbf{u} = P_1P_2 = 4\mathbf{i} + \mathbf{j} - 3\mathbf{k}$
$\mathbf{v} = P_1P_3 = 4\mathbf{i} - 1\mathbf{j} + 0\mathbf{k}$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 1 & -3 \\ 4 & -1 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -3 \\ -1 & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 4 & -3 \\ 4 & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 4 & 1 \\ 4 & -1 \end{vmatrix}\mathbf{k}$$

$$= -3\mathbf{i} - 12\mathbf{j} - 8\mathbf{k}$$

Area $= \|\mathbf{u} \times \mathbf{v}\|$
$= \sqrt{(-3)^2 + (-12)^2 + (-8)^2} = \sqrt{217}$

49. $\mathbf{u} = P_1P_2 = 0\mathbf{i} + 1\mathbf{j} + 1\mathbf{k}$
$\mathbf{v} = P_1P_3 = -3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 1 \\ -3 & 2 & -2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 \\ 2 & -2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 0 & 1 \\ -3 & -2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 0 & 1 \\ -3 & 2 \end{vmatrix}\mathbf{k}$$

$$= -4\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$$

Area $= \|\mathbf{u} \times \mathbf{v}\| = \sqrt{(-4)^2 + (-3)^2 + 3^2} = \sqrt{34}$

50. $\mathbf{u} = P_1P_2 = 0\mathbf{i} + 2\mathbf{j} + 0\mathbf{k}$
$\mathbf{v} = P_1P_3 = -4\mathbf{i} + 3\mathbf{j} + 0\mathbf{k}$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2 & 0 \\ -4 & 3 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 0 \\ 3 & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 0 & 0 \\ -4 & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 0 & 2 \\ -4 & 3 \end{vmatrix}\mathbf{k}$$

$$= 0\mathbf{i} + 0\mathbf{j} + 8\mathbf{k}$$

Area $= \|\mathbf{u} \times \mathbf{v}\| = \sqrt{0^2 + 0^2 + 8^2} = \sqrt{64} = 8$

51. $\mathbf{u} = P_1P_2 = 3\mathbf{i} + 0\mathbf{j} - 2\mathbf{k}$
$\mathbf{v} = P_1P_3 = 5\mathbf{i} - 7\mathbf{j} + 3\mathbf{k}$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 0 & -2 \\ 5 & -7 & 3 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & -2 \\ -7 & 3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & -2 \\ 5 & 3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & 0 \\ 5 & -7 \end{vmatrix}\mathbf{k}$$

$$= -14\mathbf{i} - 19\mathbf{j} - 21\mathbf{k}$$

Area $= \|\mathbf{u} \times \mathbf{v}\|$
$= \sqrt{(-14)^2 + (-19)^2 + (-21)^2} = \sqrt{998}$

52. $\mathbf{u} = P_1P_2 = 0\mathbf{i} + 1\mathbf{j} + 1\mathbf{k}$

$\mathbf{v} = P_1P_3 = -2\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}$

$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 1 \\ -2 & 3 & -6 \end{vmatrix}$

$= \begin{vmatrix} 1 & 1 \\ 3 & -6 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 0 & 1 \\ -2 & -6 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 0 & 1 \\ -2 & 3 \end{vmatrix}\mathbf{k}$

$= -9\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$

Area $= \|\mathbf{u} \times \mathbf{v}\| = \sqrt{(-9)^2 + (-2)^2 + 2^2} = \sqrt{89}$

53. $\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & -2 \\ -2 & 1 & 3 \end{vmatrix}$

$= \begin{vmatrix} 3 & -2 \\ 1 & 3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & -2 \\ -2 & 3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 3 \\ -2 & 1 \end{vmatrix}\mathbf{k}$

$= 11\mathbf{i} + 1\mathbf{j} + 7\mathbf{k}$

$\|\mathbf{v} \times \mathbf{w}\| = \sqrt{11^2 + 1^2 + 7^2} = \sqrt{171}$

$\mathbf{u} = \pm\dfrac{\mathbf{v} \times \mathbf{w}}{\|\mathbf{v} \times \mathbf{w}\|}$

$= \pm\dfrac{11\mathbf{i} + 1\mathbf{j} + 7\mathbf{k}}{\sqrt{171}}$

$= \pm\left(\dfrac{11}{\sqrt{171}}\mathbf{i} + \dfrac{1}{\sqrt{171}}\mathbf{j} + \dfrac{7}{\sqrt{171}}\mathbf{k}\right)$

$= \pm\left(\dfrac{11\sqrt{19}}{57}\mathbf{i} + \dfrac{\sqrt{19}}{57}\mathbf{j} + \dfrac{7\sqrt{19}}{57}\mathbf{k}\right)$

So, $\dfrac{11\sqrt{19}}{57}\mathbf{i} + \dfrac{\sqrt{19}}{57}\mathbf{j} + \dfrac{7\sqrt{19}}{57}\mathbf{k}$ or

$-\dfrac{11\sqrt{19}}{57}\mathbf{i} - \dfrac{\sqrt{19}}{57}\mathbf{j} - \dfrac{7\sqrt{19}}{57}\mathbf{k}$.

54. $\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & -1 \\ -2 & -4 & -3 \end{vmatrix}$

$= \begin{vmatrix} 3 & -1 \\ -4 & -3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & -1 \\ -2 & -3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & 3 \\ -2 & -4 \end{vmatrix}\mathbf{k}$

$= -13\mathbf{i} + 8\mathbf{j} - 2\mathbf{k}$

$\|\mathbf{v} \times \mathbf{w}\| = \sqrt{(-13)^2 + 8^2 + (-2)^2} = \sqrt{237}$

$\mathbf{u} = \pm\dfrac{\mathbf{v} \times \mathbf{w}}{\|\mathbf{v} \times \mathbf{w}\|}$

$= \pm\dfrac{-13\mathbf{i} + 8\mathbf{j} - 2\mathbf{k}}{\sqrt{237}}$

$= \pm\left(-\dfrac{13}{\sqrt{237}}\mathbf{i} + \dfrac{8}{\sqrt{237}}\mathbf{j} - \dfrac{2}{\sqrt{237}}\mathbf{k}\right)$

$= \pm\left(-\dfrac{13\sqrt{237}}{237}\mathbf{i} + \dfrac{8\sqrt{237}}{237}\mathbf{j} - \dfrac{2\sqrt{237}}{237}\mathbf{k}\right)$

So, $-\dfrac{13\sqrt{237}}{237}\mathbf{i} + \dfrac{8\sqrt{237}}{237}\mathbf{j} - \dfrac{2\sqrt{237}}{237}\mathbf{k}$ or

$\dfrac{13\sqrt{237}}{237}\mathbf{i} - \dfrac{8\sqrt{237}}{237}\mathbf{j} + \dfrac{2\sqrt{237}}{237}\mathbf{k}$.

55. $\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 4 \\ 2 & 1 & -2 \end{vmatrix}$

$= \begin{vmatrix} -2 & 4 \\ 1 & -2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & 4 \\ 2 & -2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & -2 \\ 2 & 1 \end{vmatrix}\mathbf{k}$

$= 0\mathbf{i} + 14\mathbf{j} + 7\mathbf{k}$

$V = |(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}|$

$= |(0\mathbf{i} + 14\mathbf{j} + 7\mathbf{k}) \cdot (3\mathbf{i} - 6\mathbf{j} - 2\mathbf{k})|$

$= |0(3) + 14(-6) + 7(-2)|$

$= |-98|$

$= 98$ cubic units

56. $\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 6 \\ 2 & 3 & -8 \end{vmatrix}$

$= \begin{vmatrix} 0 & 6 \\ 3 & -8 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 6 \\ 2 & -8 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix}\mathbf{k}$

$= -18\mathbf{i} + 20\mathbf{j} + 3\mathbf{k}$

$V = |(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}|$

$= |(-18\mathbf{i} + 20\mathbf{j} + 3\mathbf{k}) \cdot (8\mathbf{i} - 5\mathbf{j} + 6\mathbf{k})|$

$= |-18(8) + 20(-5) + 3(6)|$

$= |-226|$

$= 226$ cubic units

57. Prove: $\left\| \mathbf{u} \times \mathbf{v} \right\|^2 = \left\| \mathbf{u} \right\|^2 \left\| \mathbf{v} \right\|^2 - \left(\mathbf{u} \bullet \mathbf{v} \right)^2$

Let $\mathbf{u} = a_1 \mathbf{i} + b_1 \mathbf{j} + c_1 \mathbf{k}$ and $\mathbf{v} = a_2 \mathbf{i} + b_2 \mathbf{j} + c_2 \mathbf{k}$.

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \mathbf{k} = \left(b_1 c_2 - b_2 c_1 \right) \mathbf{i} - \left(a_1 c_2 - a_2 c_1 \right) \mathbf{j} + \left(a_1 b_2 - a_2 b_1 \right) \mathbf{k}$$

$$\begin{aligned} \left\| \mathbf{u} \times \mathbf{v} \right\|^2 &= \left(b_1 c_2 - b_2 c_1 \right)^2 + \left(a_1 c_2 - a_2 c_1 \right)^2 + \left(a_1 b_2 - a_2 b_1 \right)^2 \\ &= b_1^2 c_2^2 - 2 b_1 b_2 c_1 c_2 + b_2^2 c_1^2 + a_1^2 c_2^2 - 2 a_1 a_2 c_1 c_2 + a_2^2 c_1^2 + a_1^2 b_2^2 - 2 a_1 a_2 b_1 b_2 + a_2^2 b_1^2 \\ &= a_1^2 b_2^2 + a_1^2 c_2^2 + a_2^2 b_1^2 + a_2^2 c_1^2 + b_1^2 c_2^2 + b_2^2 c_1^2 - 2 a_1 a_2 b_1 b_2 - 2 a_1 a_2 c_1 c_2 - 2 b_1 b_2 c_1 c_2 \end{aligned}$$

$$\left\| \mathbf{u} \right\|^2 = a_1^2 + b_1^2 + c_1^2$$

$$\left\| \mathbf{v} \right\|^2 = a_2^2 + b_2^2 + c_2^2$$

$$\left(\mathbf{u} \bullet \mathbf{v} \right)^2 = \left(a_1 a_2 + b_1 b_2 + c_1 c_2 \right)^2$$

$$\left\| \mathbf{u} \right\|^2 \left\| \mathbf{v} \right\|^2 - \left(\mathbf{u} \bullet \mathbf{v} \right)^2 = \left(a_1^2 + b_1^2 + c_1^2 \right) \left(a_2^2 + b_2^2 + c_2^2 \right) - \left(a_1 a_2 + b_1 b_2 + c_1 c_2 \right)^2$$

$$\begin{aligned} &= a_1^2 a_2^2 + a_1^2 b_2^2 + a_1^2 c_2^2 + b_1^2 a_2^2 + b_1^2 b_2^2 + b_1^2 c_2^2 + c_1^2 a_2^2 + c_1^2 b_2^2 + c_1^2 c_2^2 \\ &\quad - \left(a_1^2 a_2^2 + a_1 a_2 b_1 b_2 + a_1 a_2 c_1 c_2 + a_1 a_2 b_1 b_2 + b_1^2 b_2^2 + b_1 b_2 c_1 c_2 + a_1 a_2 c_1 c_2 + b_1 b_2 c_1 c_2 + c_1^2 c_2^2 \right) \\ &= a_1^2 b_2^2 + a_1^2 c_2^2 + a_2^2 b_1^2 + a_2^2 c_1^2 + b_1^2 c_2^2 + b_2^2 c_1^2 - 2 a_1 a_2 b_1 b_2 - 2 a_1 a_2 c_1 c_2 - 2 b_1 b_2 c_1 c_2 \end{aligned}$$

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58. We know that for any two vectors $\left\| \mathbf{u} \times \mathbf{v} \right\| = \left\| \mathbf{u} \right\| \left\| \mathbf{v} \right\| \sin \theta$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$. Now, if $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, then $\theta = 90°$, so $\left\| \mathbf{u} \times \mathbf{v} \right\| = \left\| \mathbf{u} \right\| \left\| \mathbf{v} \right\| \sin 90° = \left\| \mathbf{u} \right\| \left\| \mathbf{v} \right\|$.

59. By Problem 58, since $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, we know $\left\| \mathbf{u} \times \mathbf{v} \right\| = \left\| \mathbf{u} \right\| \left\| \mathbf{v} \right\|$. Since $\mathbf{u}$ and $\mathbf{v}$ are unit vectors, we have $\left\| \mathbf{u} \times \mathbf{v} \right\| = \left\| \mathbf{u} \right\| \left\| \mathbf{v} \right\| = 1 \cdot 1 = 1$. Thus $\mathbf{u} \times \mathbf{v}$ is a unit vector.

60. Prove: $\mathbf{u} \times \mathbf{v} = - \left(\mathbf{v} \times \mathbf{u} \right)$

Let $\mathbf{u} = a_1 \mathbf{i} + b_1 \mathbf{j} + c_1 \mathbf{k}$ and $\mathbf{v} = a_2 \mathbf{i} + b_2 \mathbf{j} + c_2 \mathbf{k}$

$$\mathbf{u} \times \mathbf{v}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

$$= \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \mathbf{k}$$

$$= \left(b_1 c_2 - b_2 c_1 \right) \mathbf{i} - \left(a_1 c_2 - a_2 c_1 \right) \mathbf{j} + \left(a_1 b_2 - a_2 b_1 \right) \mathbf{k}$$

$$= - \left(b_2 c_1 - b_1 c_2 \right) \mathbf{i} + \left(a_2 c_1 - a_1 c_2 \right) \mathbf{j} - \left(a_2 b_1 - a_1 b_2 \right) \mathbf{k}$$

$$= - \left[\left(b_2 c_1 - b_1 c_2 \right) \mathbf{i} - \left(a_2 c_1 - a_1 c_2 \right) \mathbf{j} + \left(a_2 b_1 - a_1 b_2 \right) \mathbf{k} \right]$$

$$= - \left(\begin{vmatrix} b_2 & c_2 \\ b_1 & c_1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_2 & c_2 \\ a_1 & c_1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_2 & b_2 \\ a_1 & b_1 \end{vmatrix} \mathbf{k} \right)$$

$$= - \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix}$$

$$= - \left(\mathbf{v} \times \mathbf{u} \right)$$

61. Prove: $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$

Let $\mathbf{u} = a_1 \mathbf{i} + b_1 \mathbf{j} + c_1 \mathbf{k}$, $\mathbf{v} = a_2 \mathbf{i} + b_2 \mathbf{j} + c_2 \mathbf{k}$, and $\mathbf{w} = a_3 \mathbf{i} + b_3 \mathbf{j} + c_3 \mathbf{k}$.

$\mathbf{u} \times (\mathbf{v} + \mathbf{w})$

$= \mathbf{u} \times \left[(a_2 + a_3)\mathbf{i} + (b_2 + b_3)\mathbf{j} + (c_2 + c_3)\mathbf{k} \right]$

$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & b_1 & c_1 \\ a_2 + a_3 & b_2 + b_3 & c_2 + c_3 \end{vmatrix}$

$= \begin{vmatrix} b_1 & c_1 \\ b_2 + b_3 & c_2 + c_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & c_1 \\ a_2 + a_3 & c_2 + c_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & b_1 \\ a_2 + a_3 & b_2 + b_3 \end{vmatrix} \mathbf{k}$

$= (b_1 c_2 + b_1 c_3 - b_2 c_1 - b_3 c_1)\mathbf{i} - (a_1 c_2 + a_1 c_3 - a_2 c_1 - a_3 c_1)\mathbf{j} + (a_1 b_2 + a_1 b_3 - a_2 b_1 - a_3 b_1)\mathbf{k}$

$= (b_1 c_2 - b_2 c_1)\mathbf{i} + (b_1 c_3 - b_3 c_1)\mathbf{i} - (a_1 c_2 - a_2 c_1)\mathbf{j} - (a_1 c_3 - a_3 c_1)\mathbf{j} + (a_1 b_2 - a_2 b_1)\mathbf{k} + (a_1 b_3 - a_3 b_1)\mathbf{k}$

$= (b_1 c_2 - b_2 c_1)\mathbf{i} - (a_1 c_2 - a_2 c_1)\mathbf{j} + (a_1 b_2 - a_2 b_1)\mathbf{k} + (b_1 c_3 - b_3 c_1)\mathbf{i} - (a_1 c_3 - a_3 c_1)\mathbf{j} + (a_1 b_3 - a_3 b_1)\mathbf{k}$

$= \left(\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \mathbf{k} \right) + \left(\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} \mathbf{k} \right)$

$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$= (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$

62. From Exercise 57, we have:

$\| \mathbf{u} \times \mathbf{v} \|^2 = \| \mathbf{u} \|^2 \| \mathbf{v} \|^2 - (\mathbf{u} \bullet \mathbf{v})^2$

$\qquad = \| \mathbf{u} \|^2 \| \mathbf{v} \|^2 - (\| \mathbf{u} \| \| \mathbf{v} \| \cos \theta)^2$

$\qquad = \| \mathbf{u} \|^2 \| \mathbf{v} \|^2 - \| \mathbf{u} \|^2 \| \mathbf{v} \|^2 \cos^2 \theta$

$\qquad = \| \mathbf{u} \|^2 \| \mathbf{v} \|^2 (1 - \cos^2 \theta)$

$\qquad = \| \mathbf{u} \|^2 \| \mathbf{v} \|^2 \sin^2 \theta$

$\| \mathbf{u} \times \mathbf{v} \| = \| \mathbf{u} \| \| \mathbf{v} \| \sin \theta$

63. $\mathbf{u} \bullet \mathbf{v} = 0 \implies \mathbf{u}$ and $\mathbf{v}$ are orthogonal.

$\mathbf{u} \times \mathbf{v} = 0 \implies \mathbf{u}$ and $\mathbf{v}$ are orthogonal.

Therefore, if $\mathbf{u} \bullet \mathbf{v} = 0$ and $\mathbf{u} \times \mathbf{v} = 0$, then either $\mathbf{u} = 0$ or $\mathbf{v} = 0$.

Chapter 9 Review Exercises

1. $\left(3, \dfrac{\pi}{6} \right)$
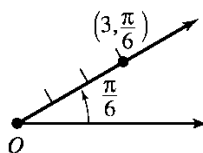
$x = 3\cos\dfrac{\pi}{6} = \dfrac{3\sqrt{3}}{2}$; $\ y = 3\sin\dfrac{\pi}{6} = \dfrac{3}{2}$

Rectangular coordinates of $\left(3, \dfrac{\pi}{6} \right)$ are $\left(\dfrac{3\sqrt{3}}{2}, \dfrac{3}{2} \right)$.
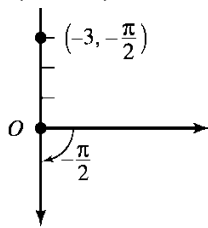
2. $\left(-2, \dfrac{4\pi}{3} \right)$
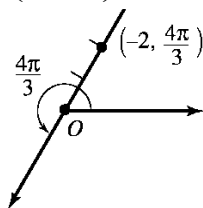
$x = -2\cos\dfrac{4\pi}{3} = 1; \quad y = -2\sin\dfrac{4\pi}{3} = \sqrt{3}$

Rectangular coordinates of $\left(-2, \dfrac{4\pi}{3}\right)$ are $\left(1, \sqrt{3}\right)$.

3. $\left(-3, -\dfrac{\pi}{2}\right)$

$x = -3\cos\left(-\dfrac{\pi}{2}\right) = 0; \quad y = -3\sin\left(-\dfrac{\pi}{2}\right) = 3$

Rectangular coordinates of $\left(-3, -\dfrac{\pi}{2}\right)$ are $(0, 3)$.

4. The point $(-3, 3)$ lies in quadrant II.

$r = \sqrt{x^2 + y^2} = \sqrt{(-3)^2 + 3^2} = 3\sqrt{2}$

$\theta = \tan^{-1}\left(\dfrac{y}{x}\right) = \tan^{-1}\left(\dfrac{3}{-3}\right) = \tan^{-1}(-1) = -\dfrac{\pi}{4}$

Polar coordinates of the point $(-3, 3)$ are

$\left(-3\sqrt{2}, -\dfrac{\pi}{4}\right)$ or $\left(3\sqrt{2}, \dfrac{3\pi}{4}\right)$.

5. The point $(0, -2)$ lies on the negative y-axis.

$r = \sqrt{x^2 + y^2} = \sqrt{0^2 + (-2)^2} = 2$

$\theta = \tan^{-1}\left(\dfrac{y}{x}\right) = \tan^{-1}\left(\dfrac{-2}{0}\right)$

$\dfrac{-2}{0}$ is undefined, so $\theta = -\dfrac{\pi}{2}$; $\quad -\dfrac{\pi}{2} + \pi = \dfrac{\pi}{2}$.

Polar coordinates of the point $(0, -2)$ are

$\left(2, -\dfrac{\pi}{2}\right)$ or $\left(-2, \dfrac{\pi}{2}\right)$.

6. The point $(3, 4)$ lies in quadrant I.

$r = \sqrt{x^2 + y^2} = \sqrt{3^2 + 4^2} = 5$

$\theta = \tan^{-1}\left(\dfrac{y}{x}\right) = \tan^{-1}\left(\dfrac{4}{3}\right) \approx 0.93$

$\tan^{-1}\left(\dfrac{4}{3}\right) + \pi \approx 4.07$

Polar coordinates of the point $(3, 4)$ are

$(5, 0.93)$ or $(-5, 4.07)$.

7. a.
$$r = 2\sin\theta$$
$$r^2 = 2r\sin\theta$$
$$x^2 + y^2 = 2y$$
$$x^2 + y^2 - 2y = 0$$
$$x^2 + y^2 - 2y + 1 = 1$$
$$x^2 + (y-1)^2 = 1^2$$

b. The graph is a circle with center $(0,1)$ and radius 1.
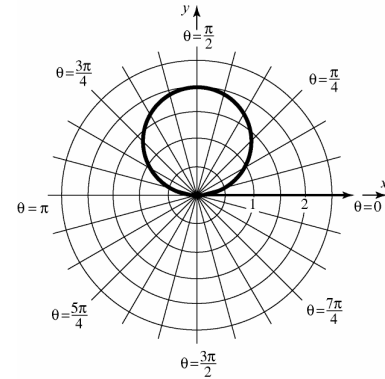
8. a.
$$r = 5$$
$$r^2 = 25$$
$$x^2 + y^2 = 5^2$$

b. The graph is a circle with center $(0,0)$ and radius 5.

9. a.
$$\theta = \dfrac{\pi}{4}$$
$$\tan\theta = \tan\left(\dfrac{\pi}{4}\right)$$
$$\dfrac{y}{x} = 1$$
$$y = x$$
$$x - y = 0$$
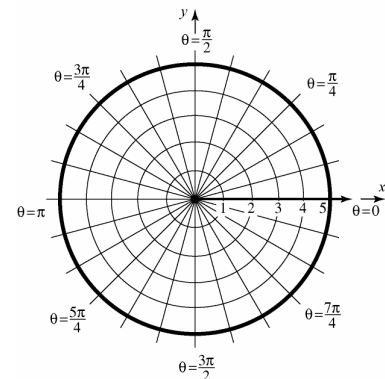
b. The graph is a line through the point $(0,0)$ with slope 1.
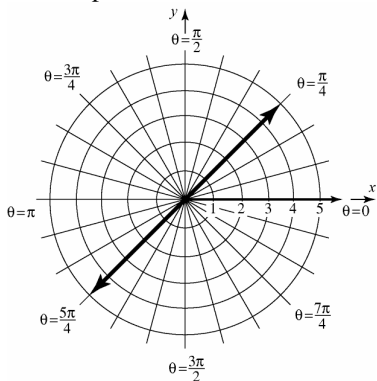
10. a.
$$r^2 + 4r\sin\theta - 8r\cos\theta = 5$$
$$x^2 + y^2 + 4y - 8x = 5$$
$$x^2 - 8x + 16 + y^2 + 4y + 4 = 5 + 16 + 4$$
$$(x-4)^2 + (y+2)^2 = 25$$

b. The graph is a circle with center $(4,-2)$ and radius 5.
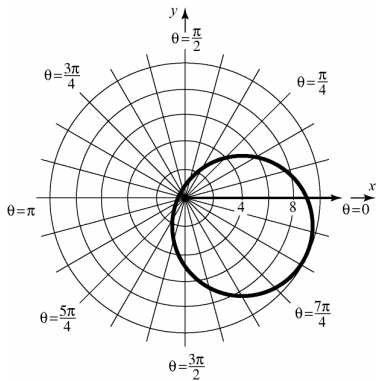
11. $r = 4\cos\theta$

The graph will be a circle with radius 2 and center $(2, 0)$.

Check for symmetry:

Polar axis: Replace θ by $-\theta$. The result is $r = 4\cos(-\theta) = 4\cos\theta$.

The graph is symmetric with respect to the polar axis.

The line $\theta = \dfrac{\pi}{2}$: Replace θ by $\pi - \theta$.
$$r = 4\cos(\pi - \theta)$$
$$= 4(\cos\pi\cos\theta + \sin\pi\sin\theta)$$
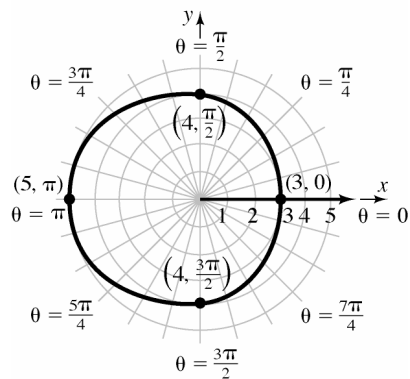$$= 4(-\cos\theta + 0)$$
$$= -4\cos\theta$$

The test fails.

The pole: Replace r by $-r$. $-r = 4\cos\theta$. The test fails.

Due to symmetry with respect to the polar axis, assign values to θ from 0 to π.

θ	$r = 4\cos\theta$
0	4
$\dfrac{\pi}{6}$	$2\sqrt{3} \approx 3.5$
$\dfrac{\pi}{3}$	2
$\dfrac{\pi}{2}$	0
$\dfrac{2\pi}{3}$	-2
$\dfrac{5\pi}{6}$	$-2\sqrt{3} \approx -3.5$
π	-4

12. $r = 3 - 3\sin\theta$

The graph will be a cardioid. Check for symmetry:

Polar axis: Replace θ by $-\theta$. The result is $r = 3 - 3\sin(-\theta) = 3 + 3\sin\theta$. The test fails.

The line $\theta = \dfrac{\pi}{2}$: Replace θ by $\pi - \theta$.

$r = 3 - 3\sin(\pi - \theta)$

$\quad = 3 - 3(\sin\pi\cos\theta - \cos\pi\sin\theta)$

$\quad = 3 - 3(0 + \sin\theta)$

$\quad = 3 - 3\sin\theta$

The graph is symmetric with respect to the line $\theta = \dfrac{\pi}{2}$.

The pole: Replace r by $-r$. $-r = 3 - 3\sin\theta$. The test fails.

Due to symmetry with respect to the line $\theta = \dfrac{\pi}{2}$, assign values to θ from $-\dfrac{\pi}{2}$ to $\dfrac{\pi}{2}$.

θ	$r = 3 - 3\sin\theta$
$-\dfrac{\pi}{2}$	6
$-\dfrac{\pi}{3}$	$3 + \dfrac{3\sqrt{3}}{2} \approx 5.6$
$-\dfrac{\pi}{6}$	$\dfrac{9}{2}$
0	3
$\dfrac{\pi}{6}$	$\dfrac{3}{2}$
$\dfrac{\pi}{3}$	$3 - \dfrac{3\sqrt{3}}{2} \approx 0.4$
$\dfrac{\pi}{2}$	0

13. $r = 4 - \cos\theta$

The graph will be a limaçon without inner loop. Check for symmetry:

Polar axis: Replace θ by $-\theta$. The result is $r = 4 - \cos(-\theta) = 4 - \cos\theta$.

The graph is symmetric with respect to the polar axis.

The line $\theta = \dfrac{\pi}{2}$: Replace θ by $\pi - \theta$.

$r = 4 - \cos(\pi - \theta)$

$\quad = 4 - (\cos\pi\cos\theta + \sin\pi\sin\theta)$

$\quad = 4 - (-\cos\theta + 0)$

$\quad = 4 + \cos\theta$

The test fails.

The pole: Replace r by $-r$. $-r = 4 - \cos\theta$. The test fails.

Due to symmetry with respect to the polar axis, assign values to θ from 0 to π.

θ	$r = 4 - \cos\theta$
0	3
$\dfrac{\pi}{6}$	$4 - \dfrac{\sqrt{3}}{2} \approx 3.1$
$\dfrac{\pi}{3}$	$\dfrac{7}{2}$
$\dfrac{\pi}{2}$	4
$\dfrac{2\pi}{3}$	$\dfrac{9}{2}$
$\dfrac{5\pi}{6}$	$4 + \dfrac{\sqrt{3}}{2} \approx 4.9$
π	5

14. $r = \sqrt{x^2 + y^2} = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$

$\tan \theta = \dfrac{y}{x} = \dfrac{-1}{-1} = 1$

$\theta = 225°$

The polar form of $z = -1 - i$ is

$z = r(\cos \theta + i \sin \theta) = \sqrt{2}(\cos 225° + i \sin 225°)$.

15. $r = \sqrt{x^2 + y^2} = \sqrt{4^2 + (-3)^2} = \sqrt{25} = 5$

$\tan \theta = \dfrac{y}{x} = -\dfrac{3}{4}$

$\theta \approx 323.1°$

The polar form of $z = 4 - 3i$ is

$z = r(\cos \theta + i \sin \theta) = 5(\cos 323.1° + i \sin 323.1°)$.

16. $2(\cos 150° + i \sin 150°) = 2\left(-\dfrac{\sqrt{3}}{2} + \dfrac{1}{2}i\right)$

$= -\sqrt{3} + i$
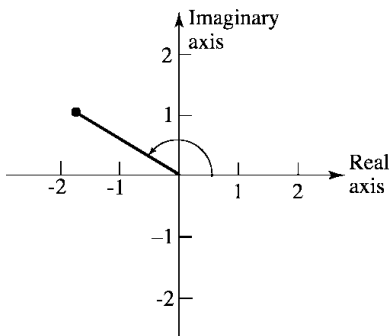
17. $3\left(\cos \dfrac{2\pi}{3} + i \sin \dfrac{2\pi}{3}\right) = 3\left(-\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i\right)$

$= -\dfrac{3}{2} + \dfrac{3\sqrt{3}}{2}i$
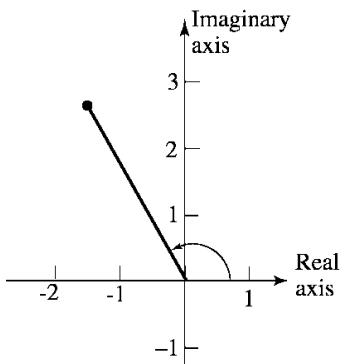
18. $0.1(\cos 350° + i \sin 350°) \approx 0.1(0.9848 - 0.1736i)$

$\approx 0.10 - 0.02i$
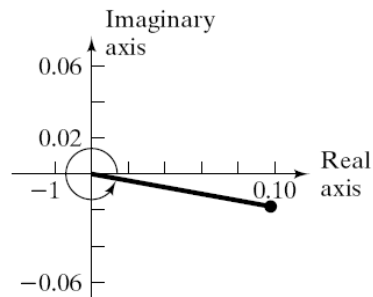
19. $z \cdot w = (\cos 80° + i \sin 80°) \cdot (\cos 50° + i \sin 50°)$

$= 1 \cdot 1[\cos(80° + 50°) + i \sin(80° + 50°)]$

$= \cos 130° + i \sin 130°$

$\dfrac{z}{w} = \dfrac{\cos 80° + i \sin 80°}{\cos 50° + i \sin 50°}$

$= \dfrac{1}{1}[\cos(80° - 50°) + i \sin(80° - 50°)]$

$= \cos 30° + i \sin 30°$

20. $z \cdot w = 3\left(\cos \dfrac{9\pi}{5} + i \sin \dfrac{9\pi}{5}\right) \cdot 2\left(\cos \dfrac{\pi}{5} + i \sin \dfrac{\pi}{5}\right)$

$= 3 \cdot 2\left[\cos\left(\dfrac{9\pi}{5} + \dfrac{\pi}{5}\right) + i \sin\left(\dfrac{9\pi}{5} + \dfrac{\pi}{5}\right)\right]$

$= 6 \cdot (\cos 2\pi + i \sin 2\pi)$

$= 6(\cos 0 + i \sin 0)$

$= 6$

$\dfrac{z}{w} = \dfrac{3 \cdot \left(\cos \dfrac{9\pi}{5} + i \sin \dfrac{9\pi}{5}\right)}{2 \cdot \left(\cos \dfrac{\pi}{5} + i \sin \dfrac{\pi}{5}\right)}$

$= \dfrac{3}{2}\left[\cos\left(\dfrac{9\pi}{5} - \dfrac{\pi}{5}\right) + i \sin\left(\dfrac{9\pi}{5} - \dfrac{\pi}{5}\right)\right]$

$= \dfrac{3}{2}\left(\cos \dfrac{8\pi}{5} + i \sin \dfrac{8\pi}{5}\right)$

21. $z \cdot w = 5(\cos 10° + i \sin 10°) \cdot (\cos 355° + i \sin 355°)$

$= 5 \cdot 1[\cos(10° + 355°) + i \sin(10° + 355°)]$

$= 5(\cos 365° + i \sin 365°)$

$= 5(\cos 5° + i \sin 5°)$

$\dfrac{z}{w} = \dfrac{5(\cos 10° + i\sin 10°)}{\cos 355° + i\sin 355°}$

$= \dfrac{5}{1}[\cos(10° - 355°) + i\sin(10° - 355°)]$

$= 5[\cos(-345°) + i\sin(-345°)]$

$= 5(\cos 15° + i\sin 15°)$

22. $[3(\cos 20° + i\sin 20°)]^3$

$= 3^3[\cos(3 \cdot 20°) + i\sin(3 \cdot 20°)]$

$= 27(\cos 60° + i\sin 60°)$

$= 27\left(\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i\right)$

$= \dfrac{27}{2} + \dfrac{27\sqrt{3}}{2}i$

23. $\left[\sqrt{2}\left(\cos\dfrac{5\pi}{8} + i\sin\dfrac{5\pi}{8}\right)\right]^4$

$= (\sqrt{2})^4\left[\cos\left(4 \cdot \dfrac{5\pi}{8}\right) + i\sin\left(4 \cdot \dfrac{5\pi}{8}\right)\right]$

$= 4 \cdot \left(\cos\dfrac{5\pi}{2} + i\sin\dfrac{5\pi}{2}\right)$

$= 4(0 + 1i)$

$= 4i$

24. $1 - \sqrt{3}i$

$r = \sqrt{1^2 + \left(-\sqrt{3}\right)^2} = 2$

$\tan\theta = \dfrac{-\sqrt{3}}{1} = -\sqrt{3}$

$\theta = 300°$

$1 - \sqrt{3}i = 2(\cos 300° + i\sin 300°)$

$\left(1 - \sqrt{3}i\right)^6 = [2(\cos 300° + i\sin 300°)]^6$

$= 2^6[\cos(6 \cdot 300°) + i\sin(6 \cdot 300°)]$

$= 64 \cdot (\cos 1800° + i\sin 1800°)$

$= 64 \cdot (\cos 0° + i\sin 0°)$

$= 64 + 0i$

$= 64$

25. $3 + 4i$

$r = \sqrt{3^2 + 4^2} = 5$

$\tan\theta = \dfrac{4}{3}$

$\theta \approx 53.13°$

$3 + 4i = 5[\cos(53.13°) + i\sin(53.13°)]$

$(3 + 4i)^4 = \left(5^4\left[\cos(4 \cdot 53.13°) + i\sin(4 \cdot 53.13°)\right]\right)^4$

$= 625[\cos(212.52°) + i\sin(212.52°)]$

$\approx 625(-0.8432 - 0.5376i)$

$= -527 - 336i$

26. $27 + 0i$

$r = \sqrt{27^2 + 0^2} = 27$

$\tan\theta = \dfrac{0}{27} = 0$

$\theta = 0°$

$27 + 0i = 27(\cos 0° + i\sin 0°)$

The three complex cube roots of

$27 + 0i = 27(\cos 0° + i\sin 0°)$ are:

$z_k = \sqrt[3]{27} \cdot \left[\cos\left(\dfrac{0°}{3} + \dfrac{360° \cdot k}{3}\right) + i\sin\left(\dfrac{0°}{3} + \dfrac{360° \cdot k}{3}\right)\right]$

$= 3[\cos(120° k) + i\sin(120° \cdot k)]$

$z_0 = 3[\cos(120° \cdot 0) + i\sin(120° \cdot 0)]$

$= 3 \cdot (\cos 0° + i\sin 0°) = 3$

$z_1 = 3[\cos(120° \cdot 1) + i\sin(120° \cdot 1)]$

$= 3 \cdot (\cos 120° + i\sin 120°) = -\dfrac{3}{2} + \dfrac{3\sqrt{3}}{2}i$

$z_2 = 3[\cos(120° \cdot 2) + i\sin(120° \cdot 2)]$

$= 3 \cdot (\cos 240° + i\sin 240°) = -\dfrac{3}{2} - \dfrac{3\sqrt{3}}{2}i$

27.

28.

$2\mathbf{u} + 3\mathbf{v}$
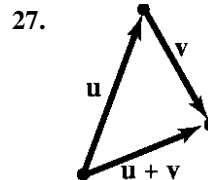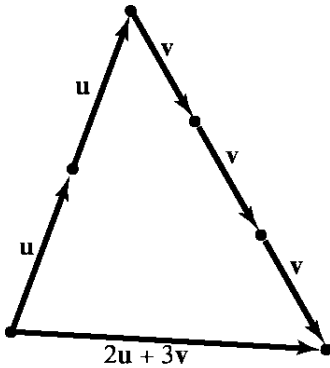
29. $P = (1, -2), Q = (3, -6)$

$\mathbf{v} = (3-1)\mathbf{i} + (-6-(-2))\mathbf{j} = 2\mathbf{i} - 4\mathbf{j}$

$\|\mathbf{v}\| = \sqrt{2^2 + (-4)^2} = \sqrt{20} = 2\sqrt{5}$

30. $P = (0, -2), Q = (-1, 1)$

$\mathbf{v} = (-1-0)\mathbf{i} + (1-(-2))\mathbf{j} = -\mathbf{i} + 3\mathbf{j}$

$\|\mathbf{v}\| = \sqrt{(-1)^2 + 3^2} = \sqrt{10}$

31. $\mathbf{v} + \mathbf{w} = (-2\mathbf{i} + \mathbf{j}) + (4\mathbf{i} - 3\mathbf{j}) = 2\mathbf{i} - 2\mathbf{j}$

32. $4\mathbf{v} - 3\mathbf{w} = 4(-2\mathbf{i} + \mathbf{j}) - 3(4\mathbf{i} - 3\mathbf{j})$

$= -8\mathbf{i} + 4\mathbf{j} - 12\mathbf{i} + 9\mathbf{j}$

$= -20\mathbf{i} + 13\mathbf{j}$

33. $\|\mathbf{v}\| = \|-2\mathbf{i} + \mathbf{j}\| = \sqrt{(-2)^2 + 1^2} = \sqrt{5}$

34. $\|\mathbf{v}\| + \|\mathbf{w}\| = \|-2\mathbf{i} + \mathbf{j}\| + \|4\mathbf{i} - 3\mathbf{j}\|$

$= \sqrt{(-2)^2 + 1^2} + \sqrt{4^2 + (-3)^2}$

$= \sqrt{5} + 5 \approx 7.24$

35. $\mathbf{u} = \dfrac{\mathbf{v}}{\|\mathbf{v}\|} = \dfrac{-2\mathbf{i} + \mathbf{j}}{\|-2\mathbf{i} + \mathbf{j}\|} = \dfrac{-2\mathbf{i} + \mathbf{j}}{\sqrt{(-2)^2 + 1^2}}$

$= \dfrac{-2\mathbf{i} + \mathbf{j}}{\sqrt{5}} = -\dfrac{2\sqrt{5}}{5}\mathbf{i} + \dfrac{\sqrt{5}}{5}\mathbf{j}$

36. Let $\mathbf{v} = x \cdot \mathbf{i} + y \cdot \mathbf{j}$, $\|\mathbf{v}\| = 3$, with the direction angle of $\mathbf{v}$ equal to $60°$.

$\|\mathbf{v}\| = 3$

$\sqrt{x^2 + y^2} = 3$

$x^2 + y^2 = 9$

The angle between $\mathbf{v}$ and $\mathbf{i}$ equals $60°$. Thus,

$\cos 60° = \dfrac{\mathbf{v} \cdot \mathbf{i}}{\|\mathbf{v}\| \cdot \|\mathbf{i}\|} = \dfrac{(x\mathbf{i} + y\mathbf{j}) \cdot \mathbf{i}}{3 \cdot 1} = \dfrac{x}{3 \cdot 1} = \dfrac{x}{3}$.

We also conclude that $\mathbf{v}$ lies in Quadrant I.

$\cos 60° = \dfrac{x}{3}$

$\dfrac{1}{2} = \dfrac{x}{3}$

$x = \dfrac{3}{2}$

$x^2 + y^2 = 9$

$\left(\dfrac{3}{2}\right)^2 + y^2 = 9$

$y^2 = 9 - \left(\dfrac{3}{2}\right)^2 = 9 - \dfrac{9}{4} = \dfrac{36-9}{4} = \dfrac{27}{4}$

$y = \pm\sqrt{\dfrac{27}{4}} = \pm\dfrac{3\sqrt{3}}{2}$

Since $\mathbf{v}$ lies in Quadrant I, $y = \dfrac{3\sqrt{3}}{2}$. So,

$\mathbf{v} = x \cdot \mathbf{i} + y \cdot \mathbf{j} = \dfrac{3}{2}\mathbf{i} + \dfrac{3\sqrt{3}}{2}\mathbf{j}$.

37. $\mathbf{v} = -\mathbf{i} + \sqrt{3}\mathbf{j}$

$= \|\mathbf{v}\|(\cos\alpha\mathbf{i} + \sin\alpha\mathbf{j})$

$\tan\alpha = -\dfrac{\sqrt{3}}{1} = -\sqrt{3}$ $\alpha = \tan^{-1}(-\sqrt{3}) = -60°$

The angle is in quadrant II, thus, $\alpha \approx 120°$.

38. $d(P_1, P_2) = \sqrt{(4-1)^2 + (-2-3)^2 + (1-(-2))^2}$

$= \sqrt{9 + 25 + 9}$

$= \sqrt{43} \approx 6.56$

39. $\mathbf{v} = (4-1)\mathbf{i} + (-2-3)\mathbf{j} + (1-(-2))\mathbf{k}$

$= 3\mathbf{i} - 5\mathbf{j} + 3\mathbf{k}$

40. $4\mathbf{v} - 3\mathbf{w} = 4(3\mathbf{i} + \mathbf{j} - 2\mathbf{k}) - 3(-3\mathbf{i} + 2\mathbf{j} - \mathbf{k})$

$= 12\mathbf{i} + 4\mathbf{j} - 8\mathbf{k} + 9\mathbf{i} - 6\mathbf{j} + 3\mathbf{k}$

$= 21\mathbf{i} - 2\mathbf{j} - 5\mathbf{k}$

41. $\|\mathbf{v}-\mathbf{w}\| = \|(3\mathbf{i}+\mathbf{j}-2\mathbf{k})-(-3\mathbf{i}+2\mathbf{j}-\mathbf{k})\|$

$\qquad = \|3\mathbf{i}+\mathbf{j}-2\mathbf{k}+3\mathbf{i}-2\mathbf{j}+\mathbf{k}\|$

$\qquad = \|6\mathbf{i}-\mathbf{j}-\mathbf{k}\|$

$\qquad = \sqrt{6^2+(-1)^2+(-1)^2}$

$\qquad = \sqrt{36+1+1}$

$\qquad = \sqrt{38}$

42. $\|\mathbf{v}\|-\|\mathbf{w}\|$

$\quad = \|3\mathbf{i}+\mathbf{j}-2\mathbf{k}\|-\|-3\mathbf{i}+2\mathbf{j}-\mathbf{k}\|$

$\quad = \sqrt{3^2+1^2+(-2)^2} - \sqrt{(-3)^2+2^2+(-1)^2}$

$\quad = \sqrt{14}-\sqrt{14}$

$\quad = 0$

43. $\mathbf{v}\times\mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & -2 \\ -3 & 2 & -1 \end{vmatrix}$

$\qquad = \begin{vmatrix} 1 & -2 \\ 2 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & -2 \\ -3 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & 1 \\ -3 & 2 \end{vmatrix}\mathbf{k}$

$\qquad = 3\mathbf{i}+9\mathbf{j}+9\mathbf{k}$

44. $\mathbf{v}\bullet(\mathbf{v}\times\mathbf{w})$

$\quad = \mathbf{v}\bullet\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & -2 \\ -3 & 2 & -1 \end{vmatrix}$

$\quad = \mathbf{v}\bullet\left(\begin{vmatrix} 1 & -2 \\ 2 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & -2 \\ -3 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & 1 \\ -3 & 2 \end{vmatrix}\mathbf{k}\right)$

$\quad = (3\mathbf{i}+\mathbf{j}-2\mathbf{k})\bullet(3\mathbf{i}+9\mathbf{j}+9\mathbf{k})$

$\quad = 3\cdot3+1\cdot9+(-2)(9)$

$\quad = 0$

45. $\pm\dfrac{\mathbf{v}\times\mathbf{w}}{\|\mathbf{v}\times\mathbf{w}\|} = \pm\dfrac{\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & -2 \\ -3 & 2 & -1 \end{vmatrix}}{\|\mathbf{v}\times\mathbf{w}\|}$

$\qquad = \pm\dfrac{\begin{vmatrix} 1 & -2 \\ 2 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & -2 \\ -3 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & 1 \\ -3 & 2 \end{vmatrix}\mathbf{k}}{\|\mathbf{v}\times\mathbf{w}\|}$

$\qquad = \pm\left(\dfrac{3\mathbf{i}+9\mathbf{j}+9\mathbf{k}}{\sqrt{3^2+9^2+9^2}}\right)$

$= \pm\left(\dfrac{3}{3\sqrt{19}}\mathbf{i}+\dfrac{9}{3\sqrt{19}}\mathbf{j}+\dfrac{9}{3\sqrt{19}}\mathbf{k}\right)$

$= \pm\left(\dfrac{1}{\sqrt{19}}\mathbf{i}+\dfrac{3}{\sqrt{19}}\mathbf{j}+\dfrac{3}{\sqrt{19}}\mathbf{k}\right)$

$= \pm\left(\dfrac{\sqrt{19}}{19}\mathbf{i}+\dfrac{3\sqrt{19}}{19}\mathbf{j}+\dfrac{3\sqrt{19}}{19}\mathbf{k}\right)$

So, $\dfrac{\sqrt{19}}{19}\mathbf{i}+\dfrac{3\sqrt{19}}{19}\mathbf{j}+\dfrac{3\sqrt{19}}{19}\mathbf{k}$ or

$-\dfrac{\sqrt{19}}{19}\mathbf{i}-\dfrac{3\sqrt{19}}{19}\mathbf{j}-\dfrac{3\sqrt{19}}{19}\mathbf{k}$

46. $\mathbf{v}=-2\mathbf{i}+\mathbf{j},\quad \mathbf{w}=4\mathbf{i}-3\mathbf{j}$

$\quad \mathbf{v}\cdot\mathbf{w} = -2(4)+1(-3) = -8-3 = -11$

$\quad \cos\theta = \dfrac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}$

$\qquad = \dfrac{-11}{\sqrt{(-2)^2+1^2}\sqrt{4^2+(-3)^2}}$

$\qquad = \dfrac{-11}{\sqrt{5}\cdot5} = \dfrac{-11}{5\sqrt{5}}$

$\quad \theta \approx 169.7°$

47. $\mathbf{v}=\mathbf{i}-3\mathbf{j},\quad \mathbf{w}=-\mathbf{i}+\mathbf{j}$

$\quad \mathbf{v}\cdot\mathbf{w} = 1(-1)+(-3)(1) = -1-3 = -4$

$\quad \cos\theta = \dfrac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}$

$\qquad = \dfrac{-4}{\sqrt{1^2+(-3)^2}\sqrt{(-1)^2+1^2}}$

$\qquad = \dfrac{-4}{\sqrt{10}\sqrt{2}} = \dfrac{-2}{\sqrt{5}} = -\dfrac{2\sqrt{5}}{5}$

$\quad \theta \approx 153.4°$

48. $\mathbf{v}\bullet\mathbf{w} = (\mathbf{i}+\mathbf{j}+\mathbf{k})\bullet(\mathbf{i}-\mathbf{j}+\mathbf{k})$

$\qquad = 1\cdot1+1(-1)+1\cdot1$

$\qquad = 1-1+1$

$\qquad = 1$

$\quad \cos\theta = \dfrac{\mathbf{v}\bullet\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}$

$\qquad = \dfrac{1}{\sqrt{1^2+1^2+1^2}\sqrt{1^2+(-1)^2+1^2}}$

$\qquad = \dfrac{1}{\sqrt{3}\sqrt{3}} = \dfrac{1}{3}$

$\quad \theta \approx 70.5°$

49. $\mathbf{v} \bullet \mathbf{w} = (4\mathbf{i} - \mathbf{j} + 2\mathbf{k}) \bullet (\mathbf{i} - 2\mathbf{j} - 3\mathbf{k})$

$\qquad = 4 \cdot 1 + (-1)(-2) + 2(-3)$

$\qquad = 4 + 2 - 6 = 0$

$\cos \theta = \dfrac{\mathbf{v} \bullet \mathbf{w}}{\| \mathbf{v} \| \| \mathbf{w} \|}$

$\qquad = \dfrac{0}{\sqrt{4^2 + (-1)^2 + 2^2} \sqrt{1^2 + (-2)^2 + (-3)^2}}$

$\qquad = 0$

$\qquad \theta = 90°$

50. $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j}, \quad \mathbf{w} = -4\mathbf{i} - 6\mathbf{j}$

$\mathbf{v} \cdot \mathbf{w} = (2)(-4) + (3)(-6) = -8 - 18 = -26$

$\cos \theta = \dfrac{\mathbf{v} \cdot \mathbf{w}}{\| \mathbf{v} \| \| \mathbf{w} \|}$

$\qquad = \dfrac{-26}{\sqrt{2^2 + 3^2} \sqrt{(-4)^2 + (-6)^2}}$

$\qquad = \dfrac{-26}{\sqrt{13}\sqrt{52}} = \dfrac{-26}{\sqrt{676}} = \dfrac{-26}{26} = -1$

$\qquad \theta = \cos^{-1}(-1) = 180°$

Thus, the vectors are parallel.

51. $\mathbf{v} = -2\mathbf{i} + 2\mathbf{j}, \quad \mathbf{w} = -3\mathbf{i} + 2\mathbf{j}$

$\mathbf{v} \cdot \mathbf{w} = (-2)(-3) + (2)(2) = 6 + 4 = 10$

$\cos \theta = \dfrac{\mathbf{v} \cdot \mathbf{w}}{\| \mathbf{v} \| \| \mathbf{w} \|}$

$\qquad = \dfrac{10}{\sqrt{(-2)^2 + 2^2} \sqrt{(-3)^2 + 2^2}}$

$\qquad = \dfrac{10}{\sqrt{8} \cdot \sqrt{13}} = \dfrac{10}{\sqrt{104}} = 0.9806$

$\theta = \cos^{-1}\left(\dfrac{10}{\sqrt{104}}\right) \approx 11.31°$

Thus, the vectors are neither parallel nor orthogonal.

52. $\mathbf{v} = 3\mathbf{i} - 2\mathbf{j}, \quad \mathbf{w} = 4\mathbf{i} + 6\mathbf{j}$

$\mathbf{v} \cdot \mathbf{w} = (3)(4) + (-2)(6) = 12 - 12 = 0$

Thus, the vectors are orthogonal.

53. $\mathbf{v} = 2\mathbf{i} + \mathbf{j}, \quad \mathbf{w} = -4\mathbf{i} + 3\mathbf{j}$

The decomposition of $\mathbf{v}$ into 2 vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ so that $\mathbf{v}_1$ is parallel to $\mathbf{w}$ and $\mathbf{v}_2$ is

perpendicular to $\mathbf{w}$ is given by: $\mathbf{v}_1 = \dfrac{\mathbf{v} \cdot \mathbf{w}}{\| \mathbf{w} \|^2} \mathbf{w}$

and $\mathbf{v}_2 = \mathbf{v} - \mathbf{v}_1$

$\mathbf{v}_1 = \dfrac{\mathbf{v} \cdot \mathbf{w}}{\| \mathbf{w} \|^2}$

$\qquad = \dfrac{(2\mathbf{i} + \mathbf{j}) \cdot (-4\mathbf{i} + 3\mathbf{j})}{\left(\sqrt{(-4)^2 + 3^2}\right)^2}(-4\mathbf{i} + 3\mathbf{j})$

$\qquad = \dfrac{(2)(-4) + (1)(3)}{25}(-4\mathbf{i} + 3\mathbf{j})$

$\qquad = -\dfrac{1}{5}(-4\mathbf{i} + 3\mathbf{j}) = \dfrac{4}{5}\mathbf{i} - \dfrac{3}{5}\mathbf{j}$

$\mathbf{v}_2 = \mathbf{v} - \mathbf{v}_1$

$\qquad = 2\mathbf{i} + \mathbf{j} - \left(\dfrac{4}{5}\mathbf{i} - \dfrac{3}{5}\mathbf{j}\right)$

$\qquad = 2\mathbf{i} + \mathbf{j} - \dfrac{4}{5}\mathbf{i} + \dfrac{3}{5}\mathbf{j}$

$\qquad = \dfrac{6}{5}\mathbf{i} + \dfrac{8}{5}\mathbf{j}$

54. $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j}, \quad \mathbf{w} = 3\mathbf{i} + \mathbf{j}$

The projection of $\mathbf{v}$ onto $\mathbf{w}$ is given by:

$\mathbf{v}_1 = \dfrac{\mathbf{v} \cdot \mathbf{w}}{\| \mathbf{w} \|^2} \mathbf{w} = \dfrac{(2\mathbf{i} + 3\mathbf{j}) \cdot (3\mathbf{i} + \mathbf{j})}{\left(\sqrt{3^2 + 1^2}\right)^2}(3\mathbf{i} + \mathbf{j})$

$\qquad = \dfrac{(2)(3) + (3)(1)}{10}(3\mathbf{i} + \mathbf{j})$

$\qquad = \dfrac{9}{10}(3\mathbf{i} + \mathbf{j}) = \dfrac{27}{10}\mathbf{i} + \dfrac{9}{10}\mathbf{j}$

$\mathbf{v}_2 = \mathbf{v} - \mathbf{v}_1$

$\qquad = 2\mathbf{i} + 3\mathbf{j} - \left(\dfrac{27}{10}\mathbf{i} + \dfrac{9}{10}\mathbf{j}\right)$

$\qquad = 2\mathbf{i} + 3\mathbf{j} - \dfrac{27}{10}\mathbf{i} - \dfrac{9}{10}\mathbf{j}$

$\qquad = -\dfrac{7}{10}\mathbf{i} + \dfrac{21}{10}\mathbf{j}$

55. $\cos\alpha = \dfrac{a}{\|\mathbf{v}\|} = \dfrac{3}{\sqrt{3^2+(-4)^2+2^2}} = \dfrac{3}{\sqrt{29}} = \dfrac{3\sqrt{29}}{29}$

$\alpha \approx 56.1°$

$\cos\beta = \dfrac{b}{\|\mathbf{v}\|} = \dfrac{-4}{\sqrt{3^2+(-4)^2+2^2}} = -\dfrac{4}{\sqrt{29}} = -\dfrac{4\sqrt{29}}{29}$

$\beta \approx 138.0°$

$\cos\gamma = \dfrac{c}{\|\mathbf{v}\|} = \dfrac{2}{\sqrt{3^2+(-4)^2+2^2}} = \dfrac{2}{\sqrt{29}} = \dfrac{2\sqrt{29}}{29}$

$\gamma \approx 68.2°$

56. $\mathbf{u} = \overrightarrow{P_1P_2} = \mathbf{i}+2\mathbf{j}+3\mathbf{k}$

$\mathbf{v} = \overrightarrow{P_1P_3} = 5\mathbf{i}+4\mathbf{j}+\mathbf{k}$

$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 5 & 4 & 1 \end{vmatrix}$

$= \begin{vmatrix} 2 & 3 \\ 4 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 3 \\ 5 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 2 \\ 5 & 4 \end{vmatrix}\mathbf{k}$

$= -10\mathbf{i}+14\mathbf{j}-6\mathbf{k}$

Area $= \|\mathbf{u}\times\mathbf{v}\|$

$= \sqrt{(-10)^2+14^2+(-6)^2}$

$= \sqrt{332} = 2\sqrt{83} \approx 18.2$ sq. units

57. $\mathbf{v}\times\mathbf{u} = -(\mathbf{u}\times\mathbf{v}) = -(2\mathbf{i}-3\mathbf{j}+\mathbf{k}) = -2\mathbf{i}+3\mathbf{j}-\mathbf{k}$

58. $\mathbf{u}\times\mathbf{v} = (3\mathbf{v})\times\mathbf{v} = 3(\mathbf{v}\times\mathbf{v}) = 3\cdot\mathbf{0} = \mathbf{0}$

59. Let the positive *x*-axis point downstream, so that the velocity of the current is $\mathbf{v}_c = 2\mathbf{i}$. Let $\mathbf{v}_w$ = the velocity of the swimmer in the water and $\mathbf{v}_g$ = the velocity of the swimmer relative to the land. Then $\mathbf{v}_g = \mathbf{v}_w + \mathbf{v}_c$. The speed of the swimmer is $\|\mathbf{v}_w\| = 5$, and the heading is directly across the river, so $\mathbf{v}_w = 5\mathbf{j}$. Then $\mathbf{v}_g = \mathbf{v}_w + \mathbf{v}_c = 5\mathbf{j}+2\mathbf{i} = 2\mathbf{i}+5\mathbf{j}$

$\|\mathbf{v}_g\| = \sqrt{2^2+5^2} = \sqrt{29} \approx 5.39$ mi/hr

Since the river is 1 mile wide, it takes the swimmer about 0.2 hour to cross the river. The swimmer will end up $(0.2)(2) = 0.4$ miles downstream.

60. Let $\mathbf{F_1}$ = the tension on the left cable, $\mathbf{F_2}$ = the tension on the right cable, and $\mathbf{F_3}$ = the force of the weight of the box.

$\mathbf{F_1} = \|\mathbf{F_1}\|\left[\cos(140°)\mathbf{i}+\sin(140°)\mathbf{j}\right]$

$\approx \|\mathbf{F_1}\|(-0.7660\mathbf{i}+0.6428\mathbf{j})$

$\mathbf{F_2} = \|\mathbf{F_2}\|\left[\cos(30°)\mathbf{i}+\sin(30°)\mathbf{j}\right]$

$\approx \|\mathbf{F_2}\|(0.8660\mathbf{i}+0.5000\mathbf{j})$

$\mathbf{F_3} = -2000\mathbf{j}$

For equilibrium, the sum of the force vectors must be zero.

$\mathbf{F_1}+\mathbf{F_2}+\mathbf{F_3}$

$= -0.766044\|\mathbf{F_1}\|\mathbf{i}+0.642788\|\mathbf{F_1}\|\mathbf{j}$

$\quad +0.866025\|\mathbf{F_2}\|\mathbf{i}+0.5\|\mathbf{F_2}\|\mathbf{j}-2000\mathbf{j}$

$= \left(-0.766044\|\mathbf{F_1}\|+0.866025\|\mathbf{F_2}\|\right)\mathbf{i}$

$\quad +\left(0.642788\|\mathbf{F_1}\|+0.5\|\mathbf{F_2}\|-2000\right)\mathbf{j}$

$= \mathbf{0}$

Set the $\mathbf{i}$ and $\mathbf{j}$ components equal to zero and solve:

$\begin{cases} -0.766044\|\mathbf{F_1}\|+0.866025\|\mathbf{F_2}\|=0 \\ 0.642788\|\mathbf{F_1}\|+0.5\|\mathbf{F_2}\|-2000=0 \end{cases}$

Solve the first equation for $\|\mathbf{F_2}\|$ and substitute the result into the second equation to solve the system:

$\|\mathbf{F_2}\| \approx \dfrac{0.766044}{0.866025}\|\mathbf{F_1}\| \approx 0.884552\|\mathbf{F_1}\|$

$0.642788\|\mathbf{F_1}\|+0.5(0.884552\|\mathbf{F_1}\|)-2000=0$

$1.085064\|\mathbf{F_1}\| = 2000$

$\|\mathbf{F_1}\| \approx 1843.21$ lb

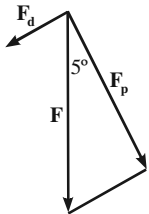$\|\mathbf{F_2}\| = 0.884552(1843.21) = 1630.41$ lb

The tension in the left cable is about 1843.21 pounds and the tension in the right cable is about 1630.41 pounds.

61. $\mathbf{F} = 5\big[\cos(60°)\mathbf{i} + \sin(60°)\mathbf{j}\big]$

$= 5\left(\dfrac{1}{2}\mathbf{i} + \dfrac{\sqrt{3}}{2}\mathbf{j}\right) = \dfrac{5}{2}\mathbf{i} + \dfrac{5\sqrt{3}}{2}\mathbf{j}$

$\overrightarrow{AB} = 20\mathbf{i}$

$W = \mathbf{F} \cdot \overrightarrow{AB} = \left(\dfrac{5}{2}\mathbf{i} + \dfrac{5\sqrt{3}}{2}\mathbf{j}\right) \cdot 20\mathbf{i}$

$= \left(\dfrac{5}{2}\right)(20) + \left(\dfrac{5\sqrt{3}}{2}\right)(0) = 50$ ft-lb

62. Split the force into the components going down the hill and perpendicular to the hill.

$\mathbf{F_d} = \mathbf{F}\sin 5° = 8000\sin 5° \approx 697.2$

$\mathbf{F_p} = \mathbf{F}\cos 5° = 8000\cos 5° \approx 7969.6$

The force required to keep the van from rolling down the street is about 697.2 pounds. The force perpendicular to the street is approximately 7969.6 pounds.

Chapter 9 Test

1 – 3.
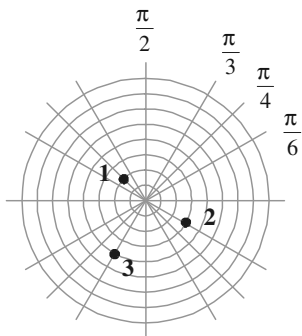
4. $x = 2$ and $y = 2\sqrt{3}$

$r = \sqrt{x^2 + y^2} = \sqrt{(2)^2 + (2\sqrt{3})^2} = \sqrt{16} = 4$ and

$\tan\theta = \dfrac{y}{x} = \dfrac{2\sqrt{3}}{2} = \sqrt{3}$

Since (x, y) is in quadrant I, we have $\theta = \dfrac{\pi}{3}$.

The point $(2, 2\sqrt{3})$ in rectangular coordinates is the point $\left(4, \dfrac{\pi}{3}\right)$ in polar coordinates.

5. $r = 7$

$\sqrt{x^2 + y^2} = 7$

$\left(\sqrt{x^2 + y^2}\right)^2 = 7^2$

$x^2 + y^2 = 49$

Thus the equation is a circle with center $(0, 0)$ and radius $= 7$.
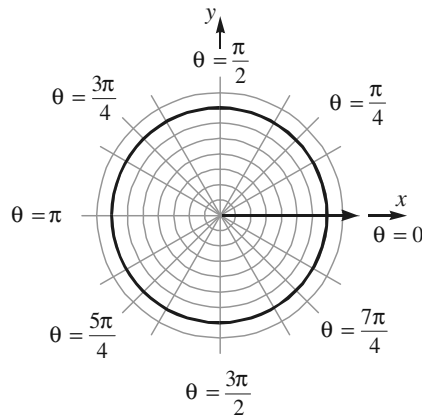
6. $\tan\theta = 3$

$\dfrac{\sin\theta}{\cos\theta} = 3$

$\dfrac{r\sin\theta}{r\cos\theta} = 3$

$\dfrac{y}{x} = 3$ or $y = 3x$

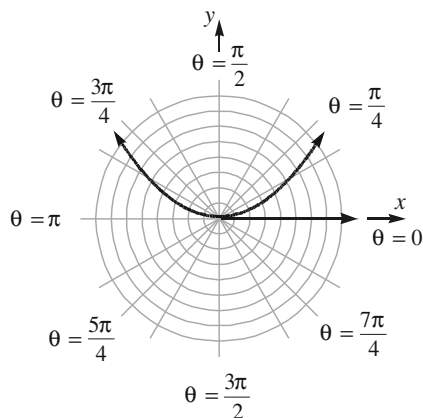Thus the equation is a straight line with $m = 3$ and $b = 0$.
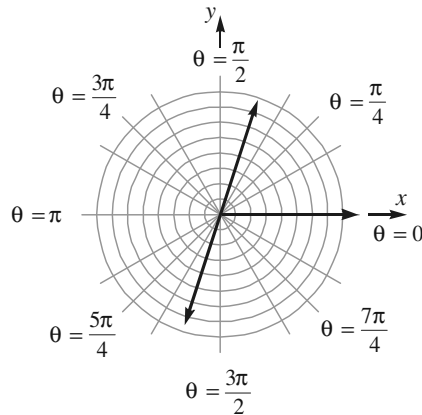
7. $r\sin^2\theta + 8\sin\theta = r$

$r^2\sin^2\theta + 8r\sin\theta = r^2$

$$y^2 + 8y = x^2 + y^2$$
$$8y = x^2 \quad \text{or} \quad 4(2)y = x^2$$

The graph is a parabola with vertex (0, 0) and focus (0, 2) in rectangular coordinates.

8. $r^2\cos\theta = 5$

Polar axis: Replace θ with $-\theta$:

$r^2\cos(-\theta) = 5$

$r^2\cos\theta = 5$

Since the resulting equation is the same as the original, the graph is symmetrical with respect to the polar axis.

The line $\theta = \frac{\pi}{2}$: Replace θ with $\pi - \theta$:

$$r^2\cos(\pi - \theta) = 5$$
$$r^2(\cos\pi\cos\theta + \sin\pi\sin\theta) = 5$$
$$r^2(-\cos\theta) = 5$$
$$-r^2\cos\theta = 5$$

Since the resulting equation is not the same as the original, the graph **may or may not be**

symmetrical with respect to the line $\theta = \frac{\pi}{2}$.

The pole: Replacing r with $-r$:

$(-r)^2\cos\theta = 5$

$r^2\cos\theta = 5$

Since the resulting equation is the same as the original, the graph is symmetrical with respect to the pole.

Note: Since we have now established symmetry about the pole and the polar axis, it can be shown that the graph must also be symmetric about the line $\theta = \frac{\pi}{2}$.

9. $r = 5\sin\theta\cos^2\theta$

The polar axis: Replace θ with $-\theta$:

$r = 5\sin(-\theta)\cos^2(-\theta)$

$r = 5(-\sin\theta)\cos^2\theta$

$r = -5\sin\theta\cos^2\theta$

Since the resulting equation is the not same as the original, the graph **may or may not be** symmetrical with respect to the polar axis.

The line $\theta = \frac{\pi}{2}$: Replace θ with $\pi - \theta$:

$r = 5\sin(\pi - \theta)\cos^2(\pi - \theta)$

$r = 5(\sin\pi\cos\theta - \cos\pi\sin\theta)(-\cos\theta)^2$

$r = 5(0\cdot\cos\theta - (-1)\cdot\sin\theta)(\cos^2\theta)$

$r = 5\sin\theta\cos^2\theta$

Since the resulting equation is the same as the original, the graph is symmetrical with respect to the line $\theta = \frac{\pi}{2}$.

The pole: Replacing r with $-r$:

$-r = 5\sin\theta\cos^2\theta$

$r = -5\sin\theta\cos^2\theta$

Since the resulting equation is not the same as the original, the graph **may or may not be** symmetrical with respect to the pole.

10. $z \cdot w = \left[2\left(\cos 85° + i\sin 85°\right)\right]\left[3\left(\cos 22° + i\sin 22°\right)\right]$

$= 2 \cdot 3[\cos(85° + 22°) + i\sin(85° + 22°)]$

$= 6(\cos 107° + i\sin 107°)$

11. $\dfrac{w}{z} = \dfrac{3\left(\cos 22° + i\sin 22°\right)}{2\left(\cos 85° + i\sin 85°\right)}$

$= \dfrac{3}{2}[\cos(22° - 85°) + i\sin(22° - 85°)]$

$= \dfrac{3}{2}[\cos(-63°) + i\sin(-63°)]$

Since $-63°$ has the same terminal side as $297°$,

we can write $\dfrac{w}{z} = \dfrac{3}{2}(\cos 297° + i\sin 297°)$

12. $w^5 = [3(\cos 22° + i\sin 22°)]^5$

$= 3^5[\cos(5 \cdot 22°) + i\sin(5 \cdot 22°)]$

$= 243(\cos 110° + i\sin 110°)$

13. Let $w = -8 + 8\sqrt{3}\,i$; then

$|w| = \sqrt{(-8)^2 + (8\sqrt{3})^2} = \sqrt{64 + 192} = \sqrt{256} = 16$

so we can write w in polar form as

$w = 16\left(-\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i\right) = 16(\cos 120° + i \sin 120°)$

Using De Moivre's Theorem, the three distinct 3rd roots of w are

$z_k = \sqrt[3]{16}\left[\cos\left(\dfrac{120°}{3} + \dfrac{360°k}{3}\right) + i \sin\left(\dfrac{120°}{3} + \dfrac{360°k}{3}\right)\right]$

$\qquad = 2\sqrt[3]{2}\left[\cos(40° + 120°k) + i \sin(40° + 120°k)\right]$

where $k = 0, 1,$ and 2. So we have

$z_0 = 2\sqrt[3]{2}\left[\cos(40° + 120° \cdot 0) + i \sin(40° + 120° \cdot 0)\right]$

$\quad = 2\sqrt[3]{2}\left(\cos 40° + i \sin 40°\right)$

$\quad \approx 1.93 + 1.62i$

$z_1 = 2\sqrt[3]{2}\left[\cos(40° + 120° \cdot 1) + i \sin(40° + 120° \cdot 1)\right]$

$\quad = 2\sqrt[3]{2}\left(\cos 160° + i \sin 160°\right)$

$\quad \approx -2.37 + 0.86i$

$z_2 = 2\sqrt[3]{2}\left[\cos(40° + 120° \cdot 2) + i \sin(40° + 120° \cdot 2)\right]$

$\quad = 2\sqrt[3]{2}\left(\cos 280° + i \sin 280°\right)$

$\quad \approx 0.44 - 2.48i$
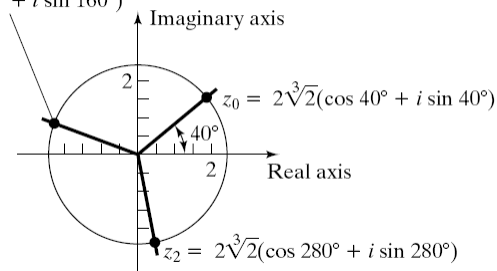
14. $v = \left\langle 8\sqrt{2} - 3\sqrt{2}, \; 2\sqrt{2} - 7\sqrt{2}\right\rangle = \left\langle 5\sqrt{2}, -5\sqrt{2}\right\rangle$

15. $\|\mathbf{v}\| = \sqrt{\left(5\sqrt{2}\right)^2 + \left(-5\sqrt{2}\right)^2} = \sqrt{100} = 10$

16. $\mathbf{u} = \dfrac{\mathbf{v}}{\|\mathbf{v}\|} = \dfrac{1}{10}\left\langle 5\sqrt{2}, -5\sqrt{2}\right\rangle = \left\langle \dfrac{\sqrt{2}}{2}, -\dfrac{\sqrt{2}}{2}\right\rangle$

17. From Exercise 16, we can write

$\mathbf{v} = \|\mathbf{v}\| \cdot \mathbf{u} = 10\left\langle \dfrac{\sqrt{2}}{2}, -\dfrac{\sqrt{2}}{2}\right\rangle$

$\quad = 10\left\langle \cos 315°, \sin 315°\right\rangle$

$\quad = 10(\cos 315° \mathbf{i} + \sin 315° \mathbf{j})$

Thus, the angle between $\mathbf{v}$ and $\mathbf{i}$ is $315°$.

18. From Exercise 17, we can write

$\mathbf{v} = 10\left\langle \cos 315° \mathbf{i} + \sin 315° \mathbf{j}\right\rangle$

$\quad = 10\left\langle \dfrac{\sqrt{2}}{2}\mathbf{i} + \left(-\dfrac{\sqrt{2}}{2}\right)\mathbf{j}\right\rangle = 5\sqrt{2}\,\mathbf{i} - 5\sqrt{2}\,\mathbf{j}$

19. $\mathbf{v}_1 + 2\mathbf{v}_2 - \mathbf{v}_3 = \left\langle 4,6\right\rangle + 2\left\langle -3,-6\right\rangle - \left\langle -8,4\right\rangle$

$\qquad = \left\langle 4 + 2(-3) - (-8), 6 + 2(-6) - 4\right\rangle$

$\qquad = \left\langle 4 - 6 + 8, 6 - 12 - 4\right\rangle$

$\qquad = \left\langle 6, -10\right\rangle$

20 – 21.

If θ_{ij} is the angle between $\mathbf{v}_i$ and $\mathbf{v}_j$, then

$\cos\theta_{ij} = \dfrac{\mathbf{v}_i \cdot \mathbf{v}_j}{\|\mathbf{v}_i\| \cdot \|\mathbf{v}_j\|}$. Thus,

$\cos\theta_{12} = \dfrac{4(-3) + 6(-6)}{\sqrt{52} \cdot \sqrt{45}} = \dfrac{-48}{6\sqrt{65}} = -\dfrac{8}{\sqrt{65}}$

$\cos\theta_{13} = \dfrac{4(-8) + 6(4)}{\sqrt{52} \cdot \sqrt{80}} = \dfrac{-8}{8\sqrt{65}} = -\dfrac{1}{\sqrt{65}}$

$\cos\theta_{14} = \dfrac{4(10) + 6(15)}{\sqrt{52} \cdot \sqrt{325}} = \dfrac{130}{130} = 1$

$\cos\theta_{23} = \dfrac{(-3)(-8) + (-6)(4)}{\sqrt{45} \cdot \sqrt{80}} = \dfrac{0}{60} = 0$

$\cos\theta_{24} = \dfrac{(-3)10 + (-6)15}{\sqrt{45} \cdot \sqrt{325}} = \dfrac{-120}{15\sqrt{65}} = -\dfrac{8}{\sqrt{65}}$

$\cos\theta_{34} = \dfrac{(-8)10 + (4)15}{\sqrt{80} \cdot \sqrt{325}} = \dfrac{-20}{20\sqrt{65}} = -\dfrac{1}{\sqrt{65}}$

The vectors $\mathbf{v}_i$ and $\mathbf{v}_j$ are parallel if $\cos\theta_{ij} = 1$ and are orthogonal if $\cos\theta_{ij} = 0$. Thus, vectors $\mathbf{v}_1$ and $\mathbf{v}_4$ are parallel and $\mathbf{v}_2$ and $\mathbf{v}_3$ are orthogonal.

22. The angle between vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ is

$\cos^{-1}\left(\dfrac{\mathbf{v}_1 \cdot \mathbf{v}_2}{\|\mathbf{v}_1\| \cdot \|\mathbf{v}_2\|}\right) = \cos^{-1}\left(\dfrac{-48}{\sqrt{52} \cdot \sqrt{45}}\right)$.

$\approx \cos^{-1}(-0.992278) \approx 172.87°$

23. $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & 1 \\ -1 & 3 & 2 \end{vmatrix}$

$= \begin{vmatrix} -3 & 1 \\ 3 & 2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & -3 \\ -1 & 3 \end{vmatrix}\mathbf{k}$

$= (-6 - 3)\mathbf{i} - (4 - (-1))\mathbf{j} + (6 - 3)\mathbf{k}$

$= -9\mathbf{i} - 5\mathbf{j} + 3\mathbf{k}$

24. To find the direction angles for $\mathbf{u}$, we first evaluate $\|\mathbf{u}\|$.

$$\|\mathbf{u}\| = \sqrt{(2)^2 + (-3)^2 + 1^2} = \sqrt{4+9+1} = \sqrt{14}$$

From the Theorem on Direction Angles, we have

$$\cos\alpha = \frac{2}{\sqrt{14}}, \quad \cos\beta = \frac{-3}{\sqrt{14}}, \quad \text{and } \cos\gamma = \frac{1}{\sqrt{14}}.$$

Thus, the direction angles can be found by using the inverse cosine function. That is,

$$\alpha = \cos^{-1}\frac{2}{\sqrt{14}} \approx 57.7°$$

$$\beta = \cos^{-1}\frac{-3}{\sqrt{14}} \approx 143.3°$$

$$\gamma = \cos^{-1}\frac{1}{\sqrt{14}} \approx 74.5°$$

25. The area of a parallelogram with $\mathbf{u}$ and $\mathbf{v}$ as adjacent sides is given by $\|\mathbf{u} \times \mathbf{v}\|$.

From problem (1), we have that
$\mathbf{u} \times \mathbf{v} = -9\mathbf{i} - 5\mathbf{j} + 3\mathbf{k}$.

Therefore, the area of the parallelogram is

$$\|\mathbf{u} \times \mathbf{v}\| = \sqrt{(-9)^2 + (-5)^2 + 3^2}$$
$$= \sqrt{81 + 25 + 9}$$
$$= \sqrt{115} \approx 10.724 \text{ square units}$$

26. We first calculate the angle α.
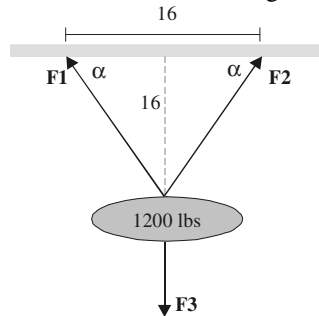
Using the right triangle in the sketch, we conclude $\tan\alpha = \dfrac{16}{8} = 2$ so that

$\alpha = \tan^{-1} 2 \approx 63.435°$.

Thus the three force vectors in the problem can be written as

$$\mathbf{F1} = \|\mathbf{F1}\|(\cos 116.565°\,\mathbf{i} + \sin 116.565°\,\mathbf{j})$$
$$= \|\mathbf{F1}\|(-0.44721\mathbf{i} + 0.89443\mathbf{j})$$

$$\mathbf{F2} = \|\mathbf{F2}\|(\cos 63.435°\,\mathbf{i} + \sin 63.435°\,\mathbf{j})$$
$$= \|\mathbf{F2}\|(0.44721\mathbf{i} + 0.89443\mathbf{j})$$

$\mathbf{F3} = -1200\mathbf{j}$

Since the system is in equilibrium, we need
$\mathbf{F1} + \mathbf{F2} + \mathbf{F3} = 0\mathbf{i} + 0\mathbf{j}$.

This means

$$\left[\|\mathbf{F1}\|(-0.44721) + \|\mathbf{F2}\|(0.44721) + 0\right] = 0 \text{ and}$$

$$\left[\|\mathbf{F1}\|(0.89443) + \|\mathbf{F2}\|(0.89443) - 1200\right] = 0.$$

The first equation gives $\|\mathbf{F1}\| = \|\mathbf{F2}\|$; if we call this common value $\|\mathbf{F}\|$ and substitute into the

second equation we get
$2 \cdot \|\mathbf{F}\| \cdot (0.89443) = 1200$

$$\|\mathbf{F}\| = \frac{1200}{2(0.89443)} \approx 670.82$$

Thus, the cable must be able to endure a tension of approximately 670.82 lbs.

Chapter 9 Cumulative Review

1.
$$e^{x^2-9} = 1$$
$$\ln\left(e^{x^2-9}\right) = \ln(1)$$
$$x^2 - 9 = 0$$
$$(x+3)(x-3) = 0$$
$$x = -3 \text{ or } x = 3$$

The solution set is $\{-3, 3\}$.

2. The line containing point (0, 0), making an angle of $30°$ with the positive x-axis has polar equation $\theta = \dfrac{\pi}{6}$.

Using rectangular coordinates:

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\tan\theta = \frac{y}{x}$$

$$\theta = \frac{\pi}{6}$$

$$\tan\theta = \tan\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{3}$$

So, $\dfrac{\sqrt{3}}{3} = \dfrac{y}{x}$

$$y = \frac{\sqrt{3}}{3}x$$

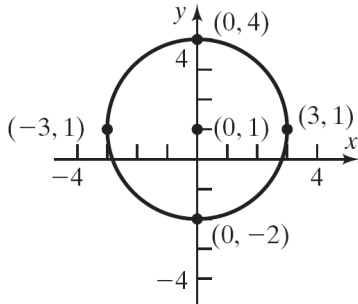**3.** The circle with center point $(h, k) = (0,1)$ and radius $r = 3$ has equation:

$(x-h)^2 + (y-k)^2 = r^2$

$(x-0)^2 + (y-1)^2 = 3^2$

$x^2 + (y-1)^2 = 9$

4. $f(x) = \ln(1-2x)$

f will be defined provided $1-2x > 0$.

$1-2x > 0$

$1 > 2x$

$\dfrac{1}{2} > x$

The domain of is $\left\{ x \mid x < \dfrac{1}{2} \right\}$ or $\left(-\infty, \dfrac{1}{2}\right)$.

5. $x^2 + y^3 = 2x^4$

Test for symmetry:

x-axis: Replace y by $-y$:

$x^2 + (-y)^3 = 2x^4$

$x^2 - y^3 = 2x^4$

which is not equivalent to $x^2 + y^3 = 2x^4$.

y-axis: Replace x by $-x$:

$(-x)^2 + y^3 = 2(-x)^4$

$x^2 + y^3 = 2x^4$

which is equivalent to $x^2 + y^3 = 2x^4$

Origin: Replace x by $-x$ and y by $-y$:

$(-x)^2 + (-y)^3 = 2(-x)^4$

$x^2 - y^3 = 2x^4$

which is not equivalent to $x^2 + y^3 = 2x^4$.

Therefore, the graph is symmetric with respect to the y-axis.

6. $y = |\ln x|$

$= \begin{cases} \ln x, & \text{when } \ln x \ge 0 \\ -\ln x, & \text{when } \ln x < 0 \end{cases}$

$= \begin{cases} \ln x, & \text{when } x \ge 1 \\ -\ln x, & \text{when } 0 < x < 1 \end{cases}$
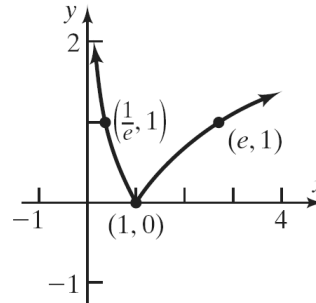
7. $y = |\sin x|$

$= \begin{cases} \sin x, & \text{when } \sin x \ge 0 \\ -\sin x, & \text{when } \sin x < 0 \end{cases}$

$= \begin{cases} \sin x, & \text{when } 0 \le x \le \pi \\ -\sin x, & \text{when } \pi < x < 2\pi \end{cases}$
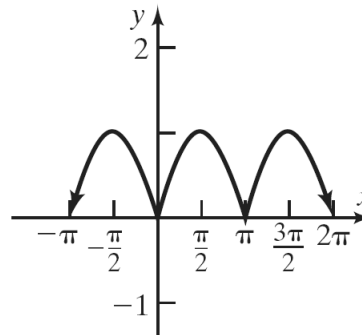
8. $y = \sin|x| = \begin{cases} \sin x, & \text{when } x \ge 0 \\ -\sin x, & \text{when } x < 0 \end{cases}$
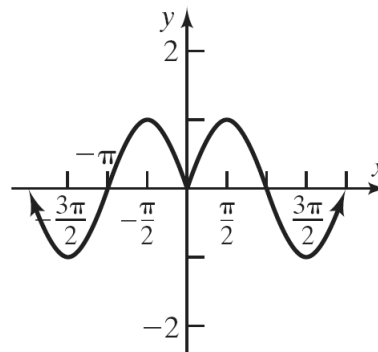
9. $\sin^{-1}\left(-\dfrac{1}{2}\right)$

We are finding the angle θ, $-\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$, whose

sine equals $-\dfrac{1}{2}$.

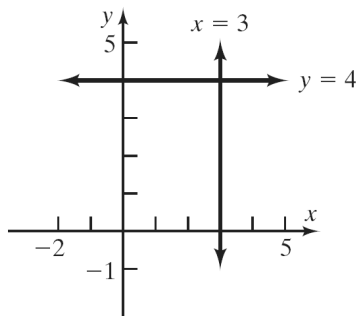$$\sin\theta = -\frac{1}{2}, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

$$\theta = -\frac{\pi}{6}$$

$$\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$$

10. Graphing $x = 3$ and $y = 4$ using rectangular coordinates:

$x = 3$ yields a vertical line passing through the point $(3, 0)$.

$y = 4$ yields a horizontal line passing through the point $(0, 4)$.

11. Graphing $r = 2$ and $\theta = \dfrac{\pi}{3}$ using polar coordinates:

$r = 2$ yields a circle, centered at $(0, 0)$, with radius $= 2$.

$\theta = \dfrac{\pi}{3}$ yields a line passing through the point $(0, 0)$,

forming an angle of $\theta = \dfrac{\pi}{3}$ with the positive x-axis.
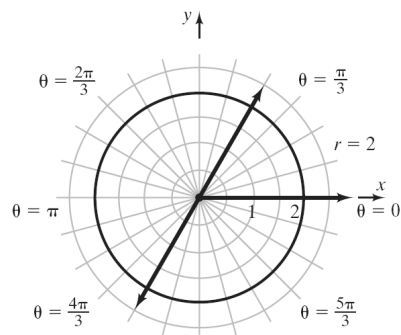
12. $y = -4\cos(\pi x)$ is of the form $y = a\cos(\omega x)$

Amplitude: $|a| = |-4| = 4$

Period: $\dfrac{2\pi}{\omega} = \dfrac{2\pi}{\pi} = 2$

Chapter 9 Projects

Project I

1. The aircraft will rise (gain altitude).

2. The aircraft will fall (lose altitude).

3. The aircraft will speed up.

4. The aircraft will slow down.

5. $\mathbf{W} = 700 \text{ lb} \times \dfrac{2.205 \text{ kg}}{1 \text{ lb}} = 1543.5 \text{ kg}$

6. $\mathbf{F} = (1543.5 \text{ kg}) \cdot (9.80 \text{ m/s}^2)$

$\qquad = 15{,}126.3 \text{ kg} \cdot \text{m/s}^2$

$\qquad = 15{,}126.3 \text{ N}$

7. The lift force of the Wright brother's plane must have exceeded 15,126.3 newtons in order for it to have gotten off the ground.

8. $\mathbf{W} = 2200 \text{ lb} \times \dfrac{2.205 \text{ kg}}{1 \text{ lb}} = 4851 \text{ kg}$

$\mathbf{F} = (4851 \text{ kg}) \cdot (9.80 \text{ m/s}^2)$

$\qquad = 47{,}539.8 \text{ kg} \cdot \text{m/s}^2$

$\qquad = 47{,}539.8 \text{ N}$

The lift force of the Cessna 170B must exceed 47,539.8 newtons in order to get off the ground.

9. $\mathbf{W} = 255{,}000 \text{ lb} \times \dfrac{2.205 \text{ kg}}{1 \text{ lb}} = 562{,}275 \text{ kg}$

$\mathbf{F} = (562{,}275 \text{ kg}) \cdot (9.80 \text{ m/s}^2)$

$\qquad = 5{,}510{,}295 \text{ kg} \cdot \text{m/s}^2$

$\qquad = 5{,}510{,}295 \text{ N}$

The lift force of the Boeing 747 must exceed 5,510,295 newtons in order to get off the ground.

Project II

1. If $h = 0$, then $\cos(2\pi h) = 1$ and $\sin(2\pi h) = 0$, then Magnitude $= \left| 1 + 0.25(1) \right| = 1.25$ (max).

If h is high enough such that $\cos(2\pi h) = \dfrac{\sqrt{2}}{2}$

and $\sin(2\pi h) = \dfrac{\sqrt{2}}{2}$, then

$$\text{Magnitude} = \left| 1 + 0.25\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i \right) \right|$$

$$= \sqrt{\left(1 + \frac{\sqrt{2}}{8} \right)^2 + \left(\frac{\sqrt{2}}{8} \right)^2}$$

$$\approx 1.19$$

If h is high enough such that $\cos(2\pi h) = 0$ and $\sin(2\pi h) = 1$, then

$$\text{Magnitude} = \left| 1 + 0.25(0 + i) \right|$$

$$= \sqrt{1 + (0.25)^2} \approx 1.03$$

If h is high enough such that $\cos(2\pi h) = 0$ and $\sin(2\pi h) = -1$, then

$$\text{Magnitude} = \left| 1 + 0.25(0 - i) \right| \approx 1.03$$

If h is high enough such that $\cos(2\pi h) = -1$ and $\sin(2\pi h) = 0$, then

$$\text{Magnitude} = \left| 1 + 0.25(-1) \right| \approx 0.75 \text{ (min)}.$$

Because sine and cosine oscillate between -1 and 1, the magnitude will oscillate between a maximum and a minimum.

$$\text{SWR} = \frac{1.25}{.75} \approx 1.67$$

2. The distance between two consecutive minima will be the period of cosine (the value of cosine is -1 once per cycle): Period $= \dfrac{2\pi}{2\pi} = 1$.

3. The distance between two consecutive maxima will be the period of cosine (the value of cosine is 1 once per cycle): Period $= \dfrac{2\pi}{2\pi} = 1$.

4. The distance between a minimum and a maximum is half a period: $0.5(1) = 0.5$.

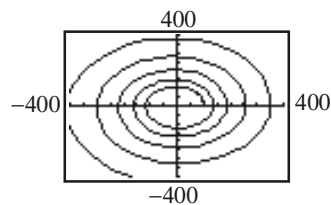**5.** The pager must be sensitive enough to receive a minimum signal strength of 0.75.
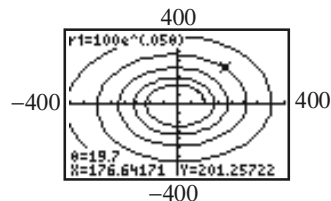
6.
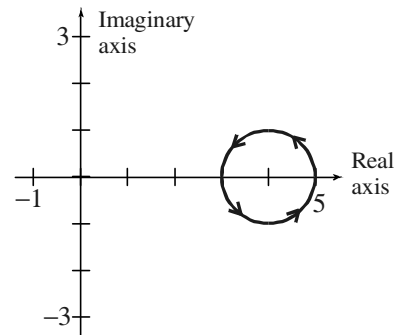
Since the magnitude of a complex number is the distance from $(0,\ 0)$ to the point, z, the minimum is 0.75 and the maximum is 1.25
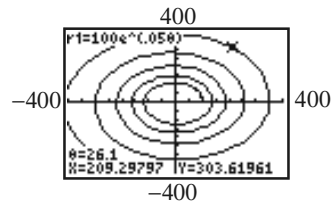
Project III

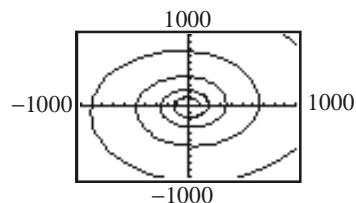**a.** $A = 100e^{0.05\theta}$, $0 \le \theta \le 30$
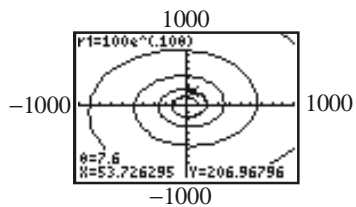

b. Approximately 19.7 years.

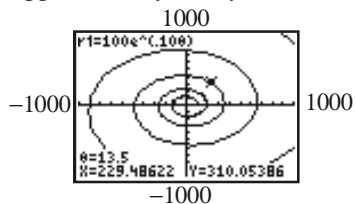**c.** Approximately 26.1 years.

d. $A = 100e^{0.10\theta}$, $0 \le \theta \le 30$

e. Approximately 7.5 years

f. Approximately 13.5 years.

g. Doubling the interest rate causes the time to double to be less than half.

$$\left(0.5 \times 19.7 - 9.85, \text{ not } 7.5\right)$$

Project IV

A.

a. $e^{i\pi} = \cos\pi + i\sin\pi = -1 + i(0) = -1$

Therefore, $e^{i\pi} + 1 = 0$.

b.
$$\frac{e^{ix} - e^{-ix}}{2i} = \frac{\cos x + i\sin x - \left[\cos(-x) + i\sin(-x)\right]}{2i}$$
$$= \frac{\cos x + i\sin x - \cos x + i\sin x}{2i} = \frac{2i\sin x}{2i}$$
$$= \sin x$$

$$\frac{e^{ix} + e^{-ix}}{2} = \frac{\cos x + i\sin x + \left[\cos(-x) + i\sin(-x)\right]}{2}$$
$$= \frac{\cos x + i\sin x + \cos x - i\sin x}{2} = \frac{2\cos x}{2}$$
$$= \cos x$$

c. $\sin(1+i) = \dfrac{e^{i(1+i)} - e^{-i(1+i)}}{2i} = \dfrac{e^{-1+i} - e^{1-i}}{2i}$

$$= \frac{e^{-1} \cdot e^{i(1)} - e \cdot e^{-i(1)}}{2i}$$

$$= \frac{e^{-1}(\cos 1 + i\sin 1) - e\left[\cos(-1) + i\sin(-1)\right]}{2i}$$

$$= \frac{0.1988 + i(0.3096) - 1.4687 + 2.2874i}{2i}$$

$$= \frac{-1.2699 + 2.597i}{2i} \cdot \frac{i}{i}$$

$$= 1.2985 + 0.6350i$$

d. $a + bi = r_1(\cos x + i\sin x) = r_1 e^{ix}$,

$$r_1 = \sqrt{a^2 + b^2}$$

$c + di = r_2(\cos y + i\sin y) = r_2 e^{iy}$,

$$r_2 = \sqrt{c^2 + d^2}$$

$$(a+bi)(c+di) = r_1 e^{ix} \cdot r_2 e^{iy}$$
$$= r_1 r_2 e^{ix} e^{iy}$$
$$= r_1 r_2 e^{ix+iy}$$
$$= r_1 r_2 e^{i(x+y)}$$
$$= r_1 r_2 \left[\cos(x+y) + i\sin(x+y)\right]$$

$$\frac{a+bi}{c+di} = \frac{r_1 e^{ix}}{r_2 e^{iy}} = \frac{r_1}{r_2} e^{ix-iy} = \frac{r_1}{r_2} e^{i(x-y)}$$

$$= \frac{r_1}{r_2}\left[\cos(x-y) + i\sin(x-y)\right]$$

B.

a. 1. $x = u + iv$

$$i(u+iv)^3 + 8 = 0$$
$$i(u^3 + 3u^2(iv) + 3u(iv)^2 + (iv)^3) + 8 = 0$$
$$i(u^3 + 3iu^2v - 3uv^2 - iv^3) + 8 = 0$$
$$(v^3 - 3u^2v + 8) + (u^3 - 3uv^2)i = 0$$

2. $v^3 - 3u^2v + 8 = 0$

$$u^2 = \frac{8+v^3}{3v}$$

$$u^3 - 3uv^2 = 0$$
$$u(u^2 - 3v^2) = 0$$
$$u = 0 \quad \text{or} \quad u^2 - 3v^2 = 0$$

If $u^2 - 3v^2 = 0$, then
$$u^2 - 3v^2 = 0$$
$$u^2 = 3v^2$$
$$3v^2 = \frac{8+v^3}{3v}$$
$$9v^3 = 8 + v^3$$
$$8v^3 = 8$$
$$v^3 = 1$$
$$v = 1$$

If $u = 0$ and $v = -2$, then $x = -2i$.

If $v = 1$, then $u = \pm\sqrt{3}$. Thus,
$$x = \sqrt{3} + i \text{ or } x = -\sqrt{3} + i$$

3. Graph $u = \pm\sqrt{\dfrac{8+v^3}{3v}}$ and $u = \pm\sqrt{3}v$
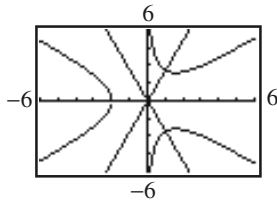
4. The points of intersection are $(1,\ 1.73)$ and $(1, -1.73)$, which matches up to the second set of intersection points. There should be three points of intersection, however, the "missing" point is complex so it will not show up on the graph.
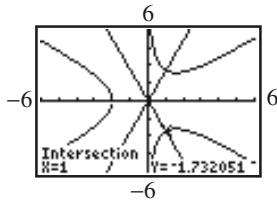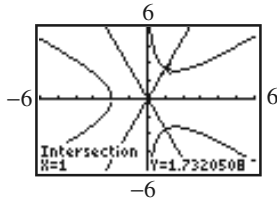
b. $ix^3 + 8 = 0$

$-x^3 + 8i = 0$

$x^3 - 8i = 0$

$x^3 = 8i$

$8i = 8(0+i) = 8(\cos 90° + i\sin 90°)$

$z_k = \sqrt[3]{8}\left[\cos\left(\dfrac{90°}{3} + \dfrac{360°}{3}k\right) + i\sin\left(\dfrac{90°}{3} + \dfrac{360°}{3}k\right)\right]$

$= 2\left[\cos(30° + 120°k) + i\sin(30° + 120°k)\right]$

$z_0 = 2\left[\cos(30°) + i\sin(30°)\right]$

$\quad = 2\left(\dfrac{\sqrt{3}}{2} + \dfrac{1}{2}i\right) = \sqrt{3} + i$

$z_1 = 2\left[\cos(30° + 120°) + i\sin(30° + 120°)\right]$

$\quad = 2\left(\dfrac{-\sqrt{3}}{2} + \dfrac{1}{2}i\right) = -\sqrt{3} + i$

$z_1 = 2\left[\cos(30° + 240°) + i\sin(30° + 240°)\right]$

$\quad = 2(0 - 1i) = -2i$

These solutions do match those in part (a).

If $u = 0$, then

$$0 = \dfrac{8+v^3}{3v}$$

$$8 + v^3 = 0$$

$$v = -2$$

$$u = 0 \quad\text{or}\quad u^2 - 3v^2 = 0$$