Chapter 7
Analytic Trigonometry

Section 7.1

1. Domain: $\{x | x \text{ is any real number}\}$;
Range: $\{y | -1 \le y \le 1\}$

2. $\{x \mid x \ge 1\}$ or $\{x \mid x \le 1\}$

3. $[3, \infty)$

4. True

5. 1; $\frac{\sqrt{3}}{2}$

6. $-\frac{1}{2}$; -1

7. $x = \sin y$

8. $0 \le x \le \pi$

9. $-\infty \le x \le \infty$

10. False. The domain of $y = \sin^{-1} x$ is $-1 \le x \le 1$.

11. True

12. True

13. $\sin^{-1} 0$

We are finding the angle θ, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, whose sine equals 0.

$$\sin\theta = 0, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$
$$\theta = 0$$
$$\sin^{-1} 0 = 0$$

14. $\cos^{-1} 1$
We are finding the angle θ, $0 \le \theta \le \pi$, whose cosine equals 1.

$$\cos\theta = 1, \quad 0 \le \theta \le \pi$$
$$\theta = 0$$
$$\cos^{-1} 1 = 0$$

15. $\sin^{-1}(-1)$

We are finding the angle θ, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, whose sine equals -1.

$$\sin\theta = -1, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$
$$\theta = -\frac{\pi}{2}$$
$$\sin^{-1}(-1) = -\frac{\pi}{2}$$

16. $\cos^{-1}(-1)$
We are finding the angle θ, $0 \le \theta \le \pi$, whose cosine equals -1.

$$\cos\theta = -1, \quad 0 \le \theta \le \pi$$
$$\theta = \pi$$
$$\cos^{-1}(-1) = \pi$$

17. $\tan^{-1} 0$

We are finding the angle θ, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, whose tangent equals 0.

$$\tan\theta = 0, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$
$$\theta = 0$$
$$\tan^{-1} 0 = 0$$

18. $\tan^{-1}(-1)$

We are finding the angle θ, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, whose tangent equals -1.

$$\tan\theta = -1, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$
$$\theta = -\frac{\pi}{4}$$
$$\tan^{-1}(-1) = -\frac{\pi}{4}$$

19. $\sin^{-1}\frac{\sqrt{2}}{2}$

We are finding the angle θ, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, whose sine equals $\frac{\sqrt{2}}{2}$.

$$\sin\theta = \frac{\sqrt{2}}{2}, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$
$$\theta = \frac{\pi}{4}$$
$$\sin^{-1}\frac{\sqrt{2}}{2} = \frac{\pi}{4}$$

20. $\tan^{-1}\frac{\sqrt{3}}{3}$

We are finding the angle θ, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, whose tangent equals $\frac{\sqrt{3}}{3}$.

$$\tan\theta = \frac{\sqrt{3}}{3}, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$
$$\theta = \frac{\pi}{6}$$
$$\tan^{-1}\frac{\sqrt{3}}{3} = \frac{\pi}{6}$$

21. $\tan^{-1}\sqrt{3}$

We are finding the angle θ, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, whose tangent equals $\sqrt{3}$.

$$\tan\theta = \sqrt{3}, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$
$$\theta = \frac{\pi}{3}$$
$$\tan^{-1}\sqrt{3} = \frac{\pi}{3}$$

22. $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

We are finding the angle θ, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, whose sine equals $-\frac{\sqrt{3}}{2}$.

$$\sin\theta = -\frac{\sqrt{3}}{2}, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$
$$\theta = -\frac{\pi}{3}$$
$$\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$$

23. $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

We are finding the angle θ, $0 \le \theta \le \pi$, whose cosine equals $-\frac{\sqrt{3}}{2}$.

$$\cos\theta = -\frac{\sqrt{3}}{2}, \quad 0 \le \theta \le \pi$$
$$\theta = \frac{5\pi}{6}$$
$$\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$$

24. $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

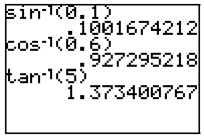
We are finding the angle θ, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, whose sine equals $-\frac{\sqrt{2}}{2}$.

$$\sin\theta = -\frac{\sqrt{2}}{2}, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$
$$\theta = -\frac{\pi}{4}$$
$$\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$$

25. $\sin^{-1} 0.1 \approx 0.10$

26. $\cos^{-1} 0.6 \approx 0.93$

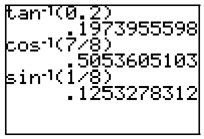
27. $\tan^{-1} 5 \approx 1.37$



28. $\tan^{-1} 0.2 \approx 0.20$

29. $\cos^{-1} \dfrac{7}{8} \approx 0.51$

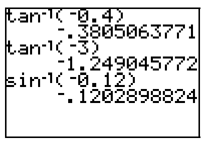
30. $\sin^{-1} \dfrac{1}{8} \approx 0.13$



31. $\tan^{-1}(-0.4) \approx -0.38$

32. $\tan^{-1}(-3) \approx -1.25$

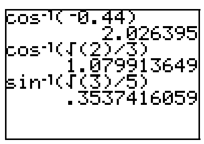
33. $\sin^{-1}(-0.12) \approx -0.12$



34. $\cos^{-1}(-0.44) \approx 2.03$

35. $\cos^{-1} \dfrac{\sqrt{2}}{3} \approx 1.08$

36. $\sin^{-1} \dfrac{\sqrt{3}}{5} \approx 0.35$



37. $\cos^{-1}\left(\cos\dfrac{4\pi}{5}\right)$ follows the form of the equation $f^{-1}\left(f\left(x\right)\right) = \cos^{-1}\left(\cos\left(x\right)\right) = x$. Since $\dfrac{4\pi}{5}$ is in the interval $\left[0, \pi\right]$, we can apply the equation directly and get $\cos^{-1}\left(\cos\dfrac{4\pi}{5}\right) = \dfrac{4\pi}{5}$.

38. $\sin^{-1}\left(\sin\left(-\dfrac{\pi}{10}\right)\right)$ follows the form of the equation $f^{-1}\left(f\left(x\right)\right) = \sin^{-1}\left(\sin\left(x\right)\right) = x$. Since $-\dfrac{\pi}{10}$ is in the interval $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$, we can apply the equation directly and get

$$\sin^{-1}\left(\sin\left(-\frac{\pi}{10}\right)\right) = -\frac{\pi}{10}.$$

39. $\tan^{-1}\left(\tan\left(-\dfrac{3\pi}{8}\right)\right)$ follows the form of the equation $f^{-1}\left(f\left(x\right)\right) = \tan^{-1}\left(\tan\left(x\right)\right) = x$. Since $-\dfrac{3\pi}{8}$ is in the interval $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$, we can apply the equation directly and get

$$\tan^{-1}\left(\tan\left(-\frac{3\pi}{8}\right)\right) = -\frac{3\pi}{8}.$$

40. $\sin^{-1}\left(\sin\left(-\dfrac{3\pi}{7}\right)\right)$ follows the form of the equation $f^{-1}\left(f\left(x\right)\right) = \sin^{-1}\left(\sin\left(x\right)\right) = x$. Since $-\dfrac{3\pi}{7}$ is in the interval $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$, we can apply the equation directly and get

$$\sin^{-1}\left(\sin\left(-\frac{3\pi}{7}\right)\right) = -\frac{3\pi}{7}.$$

41. $\sin^{-1}\left(\sin\left(\dfrac{9\pi}{8}\right)\right)$ follows the form of the equation $f^{-1}\left(f\left(x\right)\right) = \sin^{-1}\left(\sin\left(x\right)\right) = x$, but we cannot use the formula directly since $\dfrac{9\pi}{8}$ is not in the interval $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$. We need to find an angle θ in the interval $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$ for which $\sin\dfrac{9\pi}{8} = \sin\theta$. The angle $\dfrac{9\pi}{8}$ is in quadrant III so sine is negative. The reference angle of $\dfrac{9\pi}{8}$ is $\dfrac{\pi}{8}$ and we want θ to be in quadrant IV so sine will still be negative. Thus, we have $\sin\dfrac{9\pi}{8} = \sin\left(-\dfrac{\pi}{8}\right)$. Since $-\dfrac{\pi}{8}$ is in the interval $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$, we can apply the equation above and get $\sin^{-1}\left(\sin\dfrac{9\pi}{8}\right) = \sin^{-1}\left(\sin\left(-\dfrac{\pi}{8}\right)\right) = -\dfrac{\pi}{8}$.

42. $\cos^{-1}\left(\cos\left(-\frac{5\pi}{3}\right)\right)$ follows the form of the equation $f^{-1}\left(f\left(x\right)\right)=\cos^{-1}\left(\cos\left(x\right)\right)=x$, but we cannot use the formula directly since $-\frac{5\pi}{3}$ is not in the interval $\left[0,\pi\right]$. We need to find an angle θ in the interval $\left[0,\pi\right]$ for which $\cos\left(-\frac{5\pi}{3}\right)=\cos\theta$. The angle $-\frac{5\pi}{3}$ is in quadrant I so the reference angle of $-\frac{5\pi}{3}$ is $\frac{\pi}{3}$. Thus, we have $\cos\left(-\frac{5\pi}{3}\right)=\cos\frac{\pi}{3}$. Since $\frac{\pi}{3}$ is in the interval $\left[0,\pi\right]$, we can apply the equation above and get

$\cos^{-1}\left(\cos\left(-\frac{5\pi}{3}\right)\right)=\cos^{-1}\left(\cos\frac{\pi}{3}\right)=\frac{\pi}{3}$.

43. $\tan^{-1}\left(\tan\left(\frac{4\pi}{5}\right)\right)$ follows the form of the equation $f^{-1}\left(f\left(x\right)\right)=\tan^{-1}\left(\tan\left(x\right)\right)=x$, but we cannot use the formula directly since $\frac{4\pi}{5}$ is not in the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$. We need to find an angle θ in the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ for which $\tan\left(\frac{4\pi}{5}\right)=\tan\theta$. The angle $\frac{4\pi}{5}$ is in quadrant II so tangent is negative. The reference angle of $\frac{4\pi}{5}$ is $\frac{\pi}{5}$ and we want θ to be in quadrant IV so tangent will still be negative. Thus, we have $\tan\left(\frac{4\pi}{5}\right)=\tan\left(-\frac{\pi}{5}\right)$. Since $-\frac{\pi}{5}$ is in the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, we can apply the equation above and get

$\tan^{-1}\left(\tan\left(\frac{4\pi}{5}\right)\right)=\tan^{-1}\left(\tan\left(-\frac{\pi}{5}\right)\right)=-\frac{\pi}{5}$.

44. $\tan^{-1}\left(\tan\left(-\frac{2\pi}{3}\right)\right)$ follows the form of the equation $f^{-1}\left(f\left(x\right)\right)=\tan^{-1}\left(\tan\left(x\right)\right)=x$. but we cannot use the formula directly since $-\frac{2\pi}{3}$ is not in the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$. We need to find an angle θ in the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ for which $\tan\left(-\frac{2\pi}{3}\right)=\tan\theta$. The angle $-\frac{2\pi}{3}$ is in quadrant III so tangent is positive. The reference angle of $-\frac{2\pi}{3}$ is $\frac{\pi}{3}$ and we want θ to be in quadrant I so tangent will still be positive. Thus, we have $\tan\left(-\frac{2\pi}{3}\right)=\tan\left(\frac{\pi}{3}\right)$. Since $\frac{\pi}{3}$ is in the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, we can apply the equation above and get $\tan^{-1}\left(\tan\left(-\frac{2\pi}{3}\right)\right)=\tan^{-1}\left(\tan\frac{\pi}{3}\right)=\frac{\pi}{3}$.

45. $\sin\left(\sin^{-1}\frac{1}{4}\right)$ follows the form of the equation $f\left(f^{-1}\left(x\right)\right)=\sin\left(\sin^{-1}\left(x\right)\right)=x$. Since $\frac{1}{4}$ is in the interval $\left[-1,1\right]$, we can apply the equation directly and get $\sin\left(\sin^{-1}\frac{1}{4}\right)=\frac{1}{4}$.

46. $\cos\left(\cos^{-1}\left(-\frac{2}{3}\right)\right)$ follows the form of the equation $f\left(f^{-1}\left(x\right)\right)=\cos\left(\cos^{-1}\left(x\right)\right)=x$. Since $-\frac{2}{3}$ is in the interval $\left[-1,1\right]$, we can apply the equation directly and get

$\cos\left(\cos^{-1}\left(-\frac{2}{3}\right)\right)=-\frac{2}{3}$.

47. $\tan\left(\tan^{-1}4\right)$ follows the form of the equation $f\left(f^{-1}(x)\right)=\tan\left(\tan^{-1}(x)\right)=x$. Since 4 is a real number, we can apply the equation directly and get $\tan\left(\tan^{-1}4\right)=4$.

48. $\tan\left(\tan^{-1}(-2)\right)$ follows the form of the equation $f\left(f^{-1}(x)\right)=\tan\left(\tan^{-1}(x)\right)=x$. Since -2 is a real number, we can apply the equation directly and get $\tan\left(\tan^{-1}(-2)\right)=-2$.

49. Since there is no angle θ such that $\cos\theta=1.2$, the quantity $\cos^{-1}1.2$ is not defined. Thus, $\cos\left(\cos^{-1}1.2\right)$ is not defined.

50. Since there is no angle θ such that $\sin\theta=-2$, the quantity $\sin^{-1}(-2)$ is not defined. Thus, $\sin\left(\sin^{-1}(-2)\right)$ is not defined.

51. $\tan\left(\tan^{-1}\pi\right)$ follows the form of the equation $f\left(f^{-1}(x)\right)=\tan\left(\tan^{-1}(x)\right)=x$. Since π is a real number, we can apply the equation directly and get $\tan\left(\tan^{-1}\pi\right)=\pi$.

52. Since there is no angle θ such that $\sin\theta=-1.5$, the quantity $\sin^{-1}(-1.5)$ is not defined. Thus, $\sin\left(\sin^{-1}(-1.5)\right)$ is not defined.

53. $f(x)=5\sin x+2$

$y=5\sin x+2$

$x=5\sin y+2$

$5\sin y=x-2$

$\sin y=\dfrac{x-2}{5}$

$y=\sin^{-1}\dfrac{x-2}{5}=f^{-1}(x)$

The domain of $f(x)$ equals the range of $f^{-1}(x)$ and is $-\dfrac{\pi}{2}\le x\le\dfrac{\pi}{2}$ or $\left[-\dfrac{\pi}{2},\dfrac{\pi}{2}\right]$ in interval notation. To find the domain of $f^{-1}(x)$ we note that the argument of the inverse sine function is $\dfrac{x-2}{5}$ and that it must lie in the interval $[-1,1]$. That is,

$-1\le\dfrac{x-2}{5}\le1$

$-5\le x-2\le5$

$-3\le x\le7$

The domain of $f^{-1}(x)$ is $\{x\mid -3\le x\le7\}$, or $[-3,7]$ in interval notation. Recall that the domain of a function equals the range of its inverse and the range of a function equals the domain of its inverse. Thus, the range of f is also $[-3,7]$.

54. $f(x)=2\tan x-3$

$y=2\tan x-3$

$x=2\tan y-3$

$2\tan y=x+3$

$\tan y=\dfrac{x+3}{2}$

$y=\tan^{-1}\dfrac{x+3}{2}=f^{-1}(x)$

The domain of $f(x)$ equals the range of $f^{-1}(x)$ and is $-\dfrac{\pi}{2}<x<\dfrac{\pi}{2}$ or $\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)$ in interval notation. To find the domain of $f^{-1}(x)$ we note that the argument of the inverse tangent function can be any real number. Thus, the domain of $f^{-1}(x)$ is all real numbers, or $(-\infty,\infty)$ in interval notation. Recall that the domain of a function equals the range of its inverse and the range of a function equals the domain of its inverse. Thus, the range of f is $(-\infty,\infty)$.

55. $f(x)=-2\cos(3x)$

$y=-2\cos(3x)$

$x = -2\cos(3y)$

$\cos(3y) = -\dfrac{x}{2}$

$3y = \cos^{-1}\left(-\dfrac{x}{2}\right)$

$y = \dfrac{1}{3}\cos^{-1}\left(-\dfrac{x}{2}\right) = f^{-1}(x)$

The domain of $f(x)$ equals the range of $f^{-1}(x)$ and is $0 \le x \le \dfrac{\pi}{3}$, or $\left[0, \dfrac{\pi}{3}\right]$ in interval notation. To find the domain of $f^{-1}(x)$ we note that the argument of the inverse cosine function is $\dfrac{-x}{2}$ and that it must lie in the interval $[-1,1]$. That is,

$-1 \le -\dfrac{x}{2} \le 1$

$2 \ge x \ge -2$

$-2 \le x \le 2$

The domain of $f^{-1}(x)$ is $\{x \mid -2 \le x \le 2\}$, or $[-2, 2]$ in interval notation. Recall that the domain of a function equals the range of its inverse and the range of a function equals the domain of its inverse. Thus, the range of f is $[-2, 2]$.

56. $f(x) = 3\sin(2x)$

$y = 3\sin(2x)$

$x = 3\sin(2y)$

$\sin(2y) = \dfrac{x}{3}$

$2y = \sin^{-1}\dfrac{x}{3}$

$y = \dfrac{1}{2}\sin^{-1}\dfrac{x}{3} = f^{-1}(x)$

The domain of $f(x)$ equals the range of $f^{-1}(x)$ and is $-\dfrac{\pi}{4} \le x \le \dfrac{\pi}{4}$, or $\left[-\dfrac{\pi}{4}, \dfrac{\pi}{4}\right]$ in interval notation. To find the domain of $f^{-1}(x)$ we note that the argument of the inverse sine function is $\dfrac{x}{3}$ and that it must lie in the interval $[-1,1]$. That is,

$-1 \le \dfrac{x}{3} \le 1$

$-3 \le x \le 3$

The domain of $f^{-1}(x)$ is $\{x \mid -3 \le x \le 3\}$, or $[-3, 3]$ in interval notation. Recall that the domain of a function equals the range of its inverse and the range of a function equals the domain of its inverse. Thus, the range of f is $[-3, 3]$.

57. $f(x) = -\tan(x+1) - 3$

$y = -\tan(x+1) - 3$

$x = -\tan(y+1) - 3$

$\tan(y+1) = -x - 3$

$y + 1 = \tan^{-1}(-x-3)$

$y = -1 + \tan^{-1}(-x-3)$

$= -1 - \tan^{-1}(x+3) = f^{-1}(x)$

(note here we used the fact that $y = \tan^{-1} x$ is an odd function).

The domain of $f(x)$ equals the range of $f^{-1}(x)$ and is $-1 - \dfrac{\pi}{2} \le x \le \dfrac{\pi}{2} - 1$, or $\left[-1 - \dfrac{\pi}{2}, \dfrac{\pi}{2} - 1\right]$ in interval notation. To find the domain of $f^{-1}(x)$ we note that the argument of the inverse tangent function can be any real number. Thus, the domain of $f^{-1}(x)$ is all real numbers, or $(-\infty, \infty)$ in interval notation. Recall that the domain of a function equals the range of its inverse and the range of a function equals the domain of its inverse. Thus, the range of f is $(-\infty, \infty)$.

58. $f(x) = \cos(x+2) + 1$

$y = \cos(x+2) + 1$

$x = \cos(y+2) + 1$

$\cos(y+2) = x - 1$

$y + 2 = \cos^{-1}(x-1)$

$y = \cos^{-1}(x-1) - 2$

The domain of $f(x)$ equals the range of $f^{-1}(x)$ and is $-2 \le x \le \pi - 2$, or $[-2, \pi - 2]$ in interval notation. To find the domain of $f^{-1}(x)$ we note that the argument of the inverse cosine function is $x-1$ and that it must lie in the interval $[-1,1]$. That is, $-1 \le x-1 \le 1$

$0 \le x \le 2$

The domain of $f^{-1}(x)$ is $\{x \mid 0 \le x \le 2\}$, or $[0,2]$ in interval notation. Recall that the domain of a function equals the range of its inverse and the range of a function equals the domain of its inverse. Thus, the range of f is $[0,2]$.

59. $f(x) = 3\sin(2x+1)$

$$y = 3\sin(2x+1)$$
$$x = 3\sin(2y+1)$$
$$\sin(2y+1) = \frac{x}{3}$$
$$2y+1 = \sin^{-1}\frac{x}{3}$$
$$2y = \sin^{-1}\left(\frac{x}{3}\right) - 1$$
$$y = \frac{1}{2}\sin^{-1}\left(\frac{x}{3}\right) - \frac{1}{2} = f^{-1}(x)$$

The domain of $f(x)$ equals the range of $f^{-1}(x)$ and is $-\frac{1}{2} - \frac{\pi}{4} \le x \le -\frac{1}{2} + \frac{\pi}{4}$, or $\left[-\frac{1}{2} - \frac{\pi}{4}, -\frac{1}{2} + \frac{\pi}{4}\right]$ in interval notation. To find the domain of $f^{-1}(x)$ we note that the argument of the inverse sine function is $\frac{x}{3}$ and that it must lie in the interval $[-1,1]$. That is,

$$-1 \le \frac{x}{3} \le 1$$
$$-3 \le x \le 3$$

The domain of $f^{-1}(x)$ is $\{x \mid -3 \le x \le 3\}$, or $[-3,3]$ in interval notation. Recall that the domain of a function equals the range of its inverse and the range of a function equals the domain of its inverse. Thus, the range of f is $[-3,3]$.

60. $f(x) = 2\cos(3x+2)$

$$y = 2\cos(3x+2)$$
$$x = 2\cos(3y+2)$$
$$\cos(3y+2) = \frac{x}{2}$$
$$3y+2 = \cos^{-1}\left(\frac{x}{2}\right)$$
$$3y = \cos^{-1}\left(\frac{x}{2}\right) - 2$$
$$y = \frac{1}{3}\cos^{-1}\left(\frac{x}{2}\right) - \frac{2}{3} = f^{-1}(x)$$

The domain of $f(x)$ equals the range of $f^{-1}(x)$ and is $-\frac{2}{3} \le x \le -\frac{2}{3} + \frac{\pi}{3}$, or $\left[-\frac{2}{3}, -\frac{2}{3} + \frac{\pi}{3}\right]$ in interval notation. To find the domain of $f^{-1}(x)$ we note that the argument of the inverse cosine function is $\frac{x}{2}$ and that it must lie in the interval $[-1,1]$. That is,

$$-1 \le \frac{x}{2} \le 1$$
$$-2 \le x \le 2$$

The domain of $f^{-1}(x)$ is $\{x \mid -2 \le x \le 2\}$, or $[-2,2]$ in interval notation. Recall that the domain of a function equals the range of its inverse and the range of a function equals the domain of its inverse. Thus, the range of f is $[-2,2]$.

61. $4\sin^{-1} x = \pi$

$$\sin^{-1} x = \frac{\pi}{4}$$
$$x = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

The solution set is $\left\{\frac{\sqrt{2}}{2}\right\}$.

62. $2\cos^{-1} x = \pi$

$\cos^{-1} x = \frac{\pi}{2}$

$x = \cos\frac{\pi}{2} = 0$

The solution set is $\{0\}$.

63. $3\cos^{-1}(2x) = 2\pi$

$\cos^{-1}(2x) = \frac{2\pi}{3}$

$2x = \cos\frac{2\pi}{3}$

$2x = -\frac{1}{2}$

$x = -\frac{1}{4}$

The solution set is $\left\{-\frac{1}{4}\right\}$.

64. $-6\sin^{-1}(3x) = \pi$

$\sin^{-1}(3x) = -\frac{\pi}{6}$

$3x = \sin\left(-\frac{\pi}{6}\right)$

$3x = -\frac{1}{2}$

$x = -\frac{1}{6}$

The solution set is $\left\{-\frac{1}{6}\right\}$.

65. $3\tan^{-1} x = \pi$

$\tan^{-1} x = \frac{\pi}{3}$

$x = \tan\frac{\pi}{3} = \sqrt{3}$

The solution set is $\left\{\sqrt{3}\right\}$.

66. $-4\tan^{-1} x = \pi$

$\tan^{-1} x = -\frac{\pi}{4}$

$x = \tan\left(-\frac{\pi}{4}\right) = -1$

The solution set is $\{-1\}$.

67. $4\cos^{-1} x - 2\pi = 2\cos^{-1} x$

$2\cos^{-1} x - 2\pi = 0$

$2\cos^{-1} x = 2\pi$

$\cos^{-1} x = \pi$

$x = \cos\pi = -1$

The solution set is $\{-1\}$.

68. $5\sin^{-1} x - 2\pi = 2\sin^{-1} x - 3\pi$

$3\sin^{-1} x = -\pi$

$\sin^{-1} x = -\frac{\pi}{3}$

$x = \sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

The solution set is $\left\{-\frac{\sqrt{3}}{2}\right\}$.

69. Note that $\theta = 29°45' = 29.75°$.

a. $D = 24\cdot\left[1 - \frac{\cos^{-1}\left(\tan\left(23.5\cdot\frac{\pi}{180}\right)\tan\left(29.75\cdot\frac{\pi}{180}\right)\right)}{\pi}\right]$

≈ 13.92 hours or 13 hours, 55 minutes

b. $D = 24\cdot\left[1 - \frac{\cos^{-1}\left(\tan\left(0\cdot\frac{\pi}{180}\right)\tan\left(29.75\cdot\frac{\pi}{180}\right)\right)}{\pi}\right]$

≈ 12 hours

c. $D = 24\cdot\left[1 - \frac{\cos^{-1}\left(\tan\left(22.8\cdot\frac{\pi}{180}\right)\tan\left(29.75\cdot\frac{\pi}{180}\right)\right)}{\pi}\right]$

≈ 13.85 hours or 13 hours, 51 minutes

70. Note that $\theta = 40°45' = 40.75°$.

a. $D = 24\cdot\left(1 - \frac{\cos^{-1}\left(\tan\left(23.5\cdot\frac{\pi}{180}\right)\tan\left(40.75\cdot\frac{\pi}{180}\right)\right)}{\pi}\right)$

≈ 14.93 hours or 14 hours, 56 minutes

b. $D = 24\cdot\left(1 - \frac{\cos^{-1}\left(\tan\left(0\cdot\frac{\pi}{180}\right)\tan\left(40.75\cdot\frac{\pi}{180}\right)\right)}{\pi}\right)$

≈ 12 hours

c. $D = 24\cdot\left(1 - \frac{\cos^{-1}\left(\tan\left(22.8\cdot\frac{\pi}{180}\right)\tan\left(40.75\cdot\frac{\pi}{180}\right)\right)}{\pi}\right)$

≈ 14.83 hours or 14 hours, 50 minutes

71. Note that $\theta = 21°18' = 21.3°$.

a. $D = 24 \cdot \left(1 - \dfrac{\cos^{-1}\left(\tan\left(23.5 \cdot \frac{\pi}{180}\right)\tan\left(21.3 \cdot \frac{\pi}{180}\right)\right)}{\pi}\right)$

≈ 13.30 hours or 13 hours, 18 minutes

b. $D = 24 \cdot \left(1 - \dfrac{\cos^{-1}\left(\tan\left(0 \cdot \frac{\pi}{180}\right)\tan\left(21.3 \cdot \frac{\pi}{180}\right)\right)}{\pi}\right)$

≈ 12 hours

c. $D = 24 \cdot \left(1 - \dfrac{\cos^{-1}\left(\tan\left(22.8 \cdot \frac{\pi}{180}\right)\tan\left(21.3 \cdot \frac{\pi}{180}\right)\right)}{\pi}\right)$

≈ 13.26 hours or 13 hours, 15 minutes

72. Note that $\theta = 61°10' \approx 61.167°$.

a. $D = 24 \cdot \left(1 - \dfrac{\cos^{-1}\left(\tan\left(23.5 \cdot \frac{\pi}{180}\right)\tan\left(61.167 \cdot \frac{\pi}{180}\right)\right)}{\pi}\right)$

≈ 18.96 hours or 18 hours, 57 minutes

b. $D = 24 \cdot \left(1 - \dfrac{\cos^{-1}\left(\tan\left(0 \cdot \frac{\pi}{180}\right)\tan\left(61.167 \cdot \frac{\pi}{180}\right)\right)}{\pi}\right)$

≈ 12 hours

c. $D = 24 \cdot \left(1 - \dfrac{\cos^{-1}\left(\tan\left(22.8 \cdot \frac{\pi}{180}\right)\tan\left(61.167 \cdot \frac{\pi}{180}\right)\right)}{\pi}\right)$

≈ 18.64 hours or 18 hours, 38 minutes

73. a. $D = 24 \cdot \left(1 - \dfrac{\cos^{-1}\left(\tan\left(23.5 \cdot \frac{\pi}{180}\right)\tan\left(0 \cdot \frac{\pi}{180}\right)\right)}{\pi}\right)$

≈ 12 hours

b. $D = 24 \cdot \left(1 - \dfrac{\cos^{-1}\left(\tan\left(0 \cdot \frac{\pi}{180}\right)\tan\left(0 \cdot \frac{\pi}{180}\right)\right)}{\pi}\right)$

≈ 12 hours

c. $D = 24 \cdot \left(1 - \dfrac{\cos^{-1}\left(\tan\left(22.8 \cdot \frac{\pi}{180}\right)\tan\left(0 \cdot \frac{\pi}{180}\right)\right)}{\pi}\right)$

≈ 12 hours

d. There are approximately 12 hours of daylight every day at the equator.

74. Note that $\theta = 66°30' = 66.5°$.

a. $D = 24 \cdot \left(1 - \dfrac{\cos^{-1}\left(\tan\left(23.5 \cdot \frac{\pi}{180}\right)\tan\left(66.5 \cdot \frac{\pi}{180}\right)\right)}{\pi}\right)$

≈ 24 hours

b. $D = 24 \cdot \left(1 - \dfrac{\cos^{-1}\left(\tan\left(0 \cdot \frac{\pi}{180}\right)\tan\left(66.5 \cdot \frac{\pi}{180}\right)\right)}{\pi}\right)$

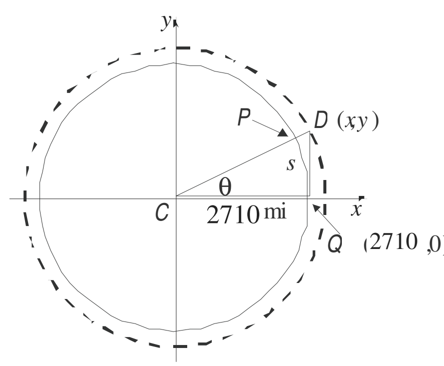
≈ 12 hours

c. $D = 24 \cdot \left(1 - \dfrac{\cos^{-1}\left(\tan\left(22.8 \cdot \frac{\pi}{180}\right)\tan\left(66.5 \cdot \frac{\pi}{180}\right)\right)}{\pi}\right)$

≈ 22.02 hours or 22 hours, 1 minute

d. The amount of daylight at this location on the winter solstice is $24 - 24 = 0$ hours. That is, on the winter solstice, there is no daylight. In general, for a location at $66°30'$ north latitude, it ranges from around-the-clock daylight to no daylight at all.

75. Let point *C* represent the point on the Earth's axis at the same latitude as Cadillac Mountain, and arrange the figure so that segment *CQ* lies along the *x*-axis (see figure).



At the latitude of Cadillac Mountain, the effective radius of the earth is 2710 miles. If point $D(x, y)$ represents the peak of Cadillac Mountain, then the length of segment *PD* is

$1530 \text{ ft} \cdot \dfrac{1 \text{ mile}}{5280 \text{ feet}} \approx 0.29 \text{ mile}$. Therefore, the point $D(x, y) = (2710, y)$ lies on a circle with radius $r = 2710.29$ miles. We now have

$$\cos\theta = \frac{x}{r} = \frac{2710}{2710.29}$$

$$\theta = \cos^{-1}\left(\frac{2710}{2710.29}\right) \approx 0.01463 \text{ radians}$$

Finally, $s = r\theta = 2710(0.01463) \approx 39.64$ miles ,

and $\dfrac{2\pi(2710)}{24} = \dfrac{39.64}{t}$, so

$$t = \frac{24(39.64)}{2\pi(2710)} \approx 0.05587 \text{ hours} \approx 3.35 \text{ minutes}$$

Therefore, a person atop Cadillac Mountain will see the first rays of sunlight about 3.35 minutes sooner than a person standing below at sea level.

76. $\theta(x) = \tan^{-1}\left(\frac{34}{x}\right) - \tan^{-1}\left(\frac{6}{x}\right)$.

a. $\theta(10) = \tan^{-1}\left(\frac{34}{10}\right) - \tan^{-1}\left(\frac{6}{10}\right) \approx 42.6°$

If you sit 10 feet from the screen, then the viewing angle is about $42.6°$.

$\theta(15) = \tan^{-1}\left(\frac{34}{15}\right) - \tan^{-1}\left(\frac{6}{15}\right) \approx 44.4°$

If you sit 15 feet from the screen, then the viewing angle is about $44.4°$.

$\theta(20) = \tan^{-1}\left(\frac{34}{20}\right) - \tan^{-1}\left(\frac{6}{20}\right) \approx 42.8°$

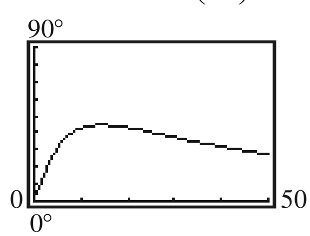
If you sit 20 feet from the screen, then the viewing angle is about $42.8°$.

b. Let *r* = the row that result in the largest viewing angle. Looking ahead to part (c), we see that the maximum viewing angle occurs when the distance from the screen is about 14.3 feet. Thus,

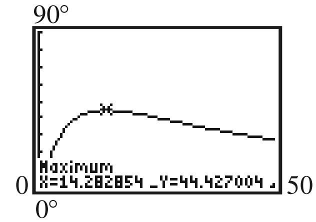
$$\begin{aligned} 5+3(r-1) &= 14.3 \\ 5+3r-3 &= 14.3 \\ 3r &= 12.3 \\ r &= 4.1 \end{aligned}$$

Sitting in the 4th row should provide the largest viewing angle.

c. Set the graphing calculator in degree mode and let $Y_1 = \tan^{-1}\left(\frac{34}{x}\right) - \tan^{-1}\left(\frac{6}{x}\right)$:



Use MAXIMUM:



The maximum viewing angle will occur when $x \approx 14.3$ feet.

77. a. $a = 0$; $b = \sqrt{3}$; The area is:

$$\begin{aligned} \tan^{-1} b - \tan^{-1} a &= \tan^{-1}\sqrt{3} - \tan^{-1} 0 \\ &= \frac{\pi}{3} - 0 \\ &= \frac{\pi}{3} \text{ square units} \end{aligned}$$

b. $a = -\frac{\sqrt{3}}{3}$; $b = 1$; The area is:

$$\begin{aligned} \tan^{-1} b - \tan^{-1} a &= \tan^{-1} 1 - \tan^{-1}\left(-\frac{\sqrt{3}}{3}\right) \\ &= \frac{\pi}{4} - \left(-\frac{\pi}{6}\right) \\ &= \frac{5\pi}{12} \text{ square units} \end{aligned}$$

78. a. $a = 0$; $b = \frac{\sqrt{3}}{2}$; The area is:

$$\begin{aligned} \sin^{-1} b - \sin^{-1} a &= \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \sin^{-1} 0 \\ &= \frac{\pi}{3} - 0 \\ &= \frac{\pi}{3} \text{ square units} \end{aligned}$$

b. $a=-\frac{1}{2}$; $b=\frac{1}{2}$; The area is:

$$\sin^{-1}b-\sin^{-1}a=\sin^{-1}\left(\frac{1}{2}\right)-\sin^{-1}\left(-\frac{1}{2}\right)=\frac{\pi}{6}-\left(-\frac{\pi}{6}\right)=\frac{\pi}{3}\text{ square units}$$

79. Here we have $\alpha_1=41°50'$, $\beta_1=-87°37'$, $\alpha_2=21°18'$, and $\beta_2=-157°50'$. Converting minutes to degrees gives $\alpha_1=\left(41\frac{5}{6}\right)°$, $\beta_1=\left(-87\frac{37}{60}\right)°$, $\alpha_2=21.3°$, and $\beta_2=\left(-157\frac{5}{6}\right)°$. Substituting these values, and $r=3960$, into our equation gives $d\approx 4250$ miles. The distance from Chicago to Honolulu is about 4250 miles.
(remember that S and W angles are negative)

80. Here we have $\alpha_1=21°18'$, $\beta_1=-157°50'$, $\alpha_2=-37°47'$, and $\beta_2=144°58'$. Converting minutes to degrees gives $\alpha_1=21.3°$, $\beta_1=\left(-157\frac{5}{6}\right)°$, $\alpha_2=\left(-37\frac{47}{60}\right)°$, and $\beta_2=\left(144\frac{29}{30}\right)°$. Substituting these values, and $r=3960$, into our equation gives $d\approx 5518$ miles. The distance from Honolulu to Melbourne is about 5518 miles.
(remember that S and W angles are negative)

Section 7.2

1. Domain: $\left\{x \,\middle|\, x\neq \text{odd integer multiples of } \frac{\pi}{2}\right\}$,
Range: $\{y \mid y\le -1 \text{ or } y\ge 1\}$

2. True

3. $\frac{1}{\sqrt{5}}=\frac{\sqrt{5}}{5}$

4. $x=\sec y$, ≥ 1, 0, π

5. cosine

6. False

7. True

8. True

9. $\cos\left(\sin^{-1}\frac{\sqrt{2}}{2}\right)$

Find the angle θ, $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$, whose sine equals $\frac{\sqrt{2}}{2}$.

$$\sin\theta=\frac{\sqrt{2}}{2},\qquad -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$$
$$\theta=\frac{\pi}{4}$$
$$\cos\left(\sin^{-1}\frac{\sqrt{2}}{2}\right)=\cos\frac{\pi}{4}=\frac{\sqrt{2}}{2}$$

10. $\sin\left(\cos^{-1}\frac{1}{2}\right)$

Find the angle θ, $0\le\theta\le\pi$, whose cosine equals $\frac{1}{2}$.

$$\cos\theta=\frac{1}{2},\qquad 0\le\theta\le\pi$$
$$\theta=\frac{\pi}{3}$$
$$\sin\left(\cos^{-1}\frac{1}{2}\right)=\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}$$

11. $\tan\left[\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right]$

Find the angle θ, $0\le\theta\le\pi$, whose cosine equals $-\frac{\sqrt{3}}{2}$.

$$\cos\theta=-\frac{\sqrt{3}}{2},\qquad 0\le\theta\le\pi$$
$$\theta=\frac{5\pi}{6}$$
$$\tan\left[\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right]=\tan\frac{5\pi}{6}=-\frac{\sqrt{3}}{3}$$

12. $\tan\left[\sin^{-1}\left(-\frac{1}{2}\right)\right]$

Find the angle θ, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, whose sine equals $-\frac{1}{2}$.

$$\sin\theta = -\frac{1}{2}, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$
$$\theta = -\frac{\pi}{6}$$
$$\tan\left[\sin^{-1}\left(-\frac{1}{2}\right)\right] = \tan\left(-\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{3}$$

13. $\sec\left(\cos^{-1}\frac{1}{2}\right)$

Find the angle θ, $0 \le \theta \le \pi$, whose cosine equals $\frac{1}{2}$.

$$\cos\theta = \frac{1}{2}, \quad 0 \le \theta \le \pi$$
$$\theta = \frac{\pi}{3}$$
$$\sec\left(\cos^{-1}\frac{1}{2}\right) = \sec\frac{\pi}{3} = 2$$

14. $\cot\left[\sin^{-1}\left(-\frac{1}{2}\right)\right]$

Find the angle θ, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, whose sine equals $-\frac{1}{2}$.

$$\sin\theta = -\frac{1}{2}, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$
$$\theta = -\frac{\pi}{6}$$
$$\cot\left[\sin^{-1}\left(-\frac{1}{2}\right)\right] = \cot\left(-\frac{\pi}{6}\right) = -\sqrt{3}$$

15. $\csc\left(\tan^{-1} 1\right)$

Find the angle θ, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, whose tangent equals 1.

$$\tan\theta = 1, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$
$$\theta = \frac{\pi}{4}$$
$$\csc\left(\tan^{-1} 1\right) = \csc\frac{\pi}{4} = \sqrt{2}$$

16. $\sec\left(\tan^{-1}\sqrt{3}\right)$

Find the angle θ, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, whose tangent equals $\sqrt{3}$.

$$\tan\theta = \sqrt{3}, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$
$$\theta = \frac{\pi}{3}$$
$$\sec\left(\tan^{-1}\sqrt{3}\right) = \sec\frac{\pi}{3} = 2$$

17. $\sin\left[\tan^{-1}(-1)\right]$

Find the angle θ, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, whose tangent equals -1.

$$\tan\theta = -1, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$
$$\theta = -\frac{\pi}{4}$$
$$\sin\left[\tan^{-1}(-1)\right] = \sin\left(-\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

18. $\cos\left[\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right]$

Find the angle θ, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, whose sine equals $-\frac{\sqrt{3}}{2}$.

$$\sin\theta = -\frac{\sqrt{3}}{2}, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$
$$\theta = -\frac{\pi}{3}$$
$$\cos\left[\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right] = \cos\left(-\frac{\pi}{3}\right) = \frac{1}{2}$$

19. $\sec\left[\sin^{-1}\left(-\frac{1}{2}\right)\right]$

Find the angle θ, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, whose sine equals $-\frac{1}{2}$.

$$\sin\theta = -\frac{1}{2}, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$
$$\theta = -\frac{\pi}{6}$$
$$\sec\left[\sin^{-1}\left(-\frac{1}{2}\right)\right] = \sec\left(-\frac{\pi}{6}\right) = \frac{2\sqrt{3}}{3}$$

20. $\csc\left[\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right]$

Find the angle θ, $0 \le \theta \le \pi$, whose cosine equals $-\frac{\sqrt{3}}{2}$.

$$\cos\theta = -\frac{\sqrt{3}}{2} \quad 0 \le \theta \le \pi$$
$$\theta = \frac{5\pi}{6}$$
$$\csc\left[\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right] = \csc\frac{5\pi}{6} = 2$$

21. $\cos^{-1}\left(\sin\frac{5\pi}{4}\right) = \cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

Find the angle θ, $0 \le \theta \le \pi$, whose cosine equals $-\frac{\sqrt{2}}{2}$.

$$\cos\theta = -\frac{\sqrt{2}}{2}, \quad 0 \le \theta \le \pi$$
$$\theta = \frac{3\pi}{4}$$
$$\cos^{-1}\left(\sin\frac{5\pi}{4}\right) = \frac{3\pi}{4}$$

22. $\tan^{-1}\left(\cot\frac{2\pi}{3}\right) = \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$

Find the angle θ, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, whose tangent equals $-\frac{1}{\sqrt{3}}$.

$$\tan\theta = -\frac{1}{\sqrt{3}}, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$
$$\theta = -\frac{\pi}{6}$$
$$\tan^{-1}\left(\cot\frac{2\pi}{3}\right) = -\frac{\pi}{6}$$

23. $\sin^{-1}\left[\cos\left(-\frac{7\pi}{6}\right)\right] = \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

Find the angle θ, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, whose sine equals $-\frac{\sqrt{3}}{2}$.

$$\sin\theta = -\frac{\sqrt{3}}{2}, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$
$$\theta = -\frac{\pi}{3}$$
$$\sin^{-1}\left[\cos\left(-\frac{7\pi}{6}\right)\right] = -\frac{\pi}{3}$$

24. $\cos^{-1}\left[\tan\left(-\frac{\pi}{3}\right)\right] = \cos^{-1}(-1)$

Find the angle θ, $0 \le \theta \le \pi$, whose cosine equals -1.

$$\cos\theta = -1, \quad 0 \le \theta \le \pi$$
$$\theta = \frac{\pi}{3}$$
$$\cos^{-1}\left[\tan\left(-\frac{\pi}{3}\right)\right] = \pi$$

25. $\tan\left(\sin^{-1}\dfrac{1}{3}\right)$

Let $\theta = \sin^{-1}\dfrac{1}{3}$. Since $\sin\theta = \dfrac{1}{3}$ and $-\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$, θ is in quadrant I, and we let $y = 1$ and $r = 3$.

Solve for x:

$$x^2 + 1 = 9$$
$$x^2 = 8$$
$$x = \pm\sqrt{8} = \pm 2\sqrt{2}$$

Since θ is in quadrant I, $x = 2\sqrt{2}$.

$$\tan\left(\sin^{-1}\frac{1}{3}\right) = \tan\theta = \frac{y}{x} = \frac{1}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{4}$$

26. $\tan\left(\cos^{-1}\dfrac{1}{3}\right)$

Let $\theta = \cos^{-1}\dfrac{1}{3}$. Since $\cos\theta = \dfrac{1}{3}$ and $0 \le \theta \le \pi$, θ is in quadrant I, and we let $x = 1$ and $r = 3$.

Solve for y:

$$1 + y^2 = 9$$
$$y^2 = 8$$
$$y = \pm\sqrt{8} = \pm 2\sqrt{2}$$

Since θ is in quadrant I, $y = 2\sqrt{2}$.

$$\tan\left(\cos^{-1}\frac{1}{3}\right) = \tan\theta = \frac{y}{x} = \frac{2\sqrt{2}}{1} = 2\sqrt{2}$$

27. $\sec\left(\tan^{-1}\dfrac{1}{2}\right)$

Let $\theta = \tan^{-1}\dfrac{1}{2}$. Since $\tan\theta = \dfrac{1}{2}$ and $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$, θ is in quadrant I, and we let $x = 2$ and $y = 1$.

Solve for r:

$$2^2 + 1 = r^2$$
$$r^2 = 5$$
$$r = \sqrt{5}$$

θ is in quadrant I.

$$\sec\left(\tan^{-1}\frac{1}{2}\right) = \sec\theta = \frac{r}{x} = \frac{\sqrt{5}}{2}$$

28. $\cos\left(\sin^{-1}\dfrac{\sqrt{2}}{3}\right)$

Let $\theta = \sin^{-1}\dfrac{\sqrt{2}}{3}$. Since $\sin\theta = \dfrac{\sqrt{2}}{3}$ and $-\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$, θ is in quadrant I, and we let $y = \sqrt{2}$ and $r = 3$.

Solve for x:

$$x^2 + 2 = 9$$
$$x^2 = 7$$
$$x = \pm\sqrt{7}$$

Since θ is in quadrant I, $x = \sqrt{7}$.

$$\cos\left(\sin^{-1}\frac{\sqrt{2}}{3}\right) = \cos\theta = \frac{x}{r} = \frac{\sqrt{7}}{3}$$

29. $\cot\left[\sin^{-1}\left(-\dfrac{\sqrt{2}}{3}\right)\right]$

Let $\theta = \sin^{-1}\left(-\dfrac{\sqrt{2}}{3}\right)$. Since $\sin\theta = -\dfrac{\sqrt{2}}{3}$ and $-\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$, θ is in quadrant IV, and we let $y = -\sqrt{2}$ and $r = 3$.

Solve for x:

$$x^2 + 2 = 9$$
$$x^2 = 7$$
$$x = \pm\sqrt{7}$$

Since θ is in quadrant IV, $x = \sqrt{7}$.

$$\cot\left[\sin^{-1}\left(-\frac{\sqrt{2}}{3}\right)\right] = \cot\theta = \frac{x}{y} = \frac{\sqrt{7}}{-\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = -\frac{\sqrt{14}}{2}$$

30. $\csc\left[\tan^{-1}(-2)\right]$

Let $\theta = \tan^{-1}(-2)$. Since $\tan\theta = -2$ and $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$, θ is in quadrant IV, and we let $x = 1$ and $y = -2$.

Solve for r:

$$1 + 4 = r^2$$
$$r^2 = 5$$
$$r = \pm\sqrt{5}$$

Since θ is in quadrant IV, $r = \sqrt{5}$.

$$\csc\left[\tan^{-1}(-2)\right] = \csc\theta = \frac{r}{y} = \frac{\sqrt{5}}{-2} = -\frac{\sqrt{5}}{2}$$

31. $\sin\left[\tan^{-1}(-3)\right]$

Let $\theta = \tan^{-1}(-3)$. Since $\tan\theta = -3$ and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, θ is in quadrant IV, and we let $x = 1$ and $y = -3$.

Solve for r:

$$1 + 9 = r^2$$
$$r^2 = 10$$
$$r = \pm\sqrt{10}$$

Since θ is in quadrant IV, $r = \sqrt{10}$.

$$\sin\left[\tan^{-1}(-3)\right] = \sin\theta = \frac{y}{r}$$
$$= \frac{-3}{\sqrt{10}} \cdot \frac{\sqrt{10}}{\sqrt{10}} = -\frac{3\sqrt{10}}{10}$$

32. $\cot\left[\cos^{-1}\left(-\frac{\sqrt{3}}{3}\right)\right]$

Let $\theta = \cos^{-1}\left(-\frac{\sqrt{3}}{3}\right)$. Since $\cos\theta = -\frac{\sqrt{3}}{3}$ and $0 \le \theta \le \pi$, θ is in quadrant II, and we let $x = -\sqrt{3}$ and $r = 3$.

Solve for y:

$$3 + y^2 = 9$$
$$y^2 = 6$$
$$y = \pm\sqrt{6}$$

Since θ is in quadrant II, $y = \sqrt{6}$.

$$\cot\left[\cos^{-1}\left(-\frac{\sqrt{3}}{3}\right)\right] = \cot\theta = \frac{x}{y}$$
$$= \frac{-\sqrt{3}}{\sqrt{6}} = \frac{-1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$$

33. $\sec\left(\sin^{-1}\frac{2\sqrt{5}}{5}\right)$

Let $\theta = \sin^{-1}\frac{2\sqrt{5}}{5}$. Since $\sin\theta = \frac{2\sqrt{5}}{5}$ and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, θ is in quadrant I, and we let $y = 2\sqrt{5}$ and $r = 5$.

Solve for x:

$$x^2 + 20 = 25$$
$$x^2 = 5$$
$$x = \pm\sqrt{5}$$

Since θ is in quadrant I, $x = \sqrt{5}$.

$$\sec\left(\sin^{-1}\frac{2\sqrt{5}}{5}\right) = \sec\theta = \frac{r}{x} = \frac{5}{\sqrt{5}} = \sqrt{5}$$

34. $\csc\left(\tan^{-1}\frac{1}{2}\right)$

Let $\theta = \tan^{-1}\frac{1}{2}$. Since $\tan\theta = \frac{1}{2}$ and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, θ is in quadrant I, and we let $x = 2$ and $y = 1$.

Solve for r:

$$2^2 + 1 = r^2$$
$$r^2 = 5$$
$$r = \sqrt{5}$$

θ is in quadrant I.

$$\csc\left(\tan^{-1}\frac{1}{2}\right) = \csc\theta = \frac{r}{y} = \frac{\sqrt{5}}{1} = \sqrt{5}$$

35. $\sin^{-1}\left(\cos\frac{3\pi}{4}\right) = \sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$

36. $\cos^{-1}\left(\sin\frac{7\pi}{6}\right) = \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$

37. $\cot^{-1}\sqrt{3}$

We are finding the angle θ, $0 < \theta < \pi$, whose cotangent equals $\sqrt{3}$.

$$\cot\theta = \sqrt{3}, \quad 0 < \theta < \pi$$
$$\theta = \frac{\pi}{6}$$
$$\cot^{-1}\sqrt{3} = \frac{\pi}{6}$$

38. $\cot^{-1} 1$

We are finding the angle θ, $0 < \theta < \pi$, whose cotangent equals 1.

$$\cot\theta = 1, \quad 0 < \theta < \pi$$
$$\theta = \frac{\pi}{4}$$
$$\cot^{-1} 1 = \frac{\pi}{4}$$

39. $\csc^{-1}(-1)$

We are finding the angle θ, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, $\theta \ne 0$, whose cosecant equals -1.

$$\csc\theta = -1, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2},\ \theta \ne 0$$

$$\theta = -\frac{\pi}{2}$$

$$\csc^{-1}(-1) = -\frac{\pi}{2}$$

40. $\csc^{-1}\sqrt{2}$

We are finding the angle θ, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, $\theta \ne 0$, whose cosecant equals $\sqrt{2}$.

$$\csc\theta = \sqrt{2}, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2},\ \theta \ne 0$$

$$\theta = \frac{\pi}{4}$$

$$\csc^{-1}\sqrt{2} = \frac{\pi}{4}$$

41. $\sec^{-1}\frac{2\sqrt{3}}{3}$

We are finding the angle θ, $0 \le \theta \le \pi$, $\theta \ne \frac{\pi}{2}$, whose secant equals $\frac{2\sqrt{3}}{3}$.

$$\sec\theta = \frac{2\sqrt{3}}{3}, \quad 0 \le \theta \le \pi,\ \theta \ne \frac{\pi}{2}$$

$$\theta = \frac{\pi}{6}$$

$$\sec^{-1}\frac{2\sqrt{3}}{3} = \frac{\pi}{6}$$

42. $\sec^{-1}(-2)$

We are finding the angle θ, $0 \le \theta \le \pi$, $\theta \ne \frac{\pi}{2}$, whose secant equals -2.

$$\sec\theta = -2, \quad 0 \le \theta \le \pi,\ \theta \ne \frac{\pi}{2}$$

$$\theta = \frac{2\pi}{3}$$

$$\sec^{-1}(-2) = \frac{2\pi}{3}$$

43. $\cot^{-1}\left(-\frac{\sqrt{3}}{3}\right)$

We are finding the angle θ, $0 < \theta < \pi$, whose cotangent equals $-\frac{\sqrt{3}}{3}$.

$$\cot\theta = -\frac{\sqrt{3}}{3}, \quad 0 < \theta < \pi$$

$$\theta = \frac{2\pi}{3}$$

$$\cot^{-1}\left(-\frac{\sqrt{3}}{3}\right) = \frac{2\pi}{3}$$

44. $\csc^{-1}\left(-\frac{2\sqrt{3}}{3}\right)$

We are finding the angle θ, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, $\theta \ne 0$, whose cosecant equals $-\frac{2\sqrt{3}}{3}$.

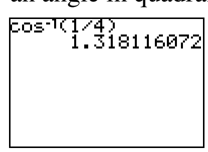
$$\csc\theta = -\frac{2\sqrt{3}}{3}, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2},\ \theta \ne 0$$

$$\theta = -\frac{\pi}{3}$$

$$\csc^{-1}\left(-\frac{2\sqrt{3}}{3}\right) = -\frac{\pi}{3}$$

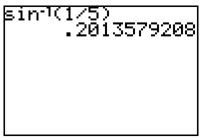
45. $\sec^{-1}4 = \cos^{-1}\frac{1}{4}$

We seek the angle θ, $0 \le \theta \le \pi$, whose cosine equals $\frac{1}{4}$. Now $\cos\theta = \frac{1}{4}$, so θ lies in quadrant I. The calculator yields $\cos^{-1}\frac{1}{4} \approx 1.32$, which is an angle in quadrant I, so $\sec^{-1}(4) \approx 1.32$.



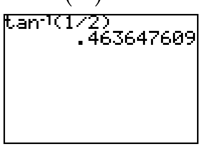
46. $\csc^{-1} 5 = \sin^{-1} \frac{1}{5}$

We seek the angle θ, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, whose sine equals $\frac{1}{5}$. Now $\sin\theta = \frac{1}{5}$, so θ lies in quadrant I. The calculator yields $\sin^{-1}\frac{1}{5} \approx 0.20$, which is an angle in quadrant I, so $\csc^{-1} 5 \approx 0.20$.



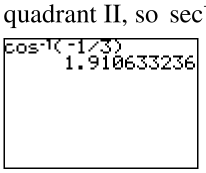
47. $\cot^{-1} 2 = \tan^{-1} \frac{1}{2}$

We seek the angle θ, $0 \le \theta \le \pi$, whose tangent equals $\frac{1}{2}$. Now $\tan\theta = \frac{1}{2}$, so θ lies in quadrant I. The calculator yields $\text{an}^{-1}\frac{1}{2} \approx 0.46$, which is an angle in quadrant I, so $\cot^{-1}(2) \approx 0.46$.



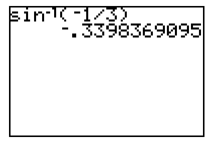
48. $\sec^{-1}(-3) = \cos^{-1}\left(-\frac{1}{3}\right)$

We seek the angle θ, $0 \le \theta \le \pi$, whose cosine equals $-\frac{1}{3}$. Now $\cos\theta = -\frac{1}{3}$, θ lies in quadrant II. The calculator yields $\cos^{-1}\left(-\frac{1}{3}\right) \approx 1.91$, which is an angle in quadrant II, so $\sec^{-1}(-3) \approx 1.91$.



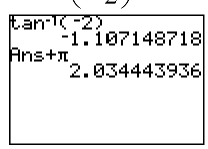
49. $\csc^{-1}(-3) = \sin^{-1}\left(-\frac{1}{3}\right)$

We seek the angle θ, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, whose sine equals $-\frac{1}{3}$. Now $\sin\theta = -\frac{1}{3}$, so θ lies in quadrant IV. The calculator yields $\sin^{-1}\left(-\frac{1}{3}\right) \approx -0.34$, which is an angle in quadrant IV, so $\csc^{-1}(-3) \approx -0.34$.



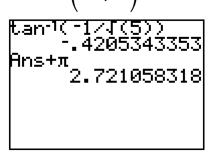
50. $\cot^{-1}\left(-\frac{1}{2}\right) = \tan^{-1}(-2)$

We seek the angle θ, $0 \le \theta \le \pi$, whose tangent equals -2. Now $\tan\theta = -2$, so θ lies in quadrant II. The calculator yields $\tan^{-1}(-2) \approx -1.11$, which is an angle in quadrant IV. Since θ lies in quadrant II, $\theta \approx -1.11 + \pi \approx 2.03$. Therefore, $\cot^{-1}\left(-\frac{1}{2}\right) \approx 2.03$.



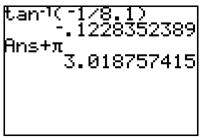
51. $\cot^{-1}\left(-\sqrt{5}\right) = \tan^{-1}\left(-\frac{1}{\sqrt{5}}\right)$

We seek the angle θ, $0 \le \theta \le \pi$, whose tangent equals $-\frac{1}{\sqrt{5}}$. Now $\tan\theta = -\frac{1}{\sqrt{5}}$, so θ lies in quadrant II. The calculator yields $\tan^{-1}\left(-\frac{1}{\sqrt{5}}\right) \approx -0.42$, which is an angle in quadrant IV. Since θ is in quadrant II, $\theta \approx -0.42 + \pi \approx 2.72$. Therefore, $\cot^{-1}\left(-\sqrt{5}\right) \approx 2.72$.



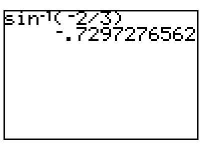
52. $\cot^{-1}(-8.1)=\tan^{-1}\left(-\frac{1}{8.1}\right)$

We seek the angle $\theta,\ 0\le\theta\le\pi,$ whose tangent equals $-\frac{1}{8.1}$. Now $\tan\theta=-\frac{1}{8.1}$, so θ lies in quadrant II. The calculator yields $\tan^{-1}\left(-\frac{1}{8.1}\right)\approx -0.12$, which is an angle in quadrant IV. Since θ is in quadrant II, $\theta\approx -0.12+\pi\approx 3.02$. Thus, $\cot^{-1}(-8.1)\approx 3.02$.



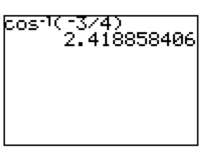
53. $\csc^{-1}\left(-\frac{3}{2}\right)=\sin^{-1}\left(-\frac{2}{3}\right)$

We seek the angle $\theta,\ -\frac{\pi}{2}\le\theta\le\frac{\pi}{2},\ \theta\ne 0$, whose sine equals $-\frac{2}{3}$. Now $\sin\theta=-\frac{2}{3}$, so θ lies in quadrant IV. The calculator yields $\sin^{-1}\left(-\frac{2}{3}\right)\approx -0.73$, which is an angle in quadrant IV, so $\csc^{-1}\left(-\frac{3}{2}\right)\approx -0.73$.



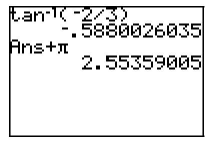
54. $\sec^{-1}\left(-\frac{4}{3}\right)=\cos^{-1}\left(-\frac{3}{4}\right)$

We are finding the angle $\theta,\ 0\le\theta\le\pi,\ \theta\ne\frac{\pi}{2}$, whose cosine equals $-\frac{3}{4}$. Now $\cos\theta=-\frac{3}{4}$, so θ lies in quadrant II. The calculator yields $\cos^{-1}\left(-\frac{3}{4}\right)\approx 2.42$, which is an angle in quadrant II, so $\sec^{-1}\left(-\frac{4}{3}\right)\approx 2.42$.



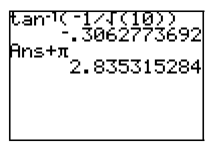
55. $\cot^{-1}\left(-\frac{3}{2}\right)=\tan^{-1}\left(-\frac{2}{3}\right)$

We are finding the angle $\theta,\ 0\le\theta\le\pi,$ whose tangent equals $-\frac{2}{3}$. Now $\tan\theta=-\frac{2}{3}$, so θ lies in quadrant II. The calculator yields $\tan^{-1}\left(-\frac{2}{3}\right)\approx -0.59$, which is an angle in quadrant IV. Since θ is in quadrant II, $\theta\approx -0.59+\pi\approx 2.55$. Thus, $\cot^{-1}\left(-\frac{3}{2}\right)\approx 2.55$.



56. $\cot^{-1}\left(-\sqrt{10}\right)=\tan^{-1}\left(-\frac{1}{\sqrt{10}}\right)$

We are finding the angle $\theta,\ 0\le\theta\le\pi,$ whose tangent equals $-\frac{1}{\sqrt{10}}$. Now $\tan\theta=-\frac{1}{\sqrt{10}}$, so θ lies in quadrant II. The calculator yields $\tan^{-1}\left(-\frac{1}{\sqrt{10}}\right)\approx -0.306$, which is an angle in quadrant IV. Since θ is in quadrant II, $\theta\approx -0.306+\pi\approx 2.84$. So, $\cot^{-1}\left(-\sqrt{10}\right)\approx 2.84$.



57. Let $\theta=\tan^{-1}u$ so that $\tan\theta=u$, $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$, $-\infty<u<\infty$. Then,

$$\cos\left(\tan^{-1}u\right)=\cos\theta=\frac{1}{\sec\theta}=\frac{1}{\sqrt{\sec^2\theta}}$$
$$=\frac{1}{\sqrt{1+\tan^2\theta}}=\frac{1}{\sqrt{1+u^2}}$$

58. Let $\theta=\cos^{-1}u$ so that $\cos\theta=u$, $0\le\theta\le\pi$, $-1\le u\le 1$. Then,

$$\sin\left(\cos^{-1}u\right)=\sin\theta=\sqrt{\sin^2\theta}$$
$$=\sqrt{1-\cos^2\theta}=\sqrt{1-u^2}$$

59. Let $\theta = \sin^{-1} u$ so that $\sin\theta = u$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, $-1 \le u \le 1$. Then,

$$\tan\left(\sin^{-1} u\right) = \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sin\theta}{\sqrt{\cos^2\theta}} = \frac{\sin\theta}{\sqrt{1-\sin^2\theta}} = \frac{u}{\sqrt{1-u^2}}$$

60. Let $\theta = \cos^{-1} u$ so that $\cos\theta = u$, $0 \le \theta \le \pi$, $-1 \le u \le 1$. Then,

$$\tan\left(\cos^{-1} u\right) = \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sqrt{\sin^2\theta}}{\cos\theta} = \frac{\sqrt{1-\cos^2\theta}}{\cos\theta} = \frac{\sqrt{1-u^2}}{u}$$

61. Let $\theta = \sec^{-1} u$ so that $\sec\theta = u$, $0 \le \theta \le \pi$ and $\theta \ne \frac{\pi}{2}$, $|u| \ge 1$. Then,

$$\sin\left(\sec^{-1} u\right) = \sin\theta = \sqrt{\sin^2\theta} = \sqrt{1-\cos^2\theta} = \sqrt{1-\frac{1}{\sec^2\theta}} = \frac{\sqrt{\sec^2\theta - 1}}{\sqrt{\sec^2\theta}} = \frac{\sqrt{u^2-1}}{|u|}$$

62. Let $\theta = \cot^{-1} u$ so that $\cot\theta = u$, $0 < \theta < \pi$, $-\infty < u < \infty$. Then,

$$\sin\left(\cot^{-1} u\right) = \sin\theta = \sqrt{\sin^2\theta} = \frac{1}{\sqrt{\csc^2\theta}} = \frac{1}{\sqrt{1+\cot^2\theta}} = \frac{1}{\sqrt{1+u^2}}$$

63. Let $\theta = \csc^{-1} u$ so that $\csc\theta = u$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, $|u| \ge 1$. Then,

$$\cos\left(\csc^{-1} u\right) = \cos\theta = \cos\theta \cdot \frac{\sin\theta}{\sin\theta} = \cot\theta\sin\theta = \frac{\cot\theta}{\csc\theta} = \frac{\sqrt{\cot^2\theta}}{\csc\theta} = \frac{\sqrt{\csc^2\theta - 1}}{\csc\theta} = \frac{\sqrt{u^2-1}}{|u|}$$

64. Let $\theta = \sec^{-1} u$ so that $\sec\theta = u$, $0 \le \theta \le \pi$ and $\theta \ne \frac{\pi}{2}$, $|u| \ge 1$. Then,

$$\cos\left(\sec^{-1} u\right) = \cos\theta = \frac{1}{\sec\theta} = \frac{1}{u}$$

65. Let $\theta = \cot^{-1} u$ so that $\cot\theta = u$, $0 < \theta < \pi$, $-\infty < u < \infty$. Then,

$$\tan\left(\cot^{-1} u\right) = \tan\theta = \frac{1}{\cot\theta} = \frac{1}{u}$$

66. Let $\theta = \sec^{-1} u$ so that $\sec\theta = u$, $0 \le \theta \le \pi$ and $\theta \ne \frac{\pi}{2}$, $|u| \ge 1$. Note that $\sin\theta \ge 0$. Then,

$$\tan\left(\sec^{-1} u\right) = \tan\theta = \sin\theta\sec\theta = \sec\theta\sqrt{1-\cos^2\theta} = u\sqrt{1-\frac{1}{u^2}} = u\sqrt{\frac{u^2-1}{u^2}}$$

The u cannot be cancelled since it can be either positive or negative.

67. $g\left(f^{-1}\left(\frac{12}{13}\right)\right) = \cos\left(\sin^{-1}\frac{12}{13}\right)$

Let $\theta = \sin^{-1}\frac{12}{13}$. Since $\sin\theta = \frac{12}{13}$ and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, θ is in quadrant I, and we let $y = 12$ and $r = 13$. Solve for x:

$$x^2 + 12^2 = 13^2$$
$$x^2 + 144 = 169$$
$$x^2 = 25 \Rightarrow x = \pm\sqrt{25} = \pm 5$$

Since θ is in quadrant I, $x = 5$.

$$g\left(f^{-1}\left(\frac{12}{13}\right)\right) = \cos\left(\sin^{-1}\frac{12}{13}\right) = \cos\theta = \frac{x}{r} = \frac{5}{13}$$

68. $f\left(g^{-1}\left(\frac{5}{13}\right)\right) = \sin\left(\cos^{-1}\frac{5}{13}\right)$

Let $\theta = \cos^{-1}\frac{5}{13}$. Since $\cos\theta = \frac{5}{13}$ and $0 \le \theta \le \pi$, θ is in quadrant I, and we let $x = 5$ and $r = 13$. Solve for *y*:

$$5^2 + y^2 = 13^2$$
$$25 + y^2 = 169$$
$$y^2 = 144$$
$$y = \pm\sqrt{144} = \pm 12$$

Since θ is in quadrant I, $y = 12$.

$$f\left(g^{-1}\left(\frac{5}{13}\right)\right) = \sin\left(\cos^{-1}\frac{5}{13}\right) = \sin\theta = \frac{y}{r} = \frac{12}{13}$$

69. $g^{-1}\left(f\left(\frac{7\pi}{4}\right)\right) = \cos^{-1}\left(\sin\frac{7\pi}{4}\right)$

$$= \cos^{-1}\left(-\frac{\sqrt{2}}{2}\right) = \frac{3\pi}{4}$$

70. $f^{-1}\left(g\left(\frac{5\pi}{6}\right)\right) = \sin^{-1}\left(\cos\frac{5\pi}{6}\right)$

$$= \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$$

71. $h\left(f^{-1}\left(-\frac{3}{5}\right)\right) = \tan\left(\sin^{-1}\left(-\frac{3}{5}\right)\right)$

Let $\theta = \sin^{-1}\left(-\frac{3}{5}\right)$. Since $\sin\theta = -\frac{3}{5}$ and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, θ is in quadrant IV, and we let $y = -3$ and $r = 5$. Solve for *x*:

$$x^2 + (-3)^2 = 5^2$$
$$x^2 + 9 = 25$$
$$x^2 = 16$$
$$x = \pm\sqrt{16} = \pm 4$$

Since θ is in quadrant IV, $x = 4$.

$$h\left(f^{-1}\left(-\frac{3}{5}\right)\right) = \tan\left(\sin^{-1}\left(-\frac{3}{5}\right)\right)$$
$$= \tan\theta = \frac{y}{x} = \frac{-3}{4} = -\frac{3}{4}$$

72. $h\left(g^{-1}\left(-\frac{4}{5}\right)\right) = \tan\left(\cos^{-1}\left(-\frac{4}{5}\right)\right)$

Let $\theta = \cos^{-1}\left(-\frac{4}{5}\right)$. Since $\cos\theta = -\frac{4}{5}$ and $0 \le \theta \le \pi$, θ is in quadrant II, and we let $x = -4$ and $r = 5$. Solve for *y*:

$$(-4)^2 + y^2 = 5^2$$
$$16 + y^2 = 25$$
$$y^2 = 9$$
$$y = \pm\sqrt{9} = \pm 3$$

Since θ is in quadrant II, $y = 3$.

$$h\left(g^{-1}\left(-\frac{4}{5}\right)\right) = \tan\left(\cos^{-1}\left(-\frac{4}{5}\right)\right)$$
$$= \tan\theta = \frac{y}{x} = \frac{-3}{4} = -\frac{3}{4}$$

73. $g\left(h^{-1}\left(\frac{12}{5}\right)\right) = \cos\left(\tan^{-1}\frac{12}{5}\right)$

Let $\theta = \tan^{-1}\frac{12}{5}$. Since $\tan\theta = \frac{12}{5}$ and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, θ is in quadrant I, and we let $x = 5$ and $y = 12$. Solve for *r*:

$$r^2 = 5^2 + 12^2$$
$$r^2 = 25 + 144 = 169$$
$$r = \pm\sqrt{169} = \pm 13$$

Now, *r* must be positive, so $r = 13$.

$$g\left(h^{-1}\left(\frac{12}{5}\right)\right) = \cos\left(\tan^{-1}\frac{12}{5}\right) = \cos\theta = \frac{x}{r} = \frac{5}{13}$$

74. $f\left(h^{-1}\left(\frac{5}{12}\right)\right) = \sin\left(\tan^{-1}\frac{5}{12}\right)$

Let $\theta = \tan^{-1}\frac{5}{12}$. Since $\tan\theta = \frac{5}{12}$ and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, θ is in quadrant I, and we let $x = 12$ and $y = 5$. Solve for *r*:

$r^2 = 12^2 + 5^2$
$r^2 = 144 + 25 = 169$
$r = \pm\sqrt{169} = \pm 13$
Now, r must be positive, so $r = 13$.
$f\left(h^{-1}\left(\frac{5}{12}\right)\right) = \sin\left(\tan^{-1}\frac{5}{12}\right) = \sin\theta = \frac{y}{r} = \frac{5}{13}$

75. $g^{-1}\left(f\left(-\frac{4\pi}{3}\right)\right) = \cos^{-1}\left(\sin\left(-\frac{4\pi}{3}\right)\right)$
$= \cos^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{6}$

76. $g^{-1}\left(f\left(-\frac{5\pi}{6}\right)\right) = \cos^{-1}\left(\sin\left(-\frac{5\pi}{6}\right)\right)$
$= \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$

77. $h\left(g^{-1}\left(-\frac{1}{4}\right)\right) = \tan\left(\cos^{-1}\left(-\frac{1}{4}\right)\right)$

Let $\theta = \cos^{-1}\left(-\frac{1}{4}\right)$. Since $\cos\theta = -\frac{1}{4}$ and $0 \le \theta \le \pi$, θ is in quadrant II, and we let $x = -1$ and $r = 4$. Solve for *y*:
$(-1)^2 + y^2 = 4^2$
$1 + y^2 = 16$
$y^2 = 15$
$y = \pm\sqrt{15}$
Since θ is in quadrant II, $y = \sqrt{15}$.
$h\left(g^{-1}\left(-\frac{1}{4}\right)\right) = \tan\left(\cos^{-1}\left(-\frac{1}{4}\right)\right)$
$= \tan\theta = \frac{y}{x} = \frac{\sqrt{15}}{-1} = -\sqrt{15}$

78. $h\left(f^{-1}\left(-\frac{2}{5}\right)\right) = \tan\left(\sin^{-1}\left(-\frac{2}{5}\right)\right)$

Let $\theta = \sin^{-1}\left(-\frac{2}{5}\right)$. Since $\sin\theta = -\frac{2}{5}$ and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, θ is in quadrant IV, and we let $y = -2$ and $r = 5$. Solve for *x*:
$x^2 + (-2)^2 = 5^2$
$x^2 + 4 = 25$
$x^2 = 21$
$x = \pm\sqrt{21}$
Since θ is in quadrant IV, $x = \sqrt{21}$.
$h\left(f^{-1}\left(-\frac{2}{5}\right)\right) = \tan\left(\sin^{-1}\left(-\frac{2}{5}\right)\right)$
$= \tan\theta = \frac{y}{x} = \frac{-2}{\sqrt{21}} = -\frac{2\sqrt{21}}{21}$

79. a. Since the diameter of the base is 45 feet, we have $r = \frac{45}{2} = 22.5$ feet. Thus,
$\theta = \cot^{-1}\left(\frac{22.5}{14}\right) = 31.89°$.

b. $\theta = \cot^{-1}\frac{r}{h}$
$\cot\theta = \frac{r}{h} \quad \rightarrow \quad r = h\cot\theta$
Here we have $\theta = 31.89°$ and $h = 17$ feet. Thus, $r = 17\cot(31.89°) = 27.32$ feet and the diameter is $2(27.32) = 54.64$ feet.

c. From part (b), we get $h = \frac{r}{\cot\theta}$.
The radius is $\frac{122}{2} = 61$ feet.
$h = \frac{r}{\cot\theta} = \frac{61}{22.5/14} \approx 37.96$ feet.
Thus, the height is 37.96 feet.

80. a. Since the diameter of the base is 6.68 feet, we have $r = \frac{6.68}{2} = 3.34$ feet. Thus,
$\theta = \cot^{-1}\left(\frac{3.34}{4}\right) = 50.14°$

b. $\theta = \cot^{-1}\frac{r}{h}$
$\cot\theta = \frac{r}{h} \quad \rightarrow \quad h = \frac{r}{\cot\theta}$
Here we have $\theta = 50.14°$ and $r = 4$ feet. Thus, $h = \frac{4}{\cot(50.14°)} = 4.79$ feet. The bunker will be 4.79 feet high.

c. $\theta_{TG} = \cot^{-1}\left(\frac{4.22}{6}\right) = 54.88°$

From part (a) we have $\theta_{USGA} = 50.14°$. For steep bunkers, a larger angle of repose is required. Therefore, the Tour Grade 50/50 sand is better suited since it has a larger angle of repose.

81. a. $\cot\theta = \frac{2x}{2y + gt^2}$

$$\theta = \cot^{-1}\left(\frac{2x}{2y + gt^2}\right)$$

The artillery shell begins at the origin and lands at the coordinates $(6175, 2450)$. Thus,

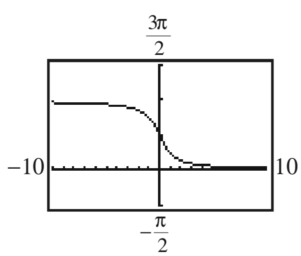
$$\theta = \cot^{-1}\left(\frac{2 \cdot 6175}{2 \cdot 2450 + 32.2(2.27)^2}\right)$$
$$\approx \cot^{-1}(2.437858) \approx 22.3°$$

The artilleryman used an angle of elevation of $22.3°$.

b. $\sec\theta = \frac{v_0 t}{x}$

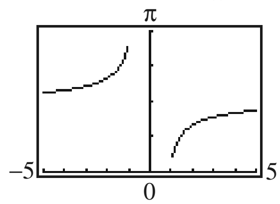
$$v_0 = \frac{x\sec\theta}{t} = \frac{(6175)\sec(22.3°)}{2.27}$$
$$= 2940.23 \text{ ft/sec}$$

82. Let. $y = \cot^{-1} x = \cos^{-1}\frac{x}{\sqrt{x^2+1}}$

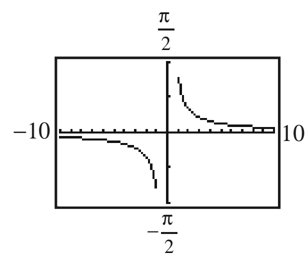


Note that the range of $y = \cot^{-1} x$ is $(0, \pi)$, so $\tan^{-1}\frac{1}{x}$ will not work.

83. $y = \sec^{-1} x = \cos^{-1}\frac{1}{x}$



84. $y = \csc^{-1} x = \sin^{-1}\frac{1}{x}$



85 – 86. Answers will vary.

Section 7.3

1. $3x - 5 = -x + 1$

$$4x = 6$$
$$x = \frac{6}{4} = \frac{3}{2}$$

The solution set is $\left\{\frac{3}{2}\right\}$.

2. $\frac{\sqrt{2}}{2}, -\frac{1}{2}$

3. $4x^2 - x - 5 = 0$

$$(4x-5)(x+1) = 0$$
$$4x - 5 = 0 \quad \text{or} \quad x + 1 = 0$$
$$x = \frac{5}{4} \quad \text{or} \quad x = -1$$

The solution set is $\left\{-1, \frac{5}{4}\right\}$.

4. $x^2 - x - 1 = 0$

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)}$$
$$= \frac{1 \pm \sqrt{1+4}}{2}$$
$$= \frac{1 \pm \sqrt{5}}{2}$$

The solution set is $\left\{\frac{1-\sqrt{5}}{2},\ \frac{1+\sqrt{5}}{2}\right\}$.

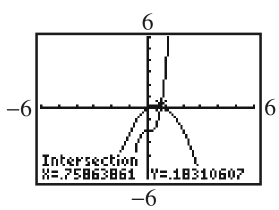
5. $(2x-1)^2 - 3(2x-1) - 4 = 0$

$$[(2x-1)+1][(2x-1)-4] = 0$$
$$2x(2x-5) = 0$$
$$2x = 0 \quad \text{or} \quad 2x - 5 = 0$$
$$x = 0 \quad \text{or} \quad x = \frac{5}{2}$$

The solution set is $\left\{0,\ \frac{5}{2}\right\}$.

6. $5x^3 - 2 = x - x^2$

Let $y_1 = 5x^3 - 2$ and $y_2 = x - x^2$. Use INTERSECT to find the solution(s):



In this case, the graphs only intersect in one location, so the equation has only one solution. Rounding as directed, the solutions set is $\{0.76\}$.

7. $\frac{\pi}{6},\ \frac{5\pi}{6}$

8. $\left\{\theta \,\middle|\, \theta = \frac{\pi}{6} + 2\pi k,\ \theta = \frac{5\pi}{6} + 2\pi k,\ k \text{ is any integer}\right\}$

9. False because of the circular nature of the functions.

10. False, 2 is outside the range of the sin function.

11. $2\sin\theta + 3 = 2$

$$2\sin\theta = -1$$
$$\sin\theta = -\frac{1}{2}$$
$$\theta = \frac{7\pi}{6} + 2k\pi \quad \text{or} \quad \theta = \frac{11\pi}{6} + 2k\pi,\ k \text{ is any integer}$$

On $0 \le \theta < 2\pi$, the solution set is $\left\{\frac{7\pi}{6},\ \frac{11\pi}{6}\right\}$.

12. $1 - \cos\theta = \frac{1}{2}$

$$1 - \cos\theta = \frac{1}{2}$$
$$\frac{1}{2} = \cos\theta$$
$$\theta = \frac{\pi}{3} + 2k\pi \quad \text{or} \quad \theta = \frac{5\pi}{3} + 2k\pi,\ k \text{ is any integer}$$

On $0 \le \theta < 2\pi$, the solution set is $\left\{\frac{\pi}{3},\ \frac{5\pi}{3}\right\}$.

13. $4\cos^2\theta = 1$

$$\cos^2\theta = \frac{1}{4}$$
$$\cos\theta = \pm\frac{1}{2}$$
$$\theta = \frac{\pi}{3} + k\pi \text{ or } \theta = \frac{2\pi}{3} + k\pi,\ k \text{ is any integer}$$

On the interval $0 \le \theta < 2\pi$, the solution set is $\left\{\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}\right\}$.

14. $\tan^2\theta = \frac{1}{3}$

$$\tan\theta = \pm\sqrt{\frac{1}{3}} = \pm\frac{\sqrt{3}}{3}$$
$$\theta = \frac{\pi}{6} + k\pi \text{ or } \theta = \frac{5\pi}{6} + k\pi,\ k \text{ is any integer}$$

On the interval $0 \le \theta < 2\pi$, the solution set is $\left\{\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}\right\}$.

15. $2\sin^2\theta - 1 = 0$

$$2\sin^2\theta = 1$$
$$\sin^2\theta = \frac{1}{2}$$
$$\sin\theta = \pm\sqrt{\frac{1}{2}} = \pm\frac{\sqrt{2}}{2}$$
$$\theta = \frac{\pi}{4} + k\pi \text{ or } \theta = \frac{3\pi}{4} + k\pi,\ k \text{ is any integer}$$

On the interval $0 \le \theta < 2\pi$, the solution set is $\left\{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\right\}$.

16. $4\cos^2\theta - 3 = 0$

$$4\cos^2\theta = 3$$
$$\cos^2\theta = \frac{3}{4}$$
$$\cos\theta = \pm\frac{\sqrt{3}}{2}$$
$$\theta = \frac{\pi}{6} + k\pi \text{ or } \theta = \frac{5\pi}{6} + k\pi,\ k \text{ is any integer}$$

On the interval $0 \le \theta < 2\pi$, the solution set is $\left\{\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}\right\}$.

17. $\sin(3\theta) = -1$

$$3\theta = \frac{3\pi}{2} + 2k\pi$$
$$\theta = \frac{\pi}{2} + \frac{2k\pi}{3},\ k \text{ is any integer}$$

On the interval $0 \le \theta < 2\pi$, the solution set is $\left\{\frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}\right\}$.

18. $\tan\left(\frac{\theta}{2}\right) = \sqrt{3}$

$$\frac{\theta}{2} = \frac{\pi}{3} + k\pi,\ k \text{ is any integer}$$
$$\theta = \frac{2\pi}{3} + 2\pi k,\ k \text{ is any integer}$$

On $0 \le \theta < 2\pi$, the solution set is $\left\{\frac{2\pi}{3}\right\}$.

19. $\cos(2\theta) = -\frac{1}{2}$

$$2\theta = \frac{2\pi}{3} + 2k\pi \text{ or } 2\theta = \frac{4\pi}{3} + 2k\pi$$
$$\theta = \frac{\pi}{3} + k\pi \text{ or } \theta = \frac{2\pi}{3} + k\pi,\ k \text{ is any integer}$$

On the interval $0 \le \theta < 2\pi$, the solution set is $\left\{\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}\right\}$.

20. $\tan(2\theta) = -1$

$$2\theta = \frac{3\pi}{4} + k\pi,\ k \text{ is any integer}$$
$$\theta = \frac{3\pi}{8} + \frac{k\pi}{2},\ k \text{ is any integer}$$

On the interval $0 \le \theta < 2\pi$, the solution set is $\left\{\frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}, \frac{15\pi}{8}\right\}$.

21. $\sec\frac{3\theta}{2} = -2$

$$\frac{3\theta}{2} = \frac{2\pi}{3} + 2k\pi \text{ or } \frac{3\theta}{2} = \frac{4\pi}{3} + 2k\pi$$
$$\theta = \frac{4\pi}{9} + \frac{4k\pi}{3} \text{ or } \theta = \frac{8\pi}{9} + \frac{4k\pi}{3},$$

k is any integer

On the interval $0 \le \theta < 2\pi$, the solution set is $\left\{\frac{4\pi}{9}, \frac{8\pi}{9}, \frac{16\pi}{9}\right\}$.

22. $\cot\frac{2\theta}{3} = -\sqrt{3}$

$$\frac{2\theta}{3} = \frac{5\pi}{6} + k\pi,\ k \text{ is any integer}$$
$$\theta = \frac{5\pi}{4} + \frac{3k\pi}{2},\ k \text{ is any integer}$$

On $0 \le \theta < 2\pi$, the solution set is $\left\{\frac{5\pi}{4}\right\}$.

23. $2\sin\theta + 1 = 0$

$$2\sin\theta = -1$$
$$\sin\theta = -\frac{1}{2}$$
$$\theta = \frac{7\pi}{6} + 2k\pi \text{ or } \theta = \frac{11\pi}{6} + 2k\pi,\ k \text{ is any integer}$$

On $0 \le \theta < 2\pi$, the solution set is $\left\{\frac{7\pi}{6}, \frac{11\pi}{6}\right\}$.

24. $\cos\theta + 1 = 0$

$\cos\theta = -1$

$\theta = \pi + 2k\pi,\ k$ is any integer

On the interval $0 \le \theta < 2\pi$, the solution set is $\{\pi\}$.

25. $\tan\theta + 1 = 0$

$\tan\theta = -1$

$\theta = \frac{3\pi}{4} + k\pi,\ k$ is any integer

On $0 \le \theta < 2\pi$, the solution set is $\left\{\frac{3\pi}{4}, \frac{7\pi}{4}\right\}$.

26. $\sqrt{3}\cot\theta + 1 = 0$

$\sqrt{3}\cot\theta = -1$

$\cot\theta = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$

$\theta = \frac{2\pi}{3} + k\pi,\ k$ is any integer

On $0 \le \theta < 2\pi$, the solution set is $\left\{\frac{2\pi}{3}, \frac{5\pi}{3}\right\}$.

27. $4\sec\theta + 6 = -2$

$4\sec\theta = -8$

$\sec\theta = -2$

$\theta = \frac{2\pi}{3} + 2k\pi$ or $\theta = \frac{4\pi}{3} + 2k\pi$, k is any integer

On $0 \le \theta < 2\pi$, the solution set is $\left\{\frac{2\pi}{3}, \frac{4\pi}{3}\right\}$.

28. $5\csc\theta - 3 = 2$

$5\csc\theta = 5$

$\csc\theta = 1$

$\theta = \frac{\pi}{2} + 2k\pi,\ k$ is any integer

On $0 \le \theta < 2\pi$, the solution set is $\left\{\frac{\pi}{2}\right\}$.

29. $3\sqrt{2}\cos\theta + 2 = -1$

$3\sqrt{2}\cos\theta = -3$

$\cos\theta = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$

$\theta = \frac{3\pi}{4} + 2k\pi$ or $\theta = \frac{5\pi}{4} + 2k\pi$, k is any integer

On $0 \le \theta < 2\pi$, the solution set is $\left\{\frac{3\pi}{4}, \frac{5\pi}{4}\right\}$.

30. $4\sin\theta + 3\sqrt{3} = \sqrt{3}$

$4\sin\theta = -2\sqrt{3}$

$\sin\theta = -\frac{2\sqrt{3}}{4} = -\frac{\sqrt{3}}{2}$

$\theta = \frac{4\pi}{3} + 2k\pi$ or $\theta = \frac{5\pi}{3} + 2k\pi$, k is any integer

On $0 \le \theta < 2\pi$, the solution set is $\left\{\frac{4\pi}{3}, \frac{5\pi}{3}\right\}$.

31. $\cos\left(2\theta - \frac{\pi}{2}\right) = -1$

$2\theta - \frac{\pi}{2} = \pi + 2k\pi$

$2\theta = \frac{3\pi}{2} + 2k\pi$

$\theta = \frac{3\pi}{4} + k\pi,\ k$ is any integer

On $0 \le \theta < 2\pi$, the solution set is $\left\{\frac{3\pi}{4}, \frac{7\pi}{4}\right\}$.

32. $\sin\left(3\theta + \frac{\pi}{18}\right) = 1$

$3\theta + \frac{\pi}{18} = \frac{\pi}{2} + 2k\pi$

$3\theta = \frac{4\pi}{9} + 2k\pi$

$\theta = \frac{4\pi}{27} + \frac{2k\pi}{3},\ k$ is any integer

On the interval $0 \le \theta < 2\pi$, the solution set is $\left\{\frac{4\pi}{27}, \frac{22\pi}{27}, \frac{40\pi}{27}\right\}$.

33. $\tan\left(\frac{\theta}{2} + \frac{\pi}{3}\right) = 1$

$\frac{\theta}{2} + \frac{\pi}{3} = \frac{\pi}{4} + k\pi$

$\frac{\theta}{2} = -\frac{\pi}{12} + k\pi$

$\theta = -\frac{\pi}{6} + 2k\pi,\ k$ is any integer

On $0 \le \theta < 2\pi$, the solution set is $\left\{\frac{11\pi}{6}\right\}$.

34. $\cos\left(\frac{\theta}{3}-\frac{\pi}{4}\right)=\frac{1}{2}$

$\frac{\theta}{3}-\frac{\pi}{4}=\frac{\pi}{3}+2k\pi$ or $\frac{\theta}{3}-\frac{\pi}{4}=\frac{5\pi}{3}+2k\pi$

$\frac{\theta}{3}=\frac{7\pi}{12}+2k\pi$ or $\frac{\theta}{3}=\frac{23\pi}{12}+2k\pi$

$\theta=\frac{7\pi}{4}+6k\pi$ or $\theta=\frac{23\pi}{4}+6k\pi$,

k is any integer.

On $0\le\theta<2\pi$, the solution set is $\left\{\frac{7\pi}{4}\right\}$.

35. $\sin\theta=\frac{1}{2}$

$\left\{\theta\,\middle|\,\theta=\frac{\pi}{6}+2k\pi \text{ or } \theta=\frac{5\pi}{6}+2k\pi\right\}$, k is any integer. Six solutions are

$\theta=\frac{\pi}{6},\frac{5\pi}{6},\frac{13\pi}{6},\frac{17\pi}{6},\frac{25\pi}{6},\frac{29\pi}{6}$.

36. $\tan\theta=1$

$\left\{\theta\,\middle|\,\theta=\frac{\pi}{4}+k\pi\right\}$, k is any integer

Six solutions are $\theta=\frac{\pi}{4},\frac{5\pi}{4},\frac{9\pi}{4},\frac{13\pi}{4},\frac{17\pi}{4},\frac{21\pi}{4}$.

37. $\tan\theta=-\frac{\sqrt{3}}{3}$

$\left\{\theta\,\middle|\,\theta=\frac{5\pi}{6}+k\pi\right\}$, k is any integer

Six solutions are

$\theta=\frac{5\pi}{6},\frac{11\pi}{6},\frac{17\pi}{6},\frac{23\pi}{6},\frac{29\pi}{6},\frac{35\pi}{6}$.

38. $\cos\theta=-\frac{\sqrt{3}}{2}$

$\left\{\theta\,\middle|\,\theta=\frac{5\pi}{6}+2k\pi \text{ or } \theta=\frac{7\pi}{6}+2k\pi\right\}$, k is any integer. Six solutions are

$\theta=\frac{5\pi}{6},\frac{7\pi}{6},\frac{17\pi}{6},\frac{19\pi}{6},\frac{29\pi}{6},\frac{31\pi}{6}$.

39. $\cos\theta=0$

$\left\{\theta\,\middle|\,\theta=\frac{\pi}{2}+2k\pi \text{ or } \theta=\frac{3\pi}{2}+2k\pi\right\}$, k is any integer

Six solutions are $\theta=\frac{\pi}{2},\frac{3\pi}{2},\frac{5\pi}{2},\frac{7\pi}{2},\frac{9\pi}{2},\frac{11\pi}{2}$.

40. $\sin\theta=\frac{\sqrt{2}}{2}$

$\left\{\theta\,\middle|\,\theta=\frac{\pi}{4}+2k\pi \text{ or } \theta=\frac{3\pi}{4}+2k\pi\right\}$, k is any integer

Six solutions are $\theta=\frac{\pi}{4},\frac{3\pi}{4},\frac{9\pi}{4},\frac{11\pi}{4},\frac{17\pi}{4},\frac{19\pi}{4}$.

41. $\cos(2\theta)=-\frac{1}{2}$

$2\theta=\frac{2\pi}{3}+2k\pi$ or $2\theta=\frac{4\pi}{3}+2k\pi$, k is any integer

$\left\{\theta\,\middle|\,\theta=\frac{\pi}{3}+k\pi \text{ or } \theta=\frac{2\pi}{3}+k\pi\right\}$, k is any integer

Six solutions are $\theta=\frac{\pi}{3},\frac{2\pi}{3},\frac{4\pi}{3},\frac{5\pi}{3},\frac{7\pi}{3},\frac{8\pi}{3}$.

42. $\sin(2\theta)=-1$

$2\theta=\frac{3\pi}{2}+2k\pi$, k is any integer

$\left\{\theta\,\middle|\,\theta=\frac{3\pi}{4}+k\pi\right\}$, k is any integer

Six solutions are

$\theta=\frac{3\pi}{4},\frac{7\pi}{4},\frac{11\pi}{4},\frac{15\pi}{4},\frac{19\pi}{4},\frac{23\pi}{4}$.

43. $\sin\frac{\theta}{2}=-\frac{\sqrt{3}}{2}$

$\frac{\theta}{2}=\frac{4\pi}{3}+2k\pi$ or $\frac{\theta}{2}=\frac{5\pi}{3}+2k\pi$, k is any integer

$\left\{\theta\,\middle|\,\theta=\frac{8\pi}{3}+4k\pi \text{ or } \theta=\frac{10\pi}{3}+4k\pi\right\}$, k is any integer. Six solutions are

$\theta=\frac{8\pi}{3},\frac{10\pi}{3},\frac{20\pi}{3},\frac{22\pi}{3},\frac{32\pi}{3},\frac{34\pi}{3}$.

44. $\tan\frac{\theta}{2}=-1$

$\frac{\theta}{2}=\frac{3\pi}{4}+k\pi$, k is any integer

$\left\{\theta\,\middle|\,\theta=\frac{3\pi}{2}+2k\pi\right\}$, k is any integer

Six solutions are

$\theta = \frac{3\pi}{2}, \frac{7\pi}{2}, \frac{11\pi}{2}, \frac{15\pi}{2}, \frac{19\pi}{2}, \frac{23\pi}{2}$.

45. $\sin\theta = 0.4$

$\theta = \sin^{-1}(0.4) \approx 0.41$

$\theta \approx 0.41$ or $\theta \approx \pi - 0.41 \approx 2.73$.

The solution set is $\{0.41,\ 2.73\}$.

46. $\cos\theta = 0.6$

$\theta = \cos^{-1}(0.6) \approx 0.93$

$\theta \approx 0.93$ or $\theta \approx 2\pi - 0.93 \approx 5.36$.

The solution set is $\{0.93,\ 5.36\}$.

47. $\tan\theta = 5$

$\theta = \tan^{-1}(5) \approx 1.37$

$\theta \approx 1.37$ or $\theta \approx \pi + 1.37 \approx 4.51$.

The solution set is $\{1.37,\ 4.51\}$.

48. $\cot\theta = 2$

$\tan\theta = \frac{1}{2}$

$\theta = \tan^{-1}\left(\frac{1}{2}\right) \approx 0.46$

$\theta \approx 0.46$ or $\theta \approx \pi + 0.46 \approx 3.61$.

The solution set is $\{0.46,\ 3.61\}$.

49. $\cos\theta = -0.9$

$\theta = \cos^{-1}(-0.9) \approx 2.69$

$\theta \approx 2.69$ or $\theta \approx 2\pi - 2.69 \approx 3.59$.

The solution set is $\{2.69,\ 3.59\}$.

50. $\sin\theta = -0.2$

$\theta = \sin^{-1}(-0.2) \approx -0.20$

$\theta \approx -0.20 + 2\pi$ or $\theta \approx \pi - (-0.20)$.

≈ 6.08 $\qquad \approx 3.34$

The solution set is $\{3.34,\ 6.08\}$.

51. $\sec\theta = -4$

$\cos\theta = -\frac{1}{4}$

$\theta = \cos^{-1}\left(-\frac{1}{4}\right) \approx 1.82$

$\theta \approx 1.82$ or $\theta \approx 2\pi - 1.82 \approx 4.46$.

The solution set is $\{1.82,\ 4.46\}$.

52. $\csc\theta = -3$

$\sin\theta = -\frac{1}{3}$

$\theta = \sin^{-1}\left(-\frac{1}{3}\right) \approx -0.34$

$\theta \approx -0.34 + 2\pi$ or $\theta \approx \pi - (-0.34)$.

≈ 5.94 $\qquad \approx 3.48$

The solution set is $\{3.48,\ 5.94\}$.

53. $5\tan\theta + 9 = 0$

$5\tan\theta = -9$

$\tan\theta = -\frac{9}{5}$

$\theta = \tan^{-1}\left(-\frac{9}{5}\right) \approx -1.064$

$\theta \approx -1.064 + \pi$ or $\theta \approx -1.064 + 2\pi$

≈ 2.08 $\qquad \approx 5.22$

The solution set is $\{2.08,\ 5.22\}$.

54. $4\cot\theta = -5$

$\cot\theta = -\frac{5}{4}$

$\tan\theta = -\frac{4}{5}$

$\theta = \tan^{-1}\left(-\frac{4}{5}\right) \approx -0.675$

$\theta \approx -0.675 + \pi$ or $\theta \approx -0.675 + 2\pi$.

≈ 2.47 $\qquad \approx 5.61$

The solution set is $\{2.47,\ 5.61\}$.

55. $3\sin\theta - 2 = 0$

$3\sin\theta = 2$

$\sin\theta = \frac{2}{3}$

$\theta = \sin^{-1}\left(\frac{2}{3}\right) \approx 0.73$

$\theta \approx 0.73$ or $\theta \approx \pi - 0.73 \approx 2.41$.

The solution set is $\{0.73,\ 2.41\}$.

56. $4\cos\theta+3=0$

$$4\cos\theta=-3$$
$$\cos\theta=-\frac{3}{4}$$
$$\theta=\cos^{-1}\left(-\frac{3}{4}\right)\approx 2.42$$

$\theta\approx 2.42$ or $\theta\approx 2\pi-2.42\approx 3.86$.

The solution set is $\{2.42,\ 3.86\}$.

57. $2\cos^2\theta+\cos\theta=0$

$$\cos\theta(2\cos\theta+1)=0$$

$\cos\theta=0$ or $2\cos\theta+1=0$

$\theta=\frac{\pi}{2},\frac{3\pi}{2}$ $\qquad 2\cos\theta=-1$

$$\cos\theta=-\frac{1}{2}$$
$$\theta=\frac{2\pi}{3},\frac{4\pi}{3}$$

The solution set is $\left\{\frac{\pi}{2},\ \frac{2\pi}{3},\ \frac{4\pi}{3},\ \frac{3\pi}{2}\right\}$.

58. $\sin^2\theta-1=0$

$$(\sin\theta+1)(\sin\theta-1)=0$$

$\sin\theta+1=0$ or $\sin\theta-1=0$

$\sin\theta=-1$ $\qquad \sin\theta=1$

$\theta=\frac{3\pi}{2}$ $\qquad \theta=\frac{\pi}{2}$

The solution set is $\left\{\frac{\pi}{2},\ \frac{3\pi}{2}\right\}$.

59. $2\sin^2\theta-\sin\theta-1=0$

$$(2\sin\theta+1)(\sin\theta-1)=0$$

$2\sin\theta+1=0$ or $\sin\theta-1=0$

$2\sin\theta=-1$ $\qquad \sin\theta=1$

$\sin\theta=-\frac{1}{2}$ $\qquad \theta=\frac{\pi}{2}$

$$\theta=\frac{7\pi}{6},\frac{11\pi}{6}$$

The solution set is $\left\{\frac{\pi}{2},\ \frac{7\pi}{6},\ \frac{11\pi}{6}\right\}$.

60. $2\cos^2\theta+\cos\theta-1=0$

$$(\cos\theta+1)(2\cos\theta-1)=0$$

$\cos\theta+1=0$ or $2\cos\theta-1=0$

$\cos\theta=-1$ $\qquad 2\cos\theta=1$

$\theta=\pi$ $\qquad \cos\theta=\frac{1}{2}$

$$\theta=\frac{\pi}{3},\frac{5\pi}{3}$$

The solution set is $\left\{\frac{\pi}{3},\ \pi,\ \frac{5\pi}{3}\right\}$.

61. $(\tan\theta-1)(\sec\theta-1)=0$

$\tan\theta-1=0$ or $\sec\theta-1=0$

$\tan\theta=1$ $\qquad \sec\theta=1$

$\theta=\frac{\pi}{4},\frac{5\pi}{4}$ $\qquad \theta=0$

The solution set is $\left\{0,\ \frac{\pi}{4},\ \frac{5\pi}{4}\right\}$.

62. $(\cot\theta+1)\left(\csc\theta-\frac{1}{2}\right)=0$

$\cot\theta+1=0$ or $\csc\theta-\frac{1}{2}=0$

$\cot\theta=-1$ $\qquad \csc\theta=\frac{1}{2}$

$\theta=\frac{3\pi}{4},\frac{7\pi}{4}$ $\qquad$ (not possible)

The solution set is $\left\{\frac{3\pi}{4},\ \frac{7\pi}{4}\right\}$.

63. $\sin^2\theta-\cos^2\theta=1+\cos\theta$

$$\left(1-\cos^2\theta\right)-\cos^2\theta=1+\cos\theta$$
$$1-2\cos^2\theta=1+\cos\theta$$
$$2\cos^2\theta+\cos\theta=0$$
$$(\cos\theta)(2\cos\theta+1)=0$$

$\cos\theta=0$ or $2\cos\theta+1=0$

$\theta=\frac{\pi}{2},\frac{3\pi}{2}$ $\qquad \cos\theta=-\frac{1}{2}$

$$\theta=\frac{2\pi}{3},\frac{4\pi}{3}$$

The solution set is $\left\{\frac{\pi}{2},\ \frac{2\pi}{3},\ \frac{4\pi}{3},\ \frac{3\pi}{2}\right\}$.

64.
$$\cos^2\theta-\sin^2\theta+\sin\theta=0$$
$$\left(1-\sin^2\theta\right)-\sin^2\theta+\sin\theta=0$$
$$1-2\sin^2\theta+\sin\theta=0$$
$$2\sin^2\theta-\sin\theta-1=0$$
$$\left(2\sin\theta+1\right)\left(\sin\theta-1\right)=0$$
$$2\sin\theta+1=0 \quad \text{or} \quad \sin\theta-1=0$$
$$\sin\theta=-\frac{1}{2} \qquad \sin\theta=1$$
$$\theta=\frac{7\pi}{6},\frac{11\pi}{6} \qquad \theta=\frac{\pi}{2}$$
The solution set is $\left\{\frac{\pi}{2},\ \frac{7\pi}{6},\ \frac{11\pi}{6}\right\}$.

65.
$$\sin^2\theta=6\left(\cos\left(-\theta\right)+1\right)$$
$$\sin^2\theta=6\left(\cos\left(\theta\right)+1\right)$$
$$1-\cos^2\theta=6\cos\theta+6$$
$$\cos^2\theta+6\cos\theta+5=0$$
$$\left(\cos\theta+5\right)\left(\cos\theta+1\right)=0$$
$$\cos\theta+5=0 \quad \text{or} \quad \cos\theta+1=0$$
$$\cos\theta=-5 \qquad \cos\theta=-1$$
$$\text{(not possible)} \qquad \theta=\pi$$
The solution set is $\{\pi\}$.

66.
$$2\sin^2\theta=3\left(1-\cos\left(-\theta\right)\right)$$
$$2\sin^2\theta=3\left(1-\cos\theta\right)$$
$$2\left(1-\cos^2\theta\right)=3\left(1-\cos\theta\right)$$
$$2-2\cos^2\theta=3-3\cos\theta$$
$$2\cos^2\theta-3\cos\theta+1=0$$
$$\left(2\cos\theta-1\right)\left(\cos\theta-1\right)=0$$
$$2\cos\theta-1=0 \quad \text{or} \quad \cos\theta-1=0$$
$$\cos\theta=\frac{1}{2} \qquad \cos\theta=1$$
$$\theta=\frac{\pi}{3},\frac{5\pi}{3} \qquad \theta=0$$
The solution set is $\left\{0,\ \frac{\pi}{3},\ \frac{5\pi}{3}\right\}$.

67.
$$\cos\theta=-\sin\left(-\theta\right)$$
$$\cos\theta=-\left(-\sin\theta\right)$$
$$\cos\theta=\sin\theta$$
$$\frac{\sin\theta}{\cos\theta}=1$$
$$\tan\theta=1$$
$$\theta=\frac{\pi}{4},\frac{5\pi}{4}$$
The solution set is $\left\{\frac{\pi}{4},\ \frac{5\pi}{4}\right\}$.

68.
$$\cos\theta-\sin\left(-\theta\right)=0$$
$$\cos\theta-\left(-\sin\left(\theta\right)\right)=0$$
$$\cos\theta+\sin\theta=0$$
$$\sin\theta=-\cos\theta$$
$$\frac{\sin\theta}{\cos\theta}=-1$$
$$\tan\theta=-1$$
$$\theta=\frac{3\pi}{4},\frac{7\pi}{4}$$
The solution set is $\left\{\frac{3\pi}{4},\ \frac{7\pi}{4}\right\}$.

69.
$$\tan\theta=2\sin\theta$$
$$\frac{\sin\theta}{\cos\theta}=2\sin\theta$$
$$\sin\theta=2\sin\theta\cos\theta$$
$$0=2\sin\theta\cos\theta-\sin\theta$$
$$0=\sin\theta(2\cos\theta-1)$$
$$2\cos\theta-1=0 \quad \text{or} \quad \sin\theta=0$$
$$\cos\theta=\frac{1}{2} \qquad \theta=0,\pi$$
$$\theta=\frac{\pi}{3},\frac{5\pi}{3}$$
The solution set is $\left\{0,\ \frac{\pi}{3},\ \pi,\ \frac{5\pi}{3}\right\}$.

70. $\tan\theta = \cot\theta$

$\tan\theta = \dfrac{1}{\tan\theta}$

$\tan^2\theta = 1$

$\tan\theta = \pm 1$

$\theta = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$

The solution set is $\left\{\dfrac{\pi}{4},\ \dfrac{3\pi}{4},\ \dfrac{5\pi}{4},\ \dfrac{7\pi}{4}\right\}$.

71. $1+\sin\theta = 2\cos^2\theta$

$1+\sin\theta = 2(1-\sin^2\theta)$

$1+\sin\theta = 2-2\sin^2\theta$

$2\sin^2\theta+\sin\theta-1=0$

$(2\sin\theta-1)(\sin\theta+1)=0$

$2\sin\theta-1=0$ or $\sin\theta+1=0$

$\sin\theta = \dfrac{1}{2}$ $\qquad \sin\theta=-1$

$\theta = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$ $\qquad \theta = \dfrac{3\pi}{2}$

The solution set is $\left\{\dfrac{\pi}{6},\ \dfrac{5\pi}{6},\ \dfrac{3\pi}{2}\right\}$.

72. $\sin^2\theta = 2\cos\theta+2$

$1-\cos^2\theta = 2\cos\theta+2$

$\cos^2\theta+2\cos\theta+1=0$

$(\cos\theta+1)^2=0$

$\cos\theta+1=0$

$\cos\theta=-1$

$\theta=\pi$

The solution set is $\{\pi\}$.

73. $2\sin^2\theta-5\sin\theta+3=0$

$(2\sin\theta-3)(\sin\theta+1)=0$

$2\sin\theta-3=0$ or $\sin\theta-1=0$

$\sin\theta=\dfrac{3}{2}$ (not possible) $\qquad \theta=\dfrac{\pi}{2}$

The solution set is $\left\{\dfrac{\pi}{2}\right\}$.

74. $2\cos^2\theta-7\cos\theta-4=0$

$(2\cos\theta+1)(\cos\theta-4)=0$

$2\cos\theta+1=0$ or $\cos\theta-4=0$

$\sin\theta=-\dfrac{1}{2}$ $\qquad \cos\theta=4$ (not possible)

$\theta=\dfrac{2\pi}{3}, \dfrac{4\pi}{3}$

The solution set is $\left\{\dfrac{2\pi}{3},\ \dfrac{4\pi}{3}\right\}$.

75. $3(1-\cos\theta)=\sin^2\theta$

$3-3\cos\theta=1-\cos^2\theta$

$\cos^2\theta-3\cos\theta+2=0$

$(\cos\theta-1)(\cos\theta-2)=0$

$\cos\theta-1=0$ or $\cos\theta-2=0$

$\cos\theta=1$ $\qquad \cos\theta=2$

$\theta=0$ $\qquad$ (not possible)

The solution set is $\{0\}$.

76. $4(1+\sin\theta)=\cos^2\theta$

$4+4\sin\theta=1-\sin^2\theta$

$\sin^2\theta+4\sin\theta+3=0$

$(\sin\theta+1)(\sin\theta+3)=0$

$\sin\theta+1=0$ or $\sin\theta+3=0$

$\sin\theta=-1$ $\qquad \sin\theta=-3$

$\theta=\dfrac{3\pi}{2}$ $\qquad$ (not possible)

The solution set is $\left\{\dfrac{3\pi}{2}\right\}$.

77. $\tan^2\theta=\dfrac{3}{2}\sec\theta$

$\sec^2\theta-1=\dfrac{3}{2}\sec\theta$

$2\sec^2\theta-2=3\sec\theta$

$2\sec^2\theta-3\sec\theta-2=0$

$(2\sec\theta+1)(\sec\theta-2)=0$

$2\sec\theta+1=0$ or $\sec\theta-2=0$

$\sec\theta=-\dfrac{1}{2}$ (not possible) $\qquad \sec\theta=2$

$\theta=\dfrac{\pi}{3}, \dfrac{5\pi}{3}$

The solution set is $\left\{\dfrac{\pi}{3},\ \dfrac{5\pi}{3}\right\}$.

78. $\csc^2\theta = \cot\theta + 1$

$1+\cot^2\theta = \cot\theta + 1$

$\cot^2\theta - \cot\theta = 0$

$\cot\theta(\cot\theta - 1) = 0$

$\cot\theta = 0$ or $\cot\theta = 1$

$\theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$ $\qquad \theta = \dfrac{\pi}{4}, \dfrac{5\pi}{4}$

The solution set is $\left\{\dfrac{\pi}{4}, \dfrac{\pi}{2}, \dfrac{5\pi}{4}, \dfrac{3\pi}{2}\right\}$.

79. $\sec^2\theta + \tan\theta = 0$

$\tan^2\theta + 1 + \tan\theta = 0$

This equation is quadratic in $\tan\theta$.

The discriminant is $b^2 - 4ac = 1 - 4 = -3 < 0$.

The equation has no real solutions.

80. $\sec\theta = \tan\theta + \cot\theta$

$\dfrac{1}{\cos\theta} = \dfrac{\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{\sin\theta}$

$\dfrac{1}{\cos\theta} = \dfrac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}$

$\dfrac{1}{\cos\theta} = \dfrac{1}{\sin\theta\cos\theta}$

$\dfrac{\sin\theta\cos\theta}{\cos\theta} = 1$

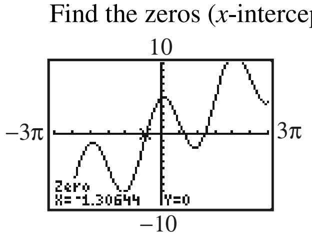
$\sin\theta = 1$

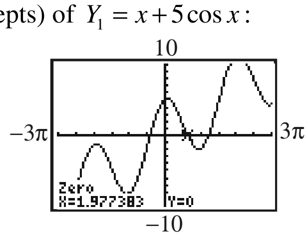
$\theta = \dfrac{\pi}{2}$

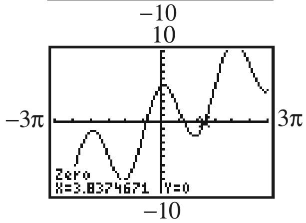
Since $\sec\left(\dfrac{\pi}{2}\right)$ and $\tan\left(\dfrac{\pi}{2}\right)$ do not exist, the equation has no real solutions.

81. $x + 5\cos x = 0$

Find the zeros (x-intercepts) of $Y_1 = x + 5\cos x$:



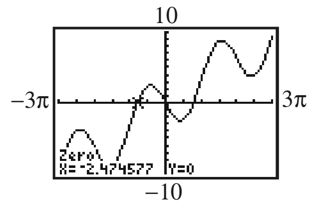


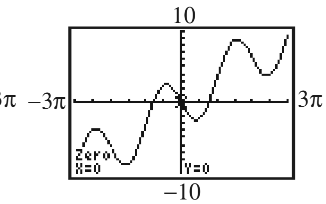


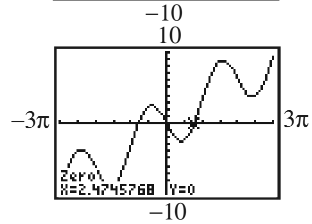
$x \approx -1.31, 1.98, 3.84$

82. $x - 4\sin x = 0$

Find the zeros (x-intercepts) of $Y_1 = x - 4\sin x$:



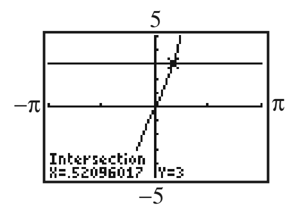




$x \approx -2.47, 0, 2.47$

83. $22x - 17\sin x = 3$

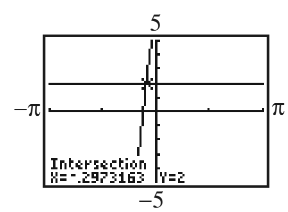
Find the intersection of $Y_1 = 22x - 17\sin x$ and $Y_2 = 3$:



$x \approx 0.52$

84. $19x + 8\cos x = 2$

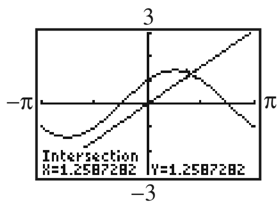
Find the intersection of $Y_1 = 19x + 8\cos x$ and $Y_2 = 2$:



$x \approx -0.30$

85. $\sin x + \cos x = x$

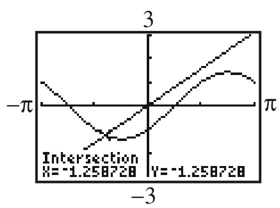
Find the intersection of $Y_1 = \sin x + \cos x$ and $Y_2 = x$:



$x \approx 1.26$

86. $\sin x - \cos x = x$

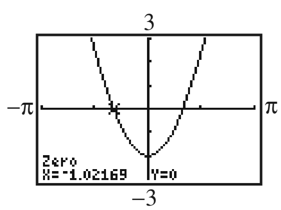
Find the intersection of $Y_1 = \sin x - \cos x$ and $Y_2 = x$:

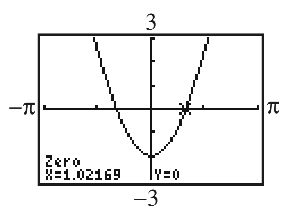


$x \approx -1.26$

87. $x^2 - 2\cos x = 0$

Find the zeros (x-intercepts) of $Y_1 = x^2 - 2\cos x$:

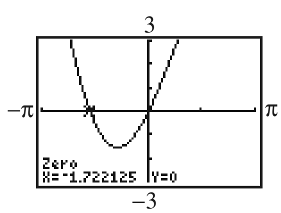


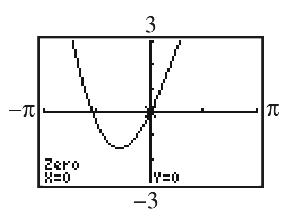


$x \approx -1.02, 1.02$

88. $x^2 + 3\sin x = 0$

Find the zeros (x-intercepts) of $Y_1 = x^2 + 3\sin x$:

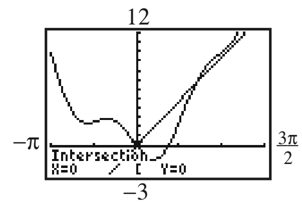


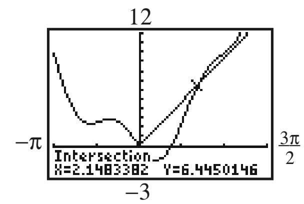


$x \approx -1.72, 0$

89. $x^2 - 2\sin(2x) = 3x$

Find the intersection of $Y_1 = x^2 - 2\sin(2x)$ and $Y_2 = 3x$:

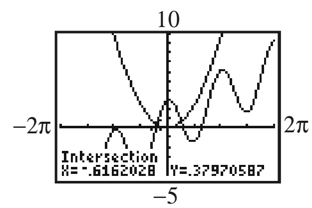


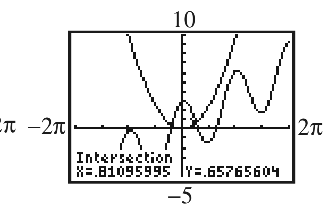


$x \approx 0, 2.15$

90. $x^2 = x + 3\cos(2x)$

Find the intersection of $Y_1 = x^2$ and $Y_2 = x + 3\cos(2x)$:

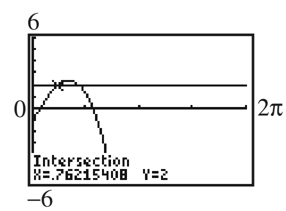


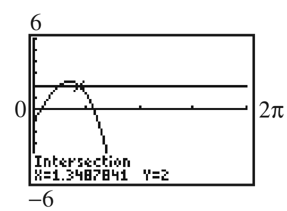


$x \approx -0.62, 0.81$

91. $6\sin x - e^x = 2, \ x > 0$

Find the intersection of $Y_1 = 6\sin x - e^x$ and $Y_2 = 2$:

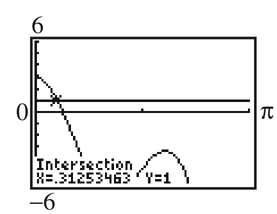




$x \approx 0.76, 1.35$

92. $4\cos(3x) - e^x = 1, \ x > 0$

Find the intersection of $Y_1 = 4\cos(3x) - e^x$ and $Y_2 = 1$:



$x \approx 0.31$

93. $f(x) = 0$

$4\sin^2 x - 3 = 0$

$4\sin^2 x = 3$

$\sin^2 x = \frac{3}{4}$

$\sin x = \pm\sqrt{\frac{3}{4}} = \pm\frac{\sqrt{3}}{2}$

$x = \frac{\pi}{3} + k\pi$ or $x = \frac{2\pi}{3} + k\pi$, k is any integer

On the interval $[0, 2\pi]$, the zeros of f are $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$.

94. $f(x) = 0$

$2\cos(3x) + 1 = 0$

$2\cos(3x) = -1$

$\cos(3x) = -\frac{1}{2}$

$3x = \frac{2\pi}{3} + 2k\pi$ or $3x = \frac{4\pi}{3} + 2k\pi$

$x = \frac{2\pi}{9} + \frac{2k\pi}{3}$ or $x = \frac{4\pi}{9} + \frac{2k\pi}{3}$,

k is any integer

On the interval $[0, \pi]$, the zeros of f are $\frac{2\pi}{9}, \frac{4\pi}{9}, \frac{8\pi}{9}$.

95. a. $f(x) = 0$

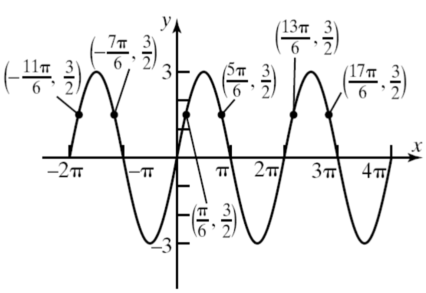
$3\sin x = 0$

$\sin x = 0$

$x = 0 + 2k\pi$ or $x = \pi + 2k\pi$, k is any integer

On the interval $[-2\pi, 4\pi]$, the zeros of f are $-2\pi, -\pi, 0, \pi, 2\pi, 3\pi, 4\pi$.

b. $f(x) = 3\sin x$



c. $f(x) = \frac{3}{2}$

$3\sin x = \frac{3}{2}$

$\sin x = \frac{1}{2}$

$x = \frac{\pi}{6} + 2k\pi$ or $x = \frac{5\pi}{6} + 2k\pi$, k is any integer

On the interval $[-2\pi, 4\pi]$, the solution set is $\left\{-\frac{11\pi}{6}, -\frac{7\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}\right\}$.

d. From the graph in part (b) and the results of part (c), the solutions of $f(x) > \frac{3}{2}$ on the interval $[-2\pi, 4\pi]$ is $\left\{x \,\middle|\, -\frac{11\pi}{6} < x < -\frac{7\pi}{6} \text{ or } \frac{\pi}{6} < x < \frac{5\pi}{6} \text{ or } \frac{13\pi}{6} < x < \frac{17\pi}{6}\right\}$.

96. a. $f(x) = 0$

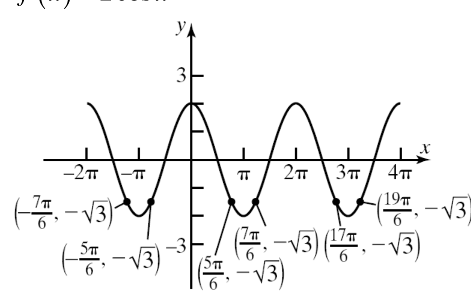
$2\cos x = 0$

$\cos x = 0$

$x = \frac{\pi}{2} + 2k\pi$ or $x = \frac{3\pi}{2} + 2k\pi$, k is any integer

On the interval $[-2\pi, 4\pi]$, the zeros of f are $-\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$.

b. $f(x) = 2\cos x$



c. $f(x) = -\sqrt{3}$

$2\cos x = -\sqrt{3}$

$\cos x = -\frac{\sqrt{3}}{2}$

$x = \frac{5\pi}{6} + 2k\pi$ or $x = \frac{7\pi}{6} + 2k\pi$, k is any integer

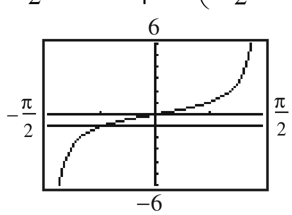
On the interval $[-2\pi, 4\pi]$, the solution set is $\left\{-\frac{7\pi}{6}, -\frac{5\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{17\pi}{6}, \frac{19\pi}{6}\right\}$.

d. From the graph in part (b) and the results of part (c), the solutions of $f(x) < -\sqrt{3}$ on the interval $[-2\pi, 4\pi]$ is $\left\{x \middle| -\frac{7\pi}{6} < x < -\frac{5\pi}{6} \right.$ or $\frac{5\pi}{6} < x < \frac{7\pi}{6}$ or $\left.\frac{17\pi}{6} < x < \frac{19\pi}{6}\right\}$.

97. $f(x) = 4\tan x$

a.
$$f(x) = -4$$
$$4\tan x = -4$$
$$\tan x = -1$$
$\left\{x \middle| x = -\frac{\pi}{4} + k\pi\right\}$, k is any integer

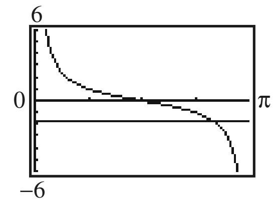
b.
$$f(x) < -4$$
$$4\tan x < -4$$
$$\tan x < -1$$
Graphing $y_1 = \tan x$ and $y_2 = -1$ on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we see that $y_1 < y_2$ for $-\frac{\pi}{2} < x < -\frac{\pi}{4}$ or $\left(-\frac{\pi}{2}, -\frac{\pi}{4}\right)$.



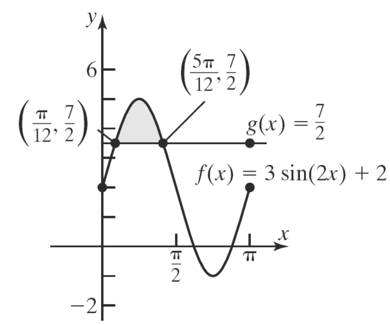
98. $f(x) = \cot x$

a.
$$f(x) = -\sqrt{3}$$
$$\cot x = -\sqrt{3}$$
$\left\{x \middle| x = \frac{5\pi}{6} + k\pi\right\}$, k is any integer

b.
$$f(x) > -\sqrt{3}$$
$$\cot x > -\sqrt{3}$$
Graphing $y_1 = \frac{1}{\tan x}$ and $y_2 = -\sqrt{3}$ on the interval $(0, \pi)$, we see that $y_1 > y_2$ for $0 < x < \frac{5\pi}{6}$ or $\left(0, \frac{5\pi}{6}\right)$.



99. a, d. $f(x) = 3\sin(2x) + 2$; $g(x) = \frac{7}{2}$

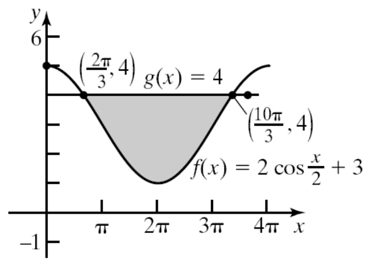


b.
$$f(x) = g(x)$$
$$3\sin(2x) + 2 = \frac{7}{2}$$
$$3\sin(2x) = \frac{3}{2}$$
$$\sin(2x) = \frac{1}{2}$$
$$2x = \frac{\pi}{6} + 2k\pi \text{ or } 2x = \frac{5\pi}{6} + 2k\pi$$
$$x = \frac{\pi}{12} + k\pi \text{ or } x = \frac{5\pi}{12} + k\pi,$$
k is any integer

On $[0, \pi]$, the solution set is $\left\{\frac{\pi}{12}, \frac{5\pi}{12}\right\}$.

c. From the graph in part (a) and the results of part (b), the solution of $f(x) > g(x)$ on $[0, \pi]$ is $\left\{x \middle| \frac{\pi}{12} < x < \frac{5\pi}{12}\right\}$ or $\left(\frac{\pi}{12}, \frac{5\pi}{12}\right)$.

100. a, d. $f(x)=2\cos\frac{x}{2}+3$; $g(x)=4$



b. $f(x)=g(x)$

$2\cos\frac{x}{2}+3=4$

$2\cos\frac{x}{2}=1$

$\cos\frac{x}{2}=\frac{1}{2}$

$\frac{x}{2}=\frac{\pi}{3}+2k\pi$ or $\frac{x}{2}=\frac{5\pi}{3}+2k\pi$

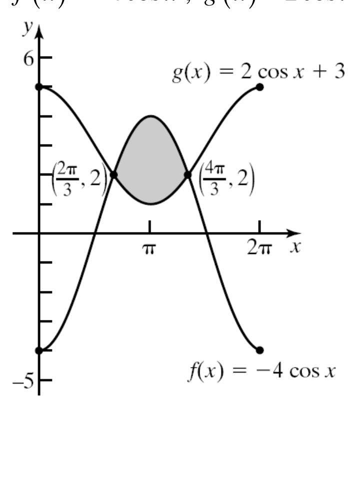
$x=\frac{2\pi}{3}+4k\pi$ or $x=\frac{10\pi}{3}+4k\pi$,

k is any integer

On $[0,4\pi]$, the solution set is $\left\{\frac{2\pi}{3},\frac{10\pi}{3}\right\}$.

c. From the graph in part (a) and the results of part (b), the solution of $f(x)<g(x)$ on $[0,4\pi]$ is $\left\{x\,\middle|\,\frac{2\pi}{3}<x<\frac{10\pi}{3}\right\}$ or $\left(\frac{2\pi}{3},\frac{10\pi}{3}\right)$.

101. a, d. $f(x)=-4\cos x$; $g(x)=2\cos x+3$



b. $f(x)=g(x)$

$-4\cos x=2\cos x+3$

$-6\cos x=3$

$\cos x=\frac{3}{-6}=-\frac{1}{2}$

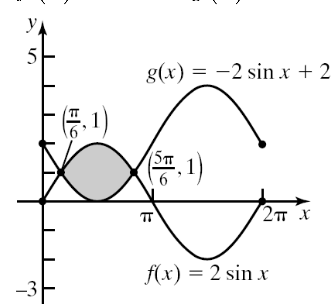
$x=\frac{2\pi}{3}+2k\pi$ or $x=\frac{4\pi}{3}+2k\pi$,

k is any integer

On $[0,2\pi]$, the solution set is $\left\{\frac{2\pi}{3},\frac{4\pi}{3}\right\}$.

c. From the graph in part (a) and the results of part (b), the solution of $f(x)>g(x)$ on $[0,2\pi]$ is $\left\{x\,\middle|\,\frac{2\pi}{3}<x<\frac{4\pi}{3}\right\}$ or $\left(\frac{2\pi}{3},\frac{4\pi}{3}\right)$.

102. a, d. $f(x)=2\sin x$; $g(x)=-2\sin x+2$



b. $f(x)=g(x)$

$2\sin x=-2\sin x+2$

$4\sin x=2$

$\sin x=\frac{2}{4}=\frac{1}{2}$

$x=\frac{\pi}{6}+2k\pi$ or $x=\frac{5\pi}{6}+2k\pi$,

k is any integer

On $[0,2\pi]$, the solution set is $\left\{\frac{\pi}{6},\frac{5\pi}{6}\right\}$.

c. From the graph in part (a) and the results of part (b), the solution of $f(x)>g(x)$ on $[0,2\pi]$ is $\left\{x\,\middle|\,\frac{\pi}{6}<x<\frac{5\pi}{6}\right\}$ or $\left(\frac{\pi}{6},\frac{5\pi}{6}\right)$.

103. $P(t)=100+20\sin\left(\frac{7\pi}{3}t\right)$

a. Solve $P(t)=100$ on the interval $[0,1]$.

$$100+20\sin\left(\frac{7\pi}{3}t\right)=100$$
$$20\sin\left(\frac{7\pi}{3}t\right)=0$$
$$\sin\left(\frac{7\pi}{3}t\right)=0$$
$$\frac{7\pi}{3}t=k\pi,\ k \text{ is any integer}$$
$$t=\frac{3}{7}k,\ k \text{ is any integer}$$

We need $0\le\frac{3}{7}k\le1$, or $0\le k\le\frac{7}{3}$.

For $k=0$, $t=0$ sec.

For $k=1$, $t=\frac{3}{7}\approx0.43$ sec.

For $k=2$, $t=\frac{6}{7}\approx0.86$ sec.

The blood pressure will be 100 mmHg after 0 seconds, 0.43 seconds, and 0.86 seconds.

b. Solve $P(t)=120$ on the interval $[0,1]$.

$$100+20\sin\left(\frac{7\pi}{3}t\right)=120$$
$$20\sin\left(\frac{7\pi}{3}t\right)=20$$
$$\sin\left(\frac{7\pi}{3}t\right)=1$$
$$\frac{7\pi}{3}t=2\pi k+\frac{\pi}{2},\ k \text{ is any integer}$$
$$t=\frac{3\left(2k+\frac{1}{2}\right)}{7},\ k \text{ is any integer}$$

We need

$$0\le\frac{3\left(2k+\frac{1}{2}\right)}{7}\le1$$
$$0\le2k+\frac{1}{2}\le\frac{7}{3}$$
$$-\frac{1}{2}\le2k\le\frac{11}{6}$$
$$-\frac{1}{4}\le k\le\frac{11}{12}$$

For $k=0$, $t=\frac{3}{14}\approx0.21$ sec

The blood pressure will be 120mmHg after 0.21 sec .

c. Solve $P(t)=105$ on the interval $[0,1]$.

$$100+20\sin\left(\frac{7\pi}{3}t\right)=105$$
$$20\sin\left(\frac{7\pi}{3}t\right)=5$$
$$\sin\left(\frac{7\pi}{3}t\right)=\frac{3}{4}$$
$$\frac{7\pi}{3}t=\sin^{-1}\left(\frac{3}{4}\right)$$
$$t=\frac{3}{7\pi}\sin^{-1}\left(\frac{3}{4}\right)$$

On the interval $[0,1]$, we get $t\approx0.03$ seconds, $t\approx0.39$ seconds, and $t\approx0.89$ seconds. Using this information, along with the results from part (a), the blood pressure will be between 100 mmHg and 105 mmHg for values of *t* (in seconds) in the interval $[0,0.03]\cup[0.39,0.43]\cup[0.86,0.89]$.

104. $h(t)=125\sin\left(0.157t-\frac{\pi}{2}\right)+125$

a. Solve $h(t)=125\sin\left(0.157t-\frac{\pi}{2}\right)+125=125$ on the interval $[0,40]$.

$$125\sin\left(0.157t-\frac{\pi}{2}\right)+125=125$$
$$125\sin\left(0.157t-\frac{\pi}{2}\right)=0$$
$$\sin\left(0.157t-\frac{\pi}{2}\right)=0$$
$$0.157t-\frac{\pi}{2}=k\pi,\ k \text{ is any integer}$$
$$0.157t=k\pi+\frac{\pi}{2},\ k \text{ is any integer}$$
$$t=\frac{k\pi+\frac{\pi}{2}}{0.157},\ k \text{ is any integer}$$

For $k=0,\ t=\frac{0+\frac{\pi}{2}}{0.157}\approx10$ seconds .

For $k=1,\ t=\frac{\pi+\frac{\pi}{2}}{0.157}\approx30$ seconds .

For $k=2,\ t=\dfrac{2\pi+\frac{\pi}{2}}{0.157}\approx 50$ seconds.

So during the first 40 seconds, an individual on the Ferris Wheel is exactly 125 feet above the ground when $t\approx 10$ seconds and again when $t\approx 30$ seconds.

b. Solve $h(t)=125\sin\left(0.157t-\dfrac{\pi}{2}\right)+125=250$ on the interval $[0,80]$.

$$125\sin\left(0.157t-\frac{\pi}{2}\right)+125=250$$
$$125\sin\left(0.157t-\frac{\pi}{2}\right)=125$$
$$\sin\left(0.157t-\frac{\pi}{2}\right)=1$$
$$0.157t-\frac{\pi}{2}=\frac{\pi}{2}+2k\pi,\quad k \text{ is any integer}$$
$$0.157t=\pi+2k\pi,\quad k \text{ is any integer}$$
$$t=\frac{\pi+2k\pi}{0.157},\quad k \text{ is any integer}$$

For $k=0,\ t=\dfrac{\pi}{0.157}\approx 20$ seconds.

For $k=1,\ t=\dfrac{\pi+2\pi}{0.157}\approx 60$ seconds.

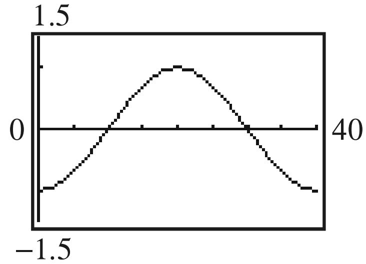
For $k=2,\ t=\dfrac{\pi+4\pi}{0.157}\approx 100$ seconds.

So during the first 80 seconds, an individual on the Ferris Wheel is exactly 250 feet above the ground when $t\approx 20$ seconds and again when $t\approx 60$ seconds.

c. Solve $h(t)=125\sin\left(0.157t-\dfrac{\pi}{2}\right)+125>125$ on the interval $[0,40]$.

$$125\sin\left(0.157t-\frac{\pi}{2}\right)+125>125$$
$$125\sin\left(0.157t-\frac{\pi}{2}\right)>0$$
$$\sin\left(0.157t-\frac{\pi}{2}\right)>0$$

Graphing $y_1=\sin\left(0.157x-\dfrac{\pi}{2}\right)$ and $y_2=0$ on the interval $[0,40]$, we see that $y_1>y_2$ for $10<x<30$.



So during the first 40 seconds, an individual on the Ferris Wheel is more than 125 feet above the ground for times between about 10 and 30 seconds. That is, on the interval $10<x<30$, or $(10,30)$.

105. $d(x)=70\sin(0.65x)+150$

a.
$$d(0)=70\sin(0.65(0))+150$$
$$=70\sin(0)+150$$
$$=150 \text{ miles}$$

b. Solve $d(x)=70\sin(0.65x)+150=100$ on the interval $[0,20]$.

$$70\sin(0.65x)+150=100$$
$$70\sin(0.65x)=-50$$
$$\sin(0.65x)=-\frac{5}{7}$$
$$0.65x=\sin^{-1}\left(-\frac{5}{7}\right)+2\pi k$$
$$x=\frac{\sin^{-1}\left(-\frac{5}{7}\right)+2\pi k}{0.65}$$
$$x\approx\frac{3.94+2\pi k}{0.65}\ \text{ or }\ x\approx\frac{5.94+2\pi k}{0.65},$$

k is any integer

For $k=0$, $x\approx\dfrac{3.94+0}{0.65}\approx 6.06$ min or $x\approx\dfrac{5.94+0}{0.65}\approx 8.44$ min

For $k=1$, $x\approx\dfrac{3.94+2\pi}{0.65}\approx 15.72$ min or $x\approx\dfrac{5.94+2\pi}{0.65}\approx 18.11$ min

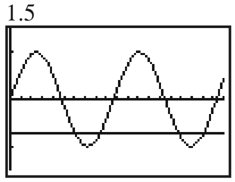
For $k=2$, $x\approx\dfrac{3.94+4\pi}{0.65}\approx 25.39$ min or $x\approx\dfrac{5.94+4\pi}{0.65}\approx 27.78$ min

So during the first 20 minutes in the holding pattern, the plane is exactly 100 miles from the airport when $x \approx 6.06$ minutes, $x \approx 8.44$ minutes, $x \approx 15.72$ minutes, and $x \approx 18.11$ minutes.

c. Solve $d(x) = 70\sin(0.65x) + 150 > 100$ on the interval $[0, 20]$.

$$70\sin(0.65x) + 150 > 100$$
$$70\sin(0.65x) > -50$$
$$\sin(0.65x) > -\frac{5}{7}$$

Graphing $y_1 = \sin(0.65x)$ and $y_2 = -\frac{5}{7}$ on the interval $[0, 20]$, we see that $y_1 > y_2$ for $0 < x < 6.06$, $8.44 < x < 15.72$, and $18.11 < x < 20$.



So during the first 20 minutes in the holding pattern, the plane is more than 100 miles from the airport before 6.06 minutes, between 8.44 and 15.72 minutes, and after 18.11 minutes.

d. No, the plane is never within 70 miles of the airport while in the holding pattern. The minimum value of $\sin(0.65x)$ is -1. Thus, the least distance that the plane is from the airport is $70(-1) + 150 = 80$ miles.

106. $R(\theta) = 672\sin(2\theta)$

a. Solve $R(\theta) = 672\sin(2\theta) = 450$ on the interval $\left[0, \frac{\pi}{2}\right)$.

$$672\sin(2\theta) = 450$$
$$\sin(2\theta) = \frac{450}{672} = \frac{225}{336}$$
$$2\theta = \sin^{-1}\left(\frac{225}{336}\right) + 2k\pi$$
$$\theta = \frac{\sin^{-1}\left(\frac{225}{336}\right) + 2k\pi}{2}$$
$$\theta \approx \frac{0.7337 + 2k\pi}{2} \text{ or } \theta \approx \frac{2.408 + 2k\pi}{2},$$

k is any integer

For $k = 0$, $\theta = \frac{0.7337 + 0}{2} \approx 0.36685 \approx 21.02°$ or $\theta = \frac{2.408 + 0}{2} \approx 1.204 \approx 68.98°$

For $k = 1$, $\theta = \frac{0.7337 + 2\pi}{2} \approx 3.508 \approx 200.99°$ or $\theta = \frac{2.408 + 2\pi}{2} \approx 4.3456 \approx 248.98°$

So the golfer should hit the ball at an angle of either $21.02°$ or $68.98°$.

b. Solve $R(\theta) = 672\sin(2\theta) = 540$ on the interval $\left[0, \frac{\pi}{2}\right)$.

$$672\sin(2\theta) = 540$$
$$\sin(2\theta) = \frac{540}{672} = \frac{135}{168}$$
$$2\theta = \sin^{-1}\left(\frac{135}{168}\right) + 2k\pi$$
$$\theta = \frac{\sin^{-1}\left(\frac{135}{168}\right) + 2k\pi}{2}$$
$$\theta \approx \frac{0.9333 + 2k\pi}{2} \text{ or } \theta \approx \frac{2.2083 + 2k\pi}{2},$$

k is any integer

For $k = 0$, $\theta = \frac{0.9330 + 0}{2} \approx 0.46665 \approx 26.74°$ or $\theta = \frac{2.2083 + 0}{2} \approx 1.10415 \approx 63.26°$

For $k=1$, $\theta=\dfrac{0.9330+2\pi}{2}$ or $\theta=\dfrac{2.2083+2\pi}{2}$

≈ 3.608 $\quad\approx 4.246$

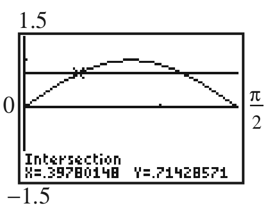
$\approx 206.72°$ $\quad\approx 243.28°$

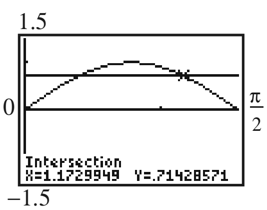
So the golfer should hit the ball at an angle of either $26.74°$ or $63.26°$.

c. Solve $R(\theta)=672\sin(2\theta)\ge 480$ on the interval $\left[0,\dfrac{\pi}{2}\right)$.

$$672\sin(2\theta)\ge 480$$
$$\sin(2\theta)\ge\frac{480}{672}$$
$$\sin(2\theta)\ge\frac{5}{7}$$

Graphing $y_1=\sin(2x)$ and $y_2=\dfrac{5}{7}$ on the interval $\left[0,\dfrac{\pi}{2}\right)$ and using INTERSECT, we see that $y_1\ge y_2$ when $0.3978\le x\le 1.1730$ radians, or $22.79°\le x\le 67.21°$.

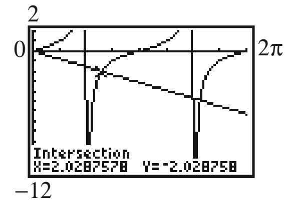


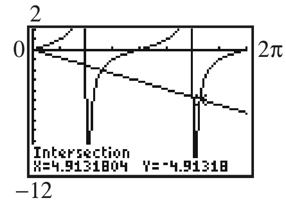


So, the golf ball will travel at least 480 feet if the angle is between about $22.79°$ and $67.21°$.

d. No; since the maximum value of the sine function is 1, the farthest the golfer can hit the ball is $672(1)=672$ feet.

107. Find the first two positive intersection points of $Y_1=-x$ and $Y_2=\tan x$.





The first two positive solutions are $x\approx 2.03$ and $x\approx 4.91$.

108. a. Let L be the length of the ladder with *x* and *y* being the lengths of the two parts in each hallway.

$$L=x+y$$

$$\cos\theta=\frac{3}{x} \qquad \sin\theta=\frac{4}{y}$$

$$x=\frac{3}{\cos\theta} \qquad y=\frac{4}{\sin\theta}$$

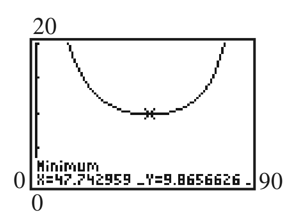
$$L(\theta)=\frac{3}{\cos\theta}+\frac{4}{\sin\theta}=3\sec\theta+4\csc\theta$$

$$3\sec\theta\tan\theta-4\csc\theta\cot\theta=0$$
$$3\sec\theta\tan\theta=4\csc\theta\cot\theta$$
$$\frac{\sec\theta\tan\theta}{\csc\theta\cot\theta}=\frac{4}{3}$$
$$\tan^3\theta=\frac{4}{3}$$
$$\tan\theta=\sqrt[3]{\frac{4}{3}}\approx 1.10064$$
$$\theta\approx 47.74°$$

b. $L(47.74°)=\dfrac{3}{\cos(47.74°)}+\dfrac{4}{\sin(47.74°)}$

≈ 9.87 feet

c. Graph $Y_1=\dfrac{3}{\cos x}+\dfrac{4}{\sin x}$ and use the MINIMUM feature:



An angle of $\theta\approx 47.74°$ minimizes the length at $L\approx 9.87$ feet.

d. For this problem, only one minimum length exists. This minimum length is 9.87 feet, and it occurs when $\theta \approx 47.74°$. No matter if we find the minimum algebraically (using calculus) or graphically, the minimum will be the same.

109. a. $$107 = \frac{(34.8)^2 \sin(2\theta)}{9.8}$$

$$\sin(2\theta) = \frac{107(9.8)}{(34.8)^2} \approx 0.8659$$

$$2\theta \approx \sin^{-1}(0.8659)$$

$$2\theta \approx 60° \text{ or } 120°$$

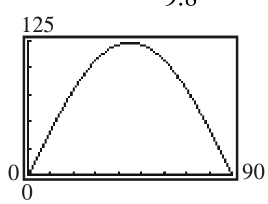
$$\theta \approx 30° \text{ or } 60°$$

b. Notice that the answers to part (a) add up to $90°$. The maximum distance will occur when the angle of elevation is $90° \div 2 = 45°$:

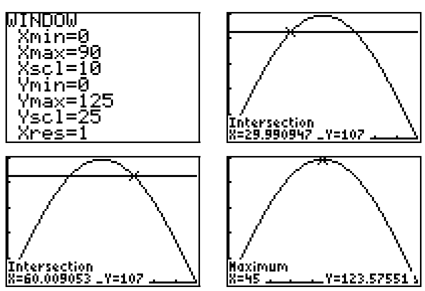
$$R(45°) = \frac{(34.8)^2 \sin\left[2(45°)\right]}{9.8} \approx 123.6$$

The maximum distance is 123.6 meters.

c. Let $Y_1 = \dfrac{(34.8)^2 \sin(2x)}{9.8}$



d.



110. a. $$110 = \frac{(40)^2 \sin(2\theta)}{9.8}$$

$$\sin(2\theta) = \frac{110 \cdot 9.8}{40^2} \approx 0.67375$$

$$2\theta \approx \sin^{-1}(0.67375)$$

$$2\theta \approx 42.4° \text{ or } 137.6°$$

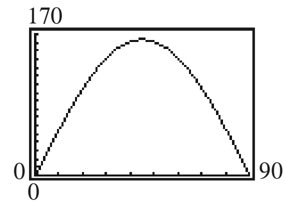
$$\theta \approx 21.2° \text{ or } 68.8°$$

b. The maximum distance will occur when the angle of elevation is $45°$:

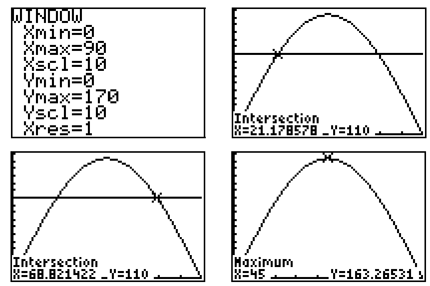
$$R(45°) = \frac{(40)^2 \sin\left[2(45°)\right]}{9.8} \approx 163.3$$

The maximum distance is approximately 163.3 meter

c. Let $Y_1 = \dfrac{(40)^2 \sin(2x)}{9.8}$:



d.



111. $$\frac{\sin 40°}{\sin \theta_2} = 1.33$$

$$1.33 \sin\theta_2 = \sin 40°$$

$$\sin\theta_2 = \frac{\sin 40°}{1.33} \approx 0.4833$$

$$\theta_2 = \sin^{-1}(0.4833) \approx 28.90°$$

112. $$\frac{\sin 50°}{\sin \theta_2} = 1.66$$

$$1.66 \sin\theta_2 = \sin 50°$$

$$\sin\theta_2 = \frac{\sin 50°}{1.66} \approx 0.4615$$

$$\theta_2 = \sin^{-1}(0.4615) \approx 27.48°$$

113. Calculate the index of refraction for each:

θ_1	θ_2	$\frac{v_1}{v_2} = \frac{\sin\theta_1}{\sin\theta_2}$
10°	8°	$\frac{\sin 10°}{\sin 8°} \approx 1.2477$
20°	$15°30' = 15.5°$	$\frac{\sin 20°}{\sin 15.5°} \approx 1.2798$
30°	$22°30' = 22.5°$	$\frac{\sin 30°}{\sin 22.5°} \approx 1.3066$
40°	$29°0' = 29°$	$\frac{\sin 40°}{\sin 29°} \approx 1.3259$
50°	$35°0' = 35°$	$\frac{\sin 50°}{\sin 35°} \approx 1.3356$
60°	$40°30' = 40.5°$	$\frac{\sin 60°}{\sin 40.5°} \approx 1.3335$
70°	$45°30' = 45.5°$	$\frac{\sin 70°}{\sin 45.5°} \approx 1.3175$
80°	$50°0' = 50°$	$\frac{\sin 80°}{\sin 50°} \approx 1.2856$

Yes, these data values agree with Snell's Law. The results vary from about 1.25 to 1.34.

114. $\frac{v_1}{v_2} = \frac{2.998\times10^8}{1.92\times10^8} \approx 1.56$

The index of refraction for this liquid is about 1.56.

115. Calculate the index of refraction:

$\theta_1 = 40°, \theta_2 = 26°;\ \frac{\sin\theta_1}{\sin\theta_2} = \frac{\sin 40°}{\sin 26°} \approx 1.47$

116. The index of refraction of crown glass is 1.52.

$$\frac{\sin 30°}{\sin\theta_2} \approx 1.52$$

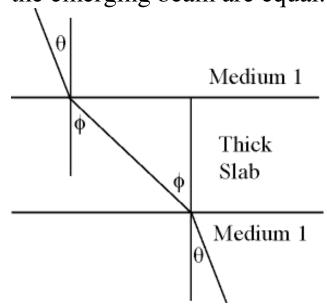
$$1.52\sin\theta_2 = \sin 30°$$

$$\sin\theta_2 = \frac{\sin 30°}{1.52} \approx 0.3289$$

$$\theta_2 \approx \sin^{-1}(0.3352) \approx 19.20°$$

The angle of refraction is about $19.20°$.

117. If θ is the original angle of incidence and ϕ is the angle of refraction, then $\frac{\sin\theta}{\sin\phi} = n_2$. The angle of incidence of the emerging beam is also ϕ, and the index of refraction is $\frac{1}{n_2}$. Thus, θ is the angle of refraction of the emerging beam. The two beams are parallel since the original angle of incidence and the angle of refraction of the emerging beam are equal.



118. Here we have $n_1 = 1.33$ and $n_2 = 1.52$.

$$n_1 \sin\theta_B = n_2 \cos_B$$

$$\frac{\sin\theta_B}{\cos\theta_B} = \frac{n_2}{n_1}$$

$$\tan\theta_B = \frac{n_2}{n_1}$$

$$\theta_B = \tan^{-1}\frac{n_2}{n_1} = \tan^{-1}\left(\frac{1.52}{1.33}\right) \approx 48.8°$$

119. Answers will vary.

120. Since the range of $y = \sin x$ is $-1 \le y \le 1$, then $y = 5\sin x + x$ cannot be equal to 3 when $x > 4\pi$ or $x < -\pi$ since you are multiplying the result by 5 and adding x.

Section 7.4

1. True
2. True
3. identity; conditional
4. -1
5. 0
6. True
7. False, you need to work with one side only.
8. True
9. $\tan\theta\cdot\csc\theta = \frac{\sin\theta}{\cos\theta}\cdot\frac{1}{\sin\theta} = \frac{1}{\cos\theta}$
10. $\cot\theta\cdot\sec\theta = \frac{\cos\theta}{\sin\theta}\cdot\frac{1}{\cos\theta} = \frac{1}{\sin\theta}$

11. $\dfrac{\cos\theta}{1-\sin\theta}\cdot\dfrac{1+\sin\theta}{1+\sin\theta}=\dfrac{\cos\theta(1+\sin\theta)}{1-\sin^2\theta}$
$=\dfrac{\cos\theta(1+\sin\theta)}{\cos^2\theta}$
$=\dfrac{1+\sin\theta}{\cos\theta}$

12. $\dfrac{\sin\theta}{1+\cos\theta}\cdot\dfrac{1-\cos\theta}{1-\cos\theta}=\dfrac{\sin\theta(1-\cos\theta)}{1-\cos^2\theta}$
$=\dfrac{\sin\theta(1-\cos\theta)}{\sin^2\theta}$
$=\dfrac{1-\cos\theta}{\sin\theta}$

13. $\dfrac{\sin\theta+\cos\theta}{\cos\theta}+\dfrac{\cos\theta-\sin\theta}{\sin\theta}$
$=\dfrac{\sin^2\theta+\sin\theta\cos\theta+\cos\theta(\cos\theta-\sin\theta)}{\sin\theta\cos\theta}$
$=\dfrac{\sin^2\theta+\sin\theta\cos\theta+\cos^2\theta-\cos\theta\sin\theta}{\sin\theta\cos\theta}$
$=\dfrac{\sin^2\theta+\cos^2\theta+\sin\theta\cos\theta-\cos\theta\sin\theta}{\sin\theta\cos\theta}$
$=\dfrac{1}{\sin\theta\cos\theta}$

14. $\dfrac{1}{1-\cos v}+\dfrac{1}{1+\cos v}=\dfrac{1+\cos v+1-\cos v}{(1-\cos v)(1+\cos v)}$
$=\dfrac{2}{1-\cos^2 v}$
$=\dfrac{2}{\sin^2 v}$

15. $\dfrac{(\sin\theta+\cos\theta)(\sin\theta+\cos\theta)-1}{\sin\theta\cos\theta}$
$=\dfrac{\sin^2\theta+2\sin\theta\cos\theta+\cos^2\theta-1}{\sin\theta\cos\theta}$
$=\dfrac{\sin^2\theta+\cos^2\theta+2\sin\theta\cos\theta-1}{\sin\theta\cos\theta}$
$=\dfrac{1+2\sin\theta\cos\theta-1}{\sin\theta\cos\theta}$
$=\dfrac{2\sin\theta\cos\theta}{\sin\theta\cos\theta}$
$=2$

16. $\dfrac{(\tan\theta+1)(\tan\theta+1)-\sec^2\theta}{\tan\theta}$
$=\dfrac{\tan^2\theta+2\tan\theta+1-\sec^2\theta}{\tan\theta}$
$=\dfrac{\tan^2\theta+1+2\tan\theta-\sec^2\theta}{\tan\theta}$
$=\dfrac{\sec^2\theta+2\tan\theta-\sec^2\theta}{\tan\theta}$
$=\dfrac{2\tan\theta}{\tan\theta}$
$=2$

17. $\dfrac{3\sin^2\theta+4\sin\theta+1}{\sin^2\theta+2\sin\theta+1}=\dfrac{(3\sin\theta+1)(\sin\theta+1)}{(\sin\theta+1)(\sin\theta+1)}$
$=\dfrac{3\sin\theta+1}{\sin\theta+1}$

18. $\dfrac{\cos^2\theta-1}{\cos^2\theta-\cos\theta}=\dfrac{(\cos\theta+1)(\cos\theta-1)}{\cos\theta(\cos\theta-1)}$
$=\dfrac{\cos\theta+1}{\cos\theta}$

19. $\csc\theta\cdot\cos\theta=\dfrac{1}{\sin\theta}\cdot\cos\theta=\dfrac{\cos\theta}{\sin\theta}=\cot\theta$

20. $\sec\theta\cdot\sin\theta=\dfrac{1}{\cos\theta}\cdot\sin\theta=\dfrac{\sin\theta}{\cos\theta}=\tan\theta$

21. $1+\tan^2(-\theta)=1+(-\tan\theta)^2=1+\tan^2\theta=\sec^2\theta$

22. $1+\cot^2(-\theta)=1+(-\cot\theta)^2=1+\cot^2\theta=\csc^2\theta$

23. $\cos\theta(\tan\theta+\cot\theta)=\cos\theta\left(\dfrac{\sin\theta}{\cos\theta}+\dfrac{\cos\theta}{\sin\theta}\right)$
$=\cos\theta\left(\dfrac{\sin^2\theta+\cos^2\theta}{\cos\theta\sin\theta}\right)$
$=\cos\theta\left(\dfrac{1}{\cos\theta\sin\theta}\right)$
$=\dfrac{1}{\sin\theta}$
$=\csc\theta$

24. $\sin\theta(\cot\theta+\tan\theta)=\sin\theta\left(\dfrac{\cos\theta}{\sin\theta}+\dfrac{\sin\theta}{\cos\theta}\right)$

$=\sin\theta\left(\dfrac{\cos^2\theta+\sin^2\theta}{\sin\theta\cos\theta}\right)$

$=\sin\theta\left(\dfrac{1}{\sin\theta\cos\theta}\right)$

$=\dfrac{1}{\cos\theta}$

$=\sec\theta$

25. $\tan u\cot u-\cos^2 u=\tan u\cdot\dfrac{1}{\tan u}-\cos^2 u$

$=1-\cos^2 u$

$=\sin^2 u$

26. $\sin u\csc u-\cos^2 u=\sin u\cdot\dfrac{1}{\sin u}-\cos^2 u$

$=1-\cos^2 u$

$=\sin^2 u$

27. $(\sec\theta-1)(\sec\theta+1)=\sec^2\theta-1=\tan^2\theta$

28. $(\csc\theta-1)(\csc\theta+1)=\csc^2\theta-1=\cot^2\theta$

29. $(\sec\theta+\tan\theta)(\sec\theta-\tan\theta)=\sec^2\theta-\tan^2\theta=1$

30. $(\csc\theta+\cot\theta)(\csc\theta-\cot\theta)=\csc^2\theta-\cot^2\theta=1$

31. $\cos^2\theta(1+\tan^2\theta)=\cos^2\theta\cdot\sec^2\theta$

$=\cos^2\theta\cdot\dfrac{1}{\cos^2\theta}$

$=1$

32. $(1-\cos^2\theta)(1+\cot^2\theta)=\sin^2\theta\cdot\csc^2\theta$

$=\sin^2\theta\cdot\dfrac{1}{\sin^2\theta}$

$=1$

33. $(\sin\theta+\cos\theta)^2+(\sin\theta-\cos\theta)^2$

$=\sin^2\theta+2\sin\theta\cos\theta+\cos^2\theta$

$\quad+\sin^2\theta-2\sin\theta\cos\theta+\cos^2\theta$

$=2\sin^2\theta+2\cos^2\theta$

$=2(\sin^2\theta+\cos^2\theta)$

$=2\cdot 1$

$=2$

34. $\tan^2\theta\cos^2\theta+\cot^2\theta\sin^2\theta$

$=\dfrac{\sin^2\theta}{\cos^2\theta}\cdot\cos^2\theta+\dfrac{\cos^2\theta}{\sin^2\theta}\cdot\sin^2\theta$

$=\sin^2\theta+\cos^2\theta$

$=1$

35. $\sec^4\theta-\sec^2\theta=\sec^2\theta(\sec^2\theta-1)$

$=(\tan^2\theta+1)\tan^2\theta$

$=\tan^4\theta+\tan^2\theta$

36. $\csc^4\theta-\csc^2\theta=\csc^2\theta(\csc^2\theta-1)$

$=(\cot^2\theta+1)\cot^2\theta$

$=\cot^4\theta+\cot^2\theta$

37. $\sec u-\tan u=\dfrac{1}{\cos u}-\dfrac{\sin u}{\cos u}$

$=\left(\dfrac{1-\sin u}{\cos u}\right)\cdot\left(\dfrac{1+\sin u}{1+\sin u}\right)$

$=\dfrac{1-\sin^2 u}{\cos u(1+\sin u)}$

$=\dfrac{\cos^2 u}{\cos u(1+\sin u)}$

$=\dfrac{\cos u}{1+\sin u}$

38. $\csc u-\cot u=\dfrac{1}{\sin u}-\dfrac{\cos u}{\sin u}$

$=\left(\dfrac{1-\cos u}{\sin u}\right)\cdot\left(\dfrac{1+\cos u}{1+\cos u}\right)$

$=\dfrac{1-\cos^2 u}{\sin u(1+\cos u)}$

$=\dfrac{\sin^2 u}{\sin u(1+\cos u)}$

$=\dfrac{\sin u}{1+\cos u}$

39. $3\sin^2\theta+4\cos^2\theta=3\sin^2\theta+3\cos^2\theta+\cos^2\theta$

$=3(\sin^2\theta+\cos^2\theta)+\cos^2\theta$

$=3\cdot 1+\cos^2\theta$

$=3+\cos^2\theta$

40. $9\sec^2\theta-5\tan^2\theta=4\sec^2\theta+5\sec^2\theta-5\tan^2\theta$
$$=4\sec^2\theta+5(\sec^2\theta-\tan^2\theta)$$
$$=4\sec^2\theta+5\cdot 1$$
$$=5+4\sec^2\theta$$

41. $1-\dfrac{\cos^2\theta}{1+\sin\theta}=1-\dfrac{1-\sin^2\theta}{1+\sin\theta}$
$$=1-\frac{(1-\sin\theta)(1+\sin\theta)}{1+\sin\theta}$$
$$=1-(1-\sin\theta)$$
$$=1-1+\sin\theta$$
$$=\sin\theta$$

42. $1-\dfrac{\sin^2\theta}{1-\cos\theta}=1-\dfrac{1-\cos^2\theta}{1-\cos\theta}$
$$=1-\frac{(1-\cos\theta)(1+\cos\theta)}{1-\cos\theta}$$
$$=1-(1+\cos\theta)$$
$$=1-1-\cos\theta$$
$$=-\cos\theta$$

43. $\dfrac{1+\tan v}{1-\tan v}=\dfrac{1+\dfrac{1}{\cot v}}{1-\dfrac{1}{\cot v}}$
$$=\frac{\left(1+\dfrac{1}{\cot v}\right)\cot v}{\left(1-\dfrac{1}{\cot v}\right)\cot v}$$
$$=\frac{\cot v+1}{\cot v-1}$$

44. $\dfrac{\csc v-1}{\csc v+1}=\dfrac{\dfrac{1}{\sin v}-1}{\dfrac{1}{\sin v}+1}$
$$=\frac{\left(\dfrac{1}{\sin v}-1\right)\sin v}{\left(\dfrac{1}{\sin v}+1\right)\sin v}$$
$$=\frac{1-\sin v}{1+\sin v}$$

45. $\dfrac{\sec\theta}{\csc\theta}+\dfrac{\sin\theta}{\cos\theta}=\dfrac{\dfrac{1}{\cos\theta}}{\dfrac{1}{\sin\theta}}+\dfrac{\sin\theta}{\cos\theta}$
$$=\frac{\sin\theta}{\cos\theta}+\frac{\sin\theta}{\cos\theta}$$
$$=\tan\theta+\tan\theta$$
$$=2\tan\theta$$

46. $\dfrac{\csc\theta-1}{\cot\theta}=\dfrac{\csc\theta-1}{\cot\theta}\cdot\dfrac{\csc\theta+1}{\csc\theta+1}$
$$=\frac{\csc^2\theta-1}{\cot\theta(\csc\theta+1)}$$
$$=\frac{\cot^2\theta}{\cot\theta(\csc\theta+1)}$$
$$=\frac{\cot\theta}{\csc\theta+1}$$

47. $\dfrac{1+\sin\theta}{1-\sin\theta}=\dfrac{1+\dfrac{1}{\csc\theta}}{1-\dfrac{1}{\csc\theta}}$
$$=\frac{\dfrac{\csc\theta+1}{\csc\theta}}{\dfrac{\csc\theta-1}{\csc\theta}}$$
$$=\frac{\csc\theta+1}{\csc\theta}\cdot\frac{\csc\theta}{\csc\theta-1}$$
$$=\frac{\csc\theta+1}{\csc\theta-1}$$

48. $\dfrac{\cos\theta+1}{\cos\theta-1}=\dfrac{\dfrac{1}{\sec\theta}+1}{\dfrac{1}{\sec\theta}-1}$
$$=\frac{\dfrac{1+\sec\theta}{\sec\theta}}{\dfrac{1-\sec\theta}{\sec\theta}}$$
$$=\frac{1+\sec\theta}{1-\sec\theta}$$

49. $\dfrac{1-\sin v}{\cos v}+\dfrac{\cos v}{1-\sin v}=\dfrac{(1-\sin v)^2+\cos^2 v}{\cos v(1-\sin v)}$

$=\dfrac{1-2\sin v+\sin^2 v+\cos^2 v}{\cos v(1-\sin v)}$

$=\dfrac{1-2\sin v+1}{\cos v(1-\sin v)}$

$=\dfrac{2-2\sin v}{\cos v(1-\sin v)}$

$=\dfrac{2(1-\sin v)}{\cos v(1-\sin v)}$

$=\dfrac{2}{\cos v}$

$=2\sec v$

50. $\dfrac{\cos v}{1+\sin v}+\dfrac{1+\sin v}{\cos v}=\dfrac{\cos^2 v+(1+\sin v)^2}{\cos v(1+\sin v)}$

$=\dfrac{\cos^2 v+1+2\sin v+\sin^2 v}{\cos v(1+\sin v)}$

$=\dfrac{2+2\sin v}{\cos v(1+\sin v)}$

$=\dfrac{2(1+\sin v)}{\cos v(1+\sin v)}$

$=\dfrac{2}{\cos v}$

$=2\sec v$

51. $\dfrac{\sin\theta}{\sin\theta-\cos\theta}=\dfrac{\sin\theta}{\sin\theta-\cos\theta}\cdot\dfrac{\frac{1}{\sin\theta}}{\frac{1}{\sin\theta}}$

$=\dfrac{1}{1-\dfrac{\cos\theta}{\sin\theta}}$

$=\dfrac{1}{1-\cot\theta}$

52. $1-\dfrac{\sin^2\theta}{1+\cos\theta}=1-\dfrac{1-\cos^2\theta}{1+\cos\theta}$

$=1-\dfrac{(1-\cos\theta)(1+\cos\theta)}{1+\cos\theta}$

$=1-(1-\cos\theta)$

$=\cos\theta$

53. $(\sec\theta-\tan\theta)^2$

$=\sec^2\theta-2\sec\theta\tan\theta+\tan^2\theta$

$=\dfrac{1}{\cos^2\theta}-2\cdot\dfrac{1}{\cos\theta}\cdot\dfrac{\sin\theta}{\cos\theta}+\dfrac{\sin^2\theta}{\cos^2\theta}$

$=\dfrac{1-2\sin\theta+\sin^2\theta}{\cos^2\theta}$

$=\dfrac{(1-\sin\theta)(1-\sin\theta)}{1-\sin^2\theta}$

$=\dfrac{(1-\sin\theta)(1-\sin\theta)}{(1-\sin\theta)(1+\sin\theta)}$

$=\dfrac{1-\sin\theta}{1+\sin\theta}$

54. $(\csc\theta-\cot\theta)^2$

$=\csc^2\theta-2\csc\theta\cot\theta+\cot^2\theta$

$=\dfrac{1}{\sin^2\theta}-2\cdot\dfrac{1}{\sin\theta}\cdot\dfrac{\cos\theta}{\sin\theta}+\dfrac{\cos^2\theta}{\sin^2\theta}$

$=\dfrac{1-2\cos\theta+\cos^2\theta}{\sin^2\theta}$

$=\dfrac{(1-\cos\theta)(1-\cos\theta)}{1-\cos^2\theta}$

$=\dfrac{(1-\cos\theta)(1-\cos\theta)}{(1-\cos\theta)(1+\cos\theta)}$

$=\dfrac{1-\cos\theta}{1+\cos\theta}$

55. $\dfrac{\cos\theta}{1-\tan\theta}+\dfrac{\sin\theta}{1-\cot\theta}$

$=\dfrac{\cos\theta}{1-\dfrac{\sin\theta}{\cos\theta}}+\dfrac{\sin\theta}{1-\dfrac{\cos\theta}{\sin\theta}}$

$=\dfrac{\cos\theta}{\dfrac{\cos\theta-\sin\theta}{\cos\theta}}+\dfrac{\sin\theta}{\dfrac{\sin\theta-\cos\theta}{\sin\theta}}$

$=\dfrac{\cos^2\theta}{\cos\theta-\sin\theta}+\dfrac{\sin^2\theta}{\sin\theta-\cos\theta}$

$=\dfrac{\cos^2\theta-\sin^2\theta}{\cos\theta-\sin\theta}$

$=\dfrac{(\cos\theta-\sin\theta)(\cos\theta+\sin\theta)}{\cos\theta-\sin\theta}$

$=\sin\theta+\cos\theta$

56. $$\frac{\cot\theta}{1-\tan\theta}+\frac{\tan\theta}{1-\cot\theta}$$
$$=\frac{\frac{\cos\theta}{\sin\theta}}{1-\frac{\sin\theta}{\cos\theta}}+\frac{\frac{\sin\theta}{\cos\theta}}{1-\frac{\cos\theta}{\sin\theta}}$$
$$=\frac{\frac{\cos\theta}{\sin\theta}}{\frac{\cos\theta-\sin\theta}{\cos\theta}}+\frac{\frac{\sin\theta}{\cos\theta}}{\frac{\sin\theta-\cos\theta}{\sin\theta}}$$
$$=\frac{\cos^2\theta}{\sin\theta(\cos\theta-\sin\theta)}+\frac{\sin^2\theta}{\cos\theta(\sin\theta-\cos\theta)}$$
$$=\frac{-\cos^2\theta\cdot\cos\theta+\sin^2\theta\cdot\sin\theta}{\sin\theta\cos\theta(\sin\theta-\cos\theta)}$$
$$=\frac{\sin^3\theta-\cos^3\theta}{\sin\theta\cos\theta(\sin\theta-\cos\theta)}$$
$$=\frac{(\sin\theta-\cos\theta)(\sin^2\theta+\sin\theta\cos\theta+\cos^2\theta)}{\sin\theta\cos\theta(\sin\theta-\cos\theta)}$$
$$=\frac{\sin^2\theta+\sin\theta\cos\theta+\cos^2\theta}{\sin\theta\cos\theta}$$
$$=\frac{\sin^2\theta}{\sin\theta\cos\theta}+\frac{\sin\theta\cos\theta}{\sin\theta\cos\theta}+\frac{\cos^2\theta}{\sin\theta\cos\theta}$$
$$=\frac{\sin\theta}{\cos\theta}+1+\frac{\cos\theta}{\sin\theta}$$
$$=1+\tan\theta+\cot\theta$$

57. $$\tan\theta+\frac{\cos\theta}{1+\sin\theta}=\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{1+\sin\theta}$$
$$=\frac{\sin\theta(1+\sin\theta)+\cos^2\theta}{\cos\theta(1+\sin\theta)}$$
$$=\frac{\sin\theta+\sin^2\theta+\cos^2\theta}{\cos\theta(1+\sin\theta)}$$
$$=\frac{\sin\theta+1}{\cos\theta(1+\sin\theta)}$$
$$=\frac{1}{\cos\theta}$$
$$=\sec\theta$$

58. $$\frac{\sin\theta\cos\theta}{\cos^2\theta-\sin^2\theta}=\frac{(\sin\theta\cos\theta)\cdot\frac{1}{\cos^2\theta}}{(\cos^2\theta-\sin^2\theta)\cdot\frac{1}{\cos^2\theta}}$$
$$=\frac{\frac{\sin\theta}{\cos\theta}}{1-\frac{\sin^2\theta}{\cos^2\theta}}$$
$$=\frac{\tan\theta}{1-\tan^2\theta}$$

59. $$\frac{\tan\theta+\sec\theta-1}{\tan\theta-\sec\theta+1}$$
$$=\frac{\tan\theta+(\sec\theta-1)}{\tan\theta-(\sec\theta-1)}\cdot\frac{\tan\theta+(\sec\theta-1)}{\tan\theta+(\sec\theta-1)}$$
$$=\frac{\tan^2\theta+2\tan\theta(\sec\theta-1)+\sec^2\theta-2\sec\theta+1}{\tan^2\theta-(\sec^2\theta-2\sec\theta+1)}$$
$$=\frac{\sec^2\theta-1+2\tan\theta(\sec\theta-1)+\sec^2\theta-2\sec\theta+1}{\sec^2\theta-1-\sec^2\theta+2\sec\theta-1}$$
$$=\frac{2\sec^2\theta-2\sec\theta+2\tan\theta(\sec\theta-1)}{2\sec\theta-2}$$
$$=\frac{2\sec\theta(\sec\theta-1)+2\tan\theta(\sec\theta-1)}{2\sec\theta-2}$$
$$=\frac{2(\sec\theta-1)(\sec\theta+\tan\theta)}{2(\sec\theta-1)}$$
$$=\tan\theta+\sec\theta$$

60. $$\frac{\sin\theta-\cos\theta+1}{\sin\theta+\cos\theta-1}$$
$$=\frac{(\sin\theta-\cos\theta)+1}{(\sin\theta+\cos\theta)-1}\cdot\frac{(\sin\theta+\cos\theta)+1}{(\sin\theta+\cos\theta)+1}$$
$$=\frac{\sin^2\theta-\cos^2\theta+\sin\theta+\cos\theta+\sin\theta-\cos\theta+1}{(\sin\theta+\cos\theta)^2-1}$$
$$=\frac{\sin^2\theta-\cos^2\theta+2\sin\theta+1}{\sin^2\theta+2\sin\theta\cos\theta+\cos^2\theta-1}$$
$$=\frac{\sin^2\theta-(1-\sin^2\theta)+2\sin\theta+1}{2\sin\theta\cos\theta+1-1}$$
$$=\frac{2\sin^2\theta+2\sin\theta}{2\sin\theta\cos\theta}$$
$$=\frac{2\sin\theta(\sin\theta+1)}{2\sin\theta\cos\theta}$$
$$=\frac{\sin\theta+1}{\cos\theta}$$

61. $$\frac{\tan\theta-\cot\theta}{\tan\theta+\cot\theta}=\frac{\frac{\sin\theta}{\cos\theta}-\frac{\cos\theta}{\sin\theta}}{\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}}=\frac{\frac{\sin^2\theta-\cos^2\theta}{\cos\theta\sin\theta}}{\frac{\sin^2\theta+\cos^2\theta}{\cos\theta\sin\theta}}=\frac{\sin^2\theta-\cos^2\theta}{1}=\sin^2\theta-\cos^2\theta$$

62. $$\frac{\sec\theta-\cos\theta}{\sec\theta+\cos\theta}=\frac{\frac{1}{\cos\theta}-\frac{\cos^2\theta}{\cos\theta}}{\frac{1}{\cos\theta}+\frac{\cos^2\theta}{\cos\theta}}=\frac{\frac{1-\cos^2\theta}{\cos\theta}}{\frac{1+\cos^2\theta}{\cos\theta}}=\frac{1-\cos^2\theta}{1+\cos^2\theta}=\frac{\sin^2\theta}{1+\cos^2\theta}$$

63. $$\frac{\tan u-\cot u}{\tan u+\cot u}+1=\frac{\frac{\sin u}{\cos u}-\frac{\cos u}{\sin u}}{\frac{\sin u}{\cos u}+\frac{\cos u}{\sin u}}+1=\frac{\frac{\sin^2 u-\cos^2 u}{\cos u\sin u}}{\frac{\sin^2 u+\cos^2 u}{\cos u\sin u}}+1=\frac{\sin^2 u-\cos^2 u}{1}+1$$
$$=\sin^2 u-\cos^2 u+1=\sin^2 u+(1-\cos^2 u)=\sin^2 u+\sin^2 u=2\sin^2 u$$

64. $$\frac{\tan u-\cot u}{\tan u+\cot u}+2\cos^2 u=\frac{\frac{\sin u}{\cos u}-\frac{\cos u}{\sin u}}{\frac{\sin u}{\cos u}+\frac{\cos u}{\sin u}}+2\cos^2 u=\frac{\frac{\sin^2 u-\cos^2 u}{\cos u\sin u}}{\frac{\sin^2 u+\cos^2 u}{\cos u\sin u}}+2\cos^2 u=\frac{\sin^2 u-\cos^2 u}{1}+2\cos^2 u$$
$$=\sin^2 u+\cos^2 u=1$$

65. $$\frac{\sec\theta+\tan\theta}{\cot\theta+\cos\theta}=\frac{\frac{1}{\cos\theta}+\frac{\sin\theta}{\cos\theta}}{\frac{\cos\theta}{\sin\theta}+\cos\theta}=\frac{\frac{1+\sin\theta}{\cos\theta}}{\frac{\cos\theta+\cos\theta\sin\theta}{\sin\theta}}=\frac{1+\sin\theta}{\cos\theta}\cdot\frac{\sin\theta}{\cos\theta(1+\sin\theta)}=\frac{\sin\theta}{\cos\theta}\cdot\frac{1}{\cos\theta}=\tan\theta\sec\theta$$

66. $$\frac{\sec\theta}{1+\sec\theta}=\frac{\frac{1}{\cos\theta}}{1+\frac{1}{\cos\theta}}=\frac{\frac{1}{\cos\theta}}{\frac{\cos\theta+1}{\cos\theta}}=\left(\frac{1}{1+\cos\theta}\right)\cdot\left(\frac{1-\cos\theta}{1-\cos\theta}\right)=\frac{1-\cos\theta}{1-\cos^2\theta}=\frac{1-\cos\theta}{\sin^2\theta}$$

67.
$$\frac{1-\tan^2\theta}{1+\tan^2\theta}+1=\frac{1-\tan^2\theta}{1+\tan^2\theta}+\frac{1+\tan^2\theta}{1+\tan^2\theta}$$
$$=\frac{1-\tan^2\theta+1+\tan^2\theta}{1+\tan^2\theta}$$
$$=\frac{2}{1+\tan^2\theta}=\frac{2}{\sec^2\theta}$$
$$=2\cdot\frac{1}{\sec^2\theta}$$
$$=2\cos^2\theta$$

68.
$$\frac{1-\cot^2\theta}{1+\cot^2\theta}+2\cos^2\theta=\frac{1-\cot^2\theta}{\csc^2\theta}+2\cos^2\theta$$
$$=\frac{1}{\csc^2\theta}-\frac{\cot^2\theta}{\csc^2\theta}+2\cos^2\theta$$
$$=\sin^2\theta-\frac{\frac{\cos^2\theta}{\sin^2\theta}}{\frac{1}{\sin^2\theta}}+2\cos^2\theta$$
$$=\sin^2\theta-\cos^2\theta+2\cos^2\theta$$
$$=\sin^2\theta+\cos^2\theta$$
$$=1$$

69.
$$\frac{\sec\theta-\csc\theta}{\sec\theta\csc\theta}=\frac{\sec\theta}{\sec\theta\csc\theta}-\frac{\csc\theta}{\sec\theta\csc\theta}$$
$$=\frac{1}{\csc\theta}-\frac{1}{\sec\theta}$$
$$=\sin\theta-\cos\theta$$

70.
$$\frac{\sin^2\theta-\tan\theta}{\cos^2\theta-\cot\theta}$$
$$=\frac{\sin^2\theta-\frac{\sin\theta}{\cos\theta}}{\cos^2\theta-\frac{\cos\theta}{\sin\theta}}$$
$$=\frac{\frac{\sin^2\theta\cos\theta-\sin\theta}{\cos\theta}}{\frac{\cos^2\theta\sin\theta-\cos\theta}{\sin\theta}}$$
$$=\frac{\sin^2\theta\cos\theta-\sin\theta}{\cos\theta}\cdot\frac{\sin\theta}{\cos^2\theta\sin\theta-\cos\theta}$$
$$=\frac{\sin\theta(\sin\theta\cos\theta-1)}{\cos\theta}\cdot\frac{\sin\theta}{\cos\theta(\cos\theta\sin\theta-1)}$$
$$=\frac{\sin^2\theta}{\cos^2\theta}$$
$$=\tan^2\theta$$

71.
$$\sec\theta-\cos\theta=\frac{1}{\cos\theta}-\cos\theta$$
$$=\frac{1-\cos^2\theta}{\cos\theta}$$
$$=\frac{\sin^2\theta}{\cos\theta}$$
$$=\sin\theta\cdot\frac{\sin\theta}{\cos\theta}$$
$$=\sin\theta\tan\theta$$

72.
$$\tan\theta+\cot\theta=\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}$$
$$=\frac{\sin^2\theta+\cos^2\theta}{\sin\theta\cos\theta}$$
$$=\frac{1}{\sin\theta\cos\theta}$$
$$=\frac{1}{\cos\theta}\cdot\frac{1}{\sin\theta}$$
$$=\sec\theta\csc\theta$$

73.
$$\frac{1}{1-\sin\theta}+\frac{1}{1+\sin\theta}=\frac{1+\sin\theta+1-\sin\theta}{(1-\sin\theta)(1+\sin\theta)}$$
$$=\frac{2}{1-\sin^2\theta}$$
$$=\frac{2}{\cos^2\theta}$$
$$=2\sec^2\theta$$

74.
$$\frac{1+\sin\theta}{1-\sin\theta}-\frac{1-\sin\theta}{1+\sin\theta}$$
$$=\frac{(1+\sin\theta)^2-(1-\sin\theta)^2}{(1-\sin\theta)(1+\sin\theta)}$$
$$=\frac{1+2\sin\theta+\sin^2\theta-(1-2\sin\theta+\sin^2\theta)}{1-\sin^2\theta}$$
$$=\frac{4\sin\theta}{\cos^2\theta}$$
$$=4\cdot\frac{\sin\theta}{\cos\theta}\cdot\frac{1}{\cos\theta}$$
$$=4\tan\theta\sec\theta$$

75. $\dfrac{\sec\theta}{1-\sin\theta}=\left(\dfrac{\sec\theta}{1-\sin\theta}\right)\cdot\left(\dfrac{1+\sin\theta}{1+\sin\theta}\right)$

$=\dfrac{\sec\theta(1+\sin\theta)}{1-\sin^2\theta}$

$=\dfrac{\sec\theta(1+\sin\theta)}{\cos^2\theta}$

$=\dfrac{1}{\cos\theta}\cdot\dfrac{1+\sin\theta}{\cos^2\theta}$

$=\dfrac{1+\sin\theta}{\cos^3\theta}$

76. $\dfrac{1+\sin\theta}{1-\sin\theta}=\dfrac{(1+\sin\theta)(1+\sin\theta)}{(1-\sin\theta)(1+\sin\theta)}$

$=\dfrac{(1+\sin\theta)^2}{1-\sin^2\theta}$

$=\dfrac{(1+\sin\theta)^2}{\cos^2\theta}$

$=\left(\dfrac{1+\sin\theta}{\cos\theta}\right)^2$

$=\left(\dfrac{1}{\cos\theta}+\dfrac{\sin\theta}{\cos\theta}\right)^2$

$=(\sec\theta+\tan\theta)^2$

77. $\dfrac{(\sec v-\tan v)^2+1}{\csc v(\sec v-\tan v)}$

$=\dfrac{\sec^2 v-2\sec v\tan v+\tan^2 v+1}{\csc v(\sec v-\tan v)}$

$=\dfrac{\sec^2 v-2\sec v\tan v+\sec^2 v}{\csc v(\sec v-\tan v)}$

$=\dfrac{2\sec^2 v-2\sec v\tan v}{\csc v(\sec v-\tan v)}$

$=\dfrac{2\sec v(\sec v-\tan v)}{\csc v(\sec v-\tan v)}$

$=\dfrac{2\sec v}{\csc v}$

$=\dfrac{2\cdot\dfrac{1}{\cos v}}{\dfrac{1}{\sin v}}$

$=2\cdot\dfrac{1}{\cos v}\cdot\dfrac{\sin v}{1}$

$=2\cdot\dfrac{\sin v}{\cos v}$

$=2\tan v$

78. $\dfrac{\sec^2 v-\tan^2 v+\tan v}{\sec v}=\dfrac{1+\tan v}{\sec v}$

$=\dfrac{1+\dfrac{\sin v}{\cos v}}{\dfrac{1}{\cos v}}$

$=\dfrac{\dfrac{\cos v+\sin v}{\cos v}}{\dfrac{1}{\cos v}}$

$=\cos v+\sin v$

79. $\dfrac{\sin\theta+\cos\theta}{\cos\theta}-\dfrac{\sin\theta-\cos\theta}{\sin\theta}$

$=\dfrac{\sin\theta}{\cos\theta}+\dfrac{\cos\theta}{\cos\theta}-\dfrac{\sin\theta}{\sin\theta}+\dfrac{\cos\theta}{\sin\theta}$

$=\dfrac{\sin\theta}{\cos\theta}+1-1+\dfrac{\cos\theta}{\sin\theta}$

$=\dfrac{\sin^2\theta+\cos^2\theta}{\cos\theta\sin\theta}$

$=\dfrac{1}{\cos\theta\sin\theta}$

$=\sec\theta\csc\theta$

80. $\dfrac{\sin\theta+\cos\theta}{\sin\theta}-\dfrac{\cos\theta-\sin\theta}{\cos\theta}$

$=\dfrac{\sin\theta}{\sin\theta}+\dfrac{\cos\theta}{\sin\theta}-\dfrac{\cos\theta}{\cos\theta}+\dfrac{\sin\theta}{\cos\theta}$

$=1+\dfrac{\cos\theta}{\sin\theta}-1+\dfrac{\sin\theta}{\cos\theta}$

$=\dfrac{\cos^2\theta+\sin^2\theta}{\cos\theta\sin\theta}$

$=\dfrac{1}{\cos\theta\sin\theta}$

$=\sec\theta\csc\theta$

81. $\dfrac{\sin^3\theta+\cos^3\theta}{\sin\theta+\cos\theta}$

$=\dfrac{(\sin\theta+\cos\theta)(\sin^2\theta-\sin\theta\cos\theta+\cos^2\theta)}{\sin\theta+\cos\theta}$

$=\sin^2\theta+\cos^2\theta-\sin\theta\cos\theta$

$=1-\sin\theta\cos\theta$

82. $\dfrac{\sin^3\theta+\cos^3\theta}{1-2\cos^2\theta}$

$= \dfrac{(\sin\theta+\cos\theta)(\sin^2\theta-\sin\theta\cos\theta+\cos^2\theta)}{1-\cos^2\theta-\cos^2\theta}$

$= \dfrac{(\sin\theta+\cos\theta)(\sin^2\theta+\cos^2\theta-\sin\theta\cos\theta)}{\sin^2\theta-\cos^2\theta}$

$= \dfrac{(\sin\theta+\cos\theta)(1-\sin\theta\cos\theta)}{(\sin\theta+\cos\theta)(\sin\theta-\cos\theta)}$

$= \dfrac{1-\sin\theta\cos\theta}{\sin\theta-\cos\theta}\cdot\dfrac{\frac{1}{\cos\theta}}{\frac{1}{\cos\theta}}$

$= \dfrac{\frac{1}{\cos\theta}-\sin\theta}{\frac{\sin\theta}{\cos\theta}-1}$

$= \dfrac{\sec\theta-\sin\theta}{\tan\theta-1}$

83. $\dfrac{\cos^2\theta-\sin^2\theta}{1-\tan^2\theta} = \dfrac{\cos^2\theta-\sin^2\theta}{1-\frac{\sin^2\theta}{\cos^2\theta}}$

$= \dfrac{\cos^2\theta-\sin^2\theta}{\frac{\cos^2\theta-\sin^2\theta}{\cos^2\theta}}$

$= (\cos^2\theta-\sin^2\theta)\cdot\dfrac{\cos^2\theta}{\cos^2\theta-\sin^2\theta}$

$= \cos^2\theta$

84. $\dfrac{\cos\theta+\sin\theta-\sin^3\theta}{\sin\theta} = \dfrac{\cos\theta}{\sin\theta}+\dfrac{\sin\theta}{\sin\theta}-\dfrac{\sin^3\theta}{\sin\theta}$

$= \cot\theta+1-\sin^2\theta$

$= \cot\theta+\cos^2\theta$

85. $\dfrac{(2\cos^2\theta-1)^2}{\cos^4\theta-\sin^4\theta} = \dfrac{\left[2\cos^2\theta-(\sin^2\theta+\cos^2\theta)\right]^2}{(\cos^2\theta-\sin^2\theta)(\cos^2\theta+\sin^2\theta)}$

$= \dfrac{(\cos^2\theta-\sin^2\theta)^2}{(\cos^2\theta-\sin^2\theta)(\cos^2\theta+\sin^2\theta)}$

$= \dfrac{\cos^2\theta-\sin^2\theta}{\cos^2\theta+\sin^2\theta}$

$= \cos^2\theta-\sin^2\theta$

$= 1-\sin^2\theta-\sin^2\theta$

$= 1-2\sin^2\theta$

86. $\dfrac{1-2\cos^2\theta}{\sin\theta\cos\theta} = \dfrac{1-\cos^2\theta-\cos^2\theta}{\sin\theta\cos\theta}$

$= \dfrac{\sin^2\theta-\cos^2\theta}{\sin\theta\cos\theta}$

$= \dfrac{\sin^2\theta}{\sin\theta\cos\theta}-\dfrac{\cos^2\theta}{\sin\theta\cos\theta}$

$= \dfrac{\sin\theta}{\cos\theta}-\dfrac{\cos\theta}{\sin\theta}$

$= \tan\theta-\cot\theta$

87. $\dfrac{1+\sin\theta+\cos\theta}{1+\sin\theta-\cos\theta}$

$= \dfrac{(1+\sin\theta)+\cos\theta}{(1+\sin\theta)-\cos\theta}\cdot\dfrac{(1+\sin\theta)+\cos\theta}{(1+\sin\theta)+\cos\theta}$

$= \dfrac{1+2\sin\theta+\sin^2\theta+2\cos\theta(1+\sin\theta)+\cos^2\theta}{1+2\sin\theta+\sin^2\theta-\cos^2\theta}$

$= \dfrac{1+2\sin\theta+\sin^2\theta+2\cos\theta(1+\sin\theta)+(1-\sin^2\theta)}{1+2\sin\theta+\sin^2\theta-(1-\sin^2\theta)}$

$= \dfrac{2+2\sin\theta+2\cos\theta(1+\sin\theta)}{2\sin\theta+2\sin^2\theta}$

$= \dfrac{2(1+\sin\theta)+2\cos\theta(1+\sin\theta)}{2\sin\theta(1+\sin\theta)}$

$= \dfrac{2(1+\sin\theta)(1+\cos\theta)}{2\sin\theta(1+\sin\theta)}$

$= \dfrac{1+\cos\theta}{\sin\theta}$

88. $\dfrac{1+\cos\theta+\sin\theta}{1+\cos\theta-\sin\theta}$

$= \dfrac{(1+\cos\theta)+\sin\theta}{(1+\cos\theta)-\sin\theta}\cdot\dfrac{(1+\cos\theta)+\sin\theta}{(1+\cos\theta)+\sin\theta}$

$= \dfrac{1+2\cos\theta+\cos^2\theta+2\sin\theta(1+\cos\theta)+\sin^2\theta}{1+2\cos\theta+\cos^2\theta-\sin^2\theta}$

$= \dfrac{1+2\cos\theta+\cos^2\theta+2\sin\theta(1+\cos\theta)+1-\cos^2\theta}{1+2\cos\theta+\cos^2\theta-(1-\cos^2\theta)}$

$= \dfrac{2+2\cos\theta+2\sin\theta(1+\cos\theta)}{2\cos\theta+2\cos^2\theta}$

$= \dfrac{2(1+\cos\theta)+2\sin\theta(1+\cos\theta)}{2\cos\theta(1+\cos\theta)}$

$= \dfrac{2(1+\cos\theta)(1+\sin\theta)}{2\cos\theta(1+\cos\theta)}$

$= \dfrac{1+\sin\theta}{\cos\theta}$

$= \dfrac{1}{\cos\theta}+\dfrac{\sin\theta}{\cos\theta}$

$= \sec\theta+\tan\theta$

89. $(a\sin\theta + b\cos\theta)^2 + (a\cos\theta - b\sin\theta)^2$

$$
\begin{aligned}
&= a^2\sin^2\theta + 2ab\sin\theta\cos\theta + b^2\cos^2\theta \\
&\qquad + a^2\cos^2\theta - 2ab\sin\theta\cos\theta + b^2\sin^2\theta \\
&= a^2(\sin^2\theta + \cos^2\theta) + b^2(\sin^2\theta + \cos^2\theta) \\
&= a^2 + b^2
\end{aligned}
$$

90. $(2a\sin\theta\cos\theta)^2 + a^2(\cos^2\theta - \sin^2\theta)^2$

$$
\begin{aligned}
&= 4a^2\sin^2\theta\cos^2\theta \\
&\qquad + a^2\left(\cos^4\theta - 2\cos^2\theta\sin^2\theta + \sin^4\theta\right) \\
&= a^2\left(4\sin^2\theta\cos^2\theta + \cos^4\theta - 2\cos^2\theta\sin^2\theta + \sin^4\theta\right) \\
&= a^2\left(\cos^4\theta + 2\cos^2\theta\sin^2\theta + \sin^4\theta\right) \\
&= a^2\left(\cos^2\theta + \sin^2\theta\right)^2 \\
&= a^2(1)^2 \\
&= a^2
\end{aligned}
$$

91.
$$
\begin{aligned}
\frac{\tan\alpha + \tan\beta}{\cot\alpha + \cot\beta} &= \frac{\tan\alpha + \tan\beta}{\frac{1}{\tan\alpha} + \frac{1}{\tan\beta}} \\
&= \frac{\tan\alpha + \tan\beta}{\frac{\tan\beta + \tan\alpha}{\tan\alpha\tan\beta}} \\
&= (\tan\alpha + \tan\beta)\cdot\left(\frac{\tan\alpha\tan\beta}{\tan\alpha + \tan\beta}\right) \\
&= \tan\alpha\tan\beta
\end{aligned}
$$

92. $(\tan\alpha + \tan\beta)(1 - \cot\alpha\cot\beta)$

$$
\begin{aligned}
&\qquad + (\cot\alpha + \cot\beta)(1 - \tan\alpha\tan\beta) \\
&= \tan\alpha + \tan\beta - \tan\alpha\cot\alpha\cot\beta \\
&\qquad - \tan\beta\cot\alpha\cot\beta + \cot\alpha + \cot\beta \\
&\qquad - \cot\alpha\tan\alpha\tan\beta - \cot\beta\tan\alpha\tan\beta \\
&= \tan\alpha + \tan\beta - \cot\beta - \cot\alpha + \cot\alpha \\
&\qquad + \cot\beta - \tan\beta - \tan\alpha \\
&= 0
\end{aligned}
$$

93. $(\sin\alpha + \cos\beta)^2 + (\cos\beta + \sin\alpha)(\cos\beta - \sin\alpha)$

$$
\begin{aligned}
&= \sin^2\alpha + 2\sin\alpha\cos\beta + \cos^2\beta + \cos^2\beta - \sin^2\alpha \\
&= 2\sin\alpha\cos\beta + 2\cos^2\beta \\
&= 2\cos\beta(\sin\alpha + \cos\beta)
\end{aligned}
$$

94. $(\sin\alpha - \cos\beta)^2 + (\cos\beta + \sin\alpha)(\cos\beta - \sin\alpha)$

$$
\begin{aligned}
&= \sin^2\alpha - 2\sin\alpha\cos\beta + \cos^2\beta + \cos^2\beta - \sin^2\alpha \\
&= -2\sin\alpha\cos\beta + 2\cos^2\beta = -2\cos\beta(\sin\alpha - \cos\beta)
\end{aligned}
$$

95. $\ln|\sec\theta| = \ln\left|\frac{1}{\cos\theta}\right| = \ln|\cos\theta|^{-1} = -\ln|\cos\theta|$

96. $\ln|\tan\theta| = \ln\left|\frac{\sin\theta}{\cos\theta}\right| = \ln|\sin\theta| - \ln|\cos\theta|$

97. $\ln|1 + \cos\theta| + \ln|1 - \cos\theta|$

$$
\begin{aligned}
&= \ln\left(|1 + \cos\theta|\cdot|1 - \cos\theta|\right) \\
&= \ln|1 - \cos^2\theta| \\
&= \ln|\sin^2\theta| \\
&= 2\ln|\sin\theta|
\end{aligned}
$$

98. $\ln|\sec\theta + \tan\theta| + \ln|\sec\theta - \tan\theta|$

$$
\begin{aligned}
&= \ln\left(|\sec\theta + \tan\theta|\cdot|\sec\theta - \tan\theta|\right) \\
&= \ln|\sec^2\theta - \tan^2\theta| \\
&= \ln|\tan^2\theta + 1 - \tan^2\theta| \\
&= \ln|1| \\
&= 0
\end{aligned}
$$

99.
$$
\begin{aligned}
f(x) &= \sin x \cdot \tan x \\
&= \sin x \cdot \frac{\sin x}{\cos x} \\
&= \frac{\sin^2 x}{\cos x} \\
&= \frac{1 - \cos^2 x}{\cos x} \\
&= \frac{1}{\cos x} - \frac{\cos^2 x}{\cos x} \\
&= \sec x - \cos x \\
&= g(x)
\end{aligned}
$$

100. $f(x) = \cos x \cdot \cot x$

$$= \cos x \cdot \frac{\cos x}{\sin x}$$
$$= \frac{\cos^2 x}{\sin x}$$
$$= \frac{1-\sin^2 x}{\sin x}$$
$$= \frac{1}{\sin x} - \frac{\sin^2 x}{\sin x}$$
$$= \csc x - \sin x$$
$$= g(x)$$

101. $f(\theta) = \frac{1-\sin\theta}{\cos\theta} - \frac{\cos\theta}{1+\sin\theta}$

$$= \frac{(1-\sin\theta)(1+\sin\theta)}{\cos\theta(1+\sin\theta)} - \frac{\cos\theta \cdot \cos\theta}{(1+\sin\theta)\cdot\cos\theta}$$
$$= \frac{1-\sin^2\theta - \cos^2\theta}{\cos\theta(1+\sin\theta)}$$
$$= \frac{1-(\sin^2\theta + \cos^2\theta)}{\cos\theta(1+\sin\theta)}$$
$$= \frac{1-1}{\cos\theta(1+\sin\theta)}$$
$$= \frac{0}{\cos\theta(1+\sin\theta)}$$
$$= 0$$
$$= g(\theta)$$

102. $f(\theta) = \tan\theta + \sec\theta$

$$= \frac{\sin\theta}{\cos\theta} + \frac{1}{\cos\theta}$$
$$= \frac{1+\sin\theta}{\cos\theta}$$
$$= \frac{1+\sin\theta}{\cos\theta} \cdot \frac{1-\sin\theta}{1-\sin\theta}$$
$$= \frac{1-\sin^2\theta}{\cos\theta(1-\sin\theta)}$$
$$= \frac{\cos^2\theta}{\cos\theta(1-\sin\theta)}$$
$$= \frac{\cos\theta}{1-\sin\theta}$$
$$= g(\theta)$$

103. $\sqrt{16+16\tan^2\theta} = \sqrt{16(1+\tan^2\theta)} = 4\sqrt{1+\tan^2\theta}$.

Since $\sec\theta > 0$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, then

$$4\sqrt{1+\tan^2\theta} = 4\sqrt{\sec^2\theta} = 4\sec\theta$$

104. $\sqrt{9\sec^2\theta - 9} = \sqrt{9(\sec^2\theta - 1)} = 3\sqrt{\sec^2\theta - 1}$.

Since $\tan\theta > 0$ for $\pi < \theta < \frac{3\pi}{2}$, then

$$3\sqrt{\sec^2\theta - 1} = 3\sqrt{\tan^2\theta} = 3\tan\theta$$

105. $1200\sec\theta\left(2\sec^2\theta - 1\right) = 1200\frac{1}{\cos\theta}\left(\frac{2}{\cos^2\theta} - 1\right)$

$$= 1200\frac{1}{\cos\theta}\left(\frac{2}{\cos^2\theta} - \frac{\cos^2\theta}{\cos^2\theta}\right)$$
$$= 1200\frac{1}{\cos\theta}\left(\frac{2-\cos^2\theta}{\cos^2\theta}\right)$$
$$= \frac{1200\left(1+1-\cos^2\theta\right)}{\cos^3\theta}$$
$$= \frac{1200\left(1+\sin^2\theta\right)}{\cos^3\theta}$$

106. $I_t = 4A^2\frac{(\csc\theta - 1)(\sec\theta + \tan\theta)}{\csc\theta\sec\theta}$

$$= 4A^2\frac{\csc\theta - 1}{\csc\theta} \cdot \frac{\sec\theta + \tan\theta}{\sec\theta}$$
$$= 4A^2\left(1 - \frac{1}{\csc\theta}\right)\left(1 + \frac{\tan\theta}{\sec\theta}\right)$$
$$= 4A^2(1-\sin\theta)(1+\sin\theta)$$
$$= 4A^2\left(1-\sin^2\theta\right)$$
$$= 4A^2\cos^2\theta = (2A\cos\theta)^2$$

107. Answers will vary.

108. $\sin^2\theta + \cos^2\theta = 1$

$\tan^2\theta + 1 = \sec^2\theta$

$1 + \cot^2\theta = \csc^2\theta$

109 – 110. Answers will vary.

Section 7.5

1. $\sqrt{(5-2)^2+(1-(-3))^2}$
$= \sqrt{3^2+4^2} = \sqrt{9+16} = \sqrt{25} = 5$

2. $-\frac{3}{5}$

3. a. $\frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \frac{\sqrt{2}}{4}$

b. $1-\frac{1}{2} = \frac{1}{2}$

4. $y = 4, r = 5, x = -3$ (Quadrant 2)
$\cos\alpha = \frac{x}{r} = -\frac{3}{5}$

5. –

6. –

7. False

8. False

9. False

10. True

11.
$$\sin\frac{5\pi}{12} = \sin\left(\frac{3\pi}{12}+\frac{2\pi}{12}\right) = \sin\frac{\pi}{4}\cdot\cos\frac{\pi}{6}+\cos\frac{\pi}{4}\cdot\sin\frac{\pi}{6} = \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}+\frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \frac{1}{4}\left(\sqrt{6}+\sqrt{2}\right)$$

12.
$$\sin\frac{\pi}{12} = \sin\left(\frac{3\pi}{12}-\frac{2\pi}{12}\right) = \sin\frac{\pi}{4}\cdot\cos\frac{\pi}{6}-\cos\frac{\pi}{4}\cdot\sin\frac{\pi}{6} = \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}-\frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \frac{1}{4}\left(\sqrt{6}-\sqrt{2}\right)$$

13.
$$\cos\frac{7\pi}{12} = \cos\left(\frac{4\pi}{12}+\frac{3\pi}{12}\right) = \cos\frac{\pi}{3}\cdot\cos\frac{\pi}{4}-\sin\frac{\pi}{3}\cdot\sin\frac{\pi}{4} = \frac{1}{2}\cdot\frac{\sqrt{2}}{2}-\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} = \frac{1}{4}\left(\sqrt{2}-\sqrt{6}\right)$$

14.
$$\tan\frac{7\pi}{12} = \tan\left(\frac{3\pi}{12}+\frac{4\pi}{12}\right) = \frac{\tan\frac{\pi}{4}+\tan\frac{\pi}{3}}{1-\tan\frac{\pi}{4}\cdot\tan\frac{\pi}{3}} = \frac{1+\sqrt{3}}{1-1\cdot\sqrt{3}} = \left(\frac{1+\sqrt{3}}{1-\sqrt{3}}\right)\cdot\left(\frac{1+\sqrt{3}}{1+\sqrt{3}}\right) = \frac{1+2\sqrt{3}+3}{1-3} = \frac{4+2\sqrt{3}}{-2} = -2-\sqrt{3}$$

15.
$$\cos 165^\circ = \cos\left(120^\circ+45^\circ\right) = \cos 120^\circ\cdot\cos 45^\circ-\sin 120^\circ\cdot\sin 45^\circ = -\frac{1}{2}\cdot\frac{\sqrt{2}}{2}-\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} = -\frac{1}{4}\left(\sqrt{2}+\sqrt{6}\right)$$

16.
$$\sin 105^\circ = \sin\left(60^\circ+45^\circ\right) = \sin 60^\circ\cdot\cos 45^\circ+\cos 60^\circ\cdot\sin 45^\circ = \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}+\frac{1}{2}\cdot\frac{\sqrt{2}}{2} = \frac{1}{4}\left(\sqrt{6}+\sqrt{2}\right)$$

17. $\tan 15° = \tan(45° - 30°)$

$$= \frac{\tan 45° - \tan 30°}{1 + \tan 45° \cdot \tan 30°}$$

$$= \frac{1 - \frac{\sqrt{3}}{3}}{1 + 1 \cdot \frac{\sqrt{3}}{3}} \cdot \frac{3}{3}$$

$$= \frac{3 - \sqrt{3}}{3 + \sqrt{3}} \cdot \frac{3 - \sqrt{3}}{3 - \sqrt{3}}$$

$$= \frac{9 - 6\sqrt{3} + 3}{9 - 3}$$

$$= \frac{12 - 6\sqrt{3}}{6}$$

$$= 2 - \sqrt{3}$$

18. $\tan 195° = \tan(135° + 60°)$

$$= \frac{\tan 135° + \tan 60°}{1 - \tan 135° \cdot \tan 60°}$$

$$= \frac{-1 + \sqrt{3}}{1 - (-1) \cdot \sqrt{3}}$$

$$= \frac{-1 + \sqrt{3}}{1 + \sqrt{3}} \cdot \frac{1 - \sqrt{3}}{1 - \sqrt{3}}$$

$$= \frac{-1 + 2\sqrt{3} - 3}{1 - 3}$$

$$= \frac{-4 + 2\sqrt{3}}{-2}$$

$$= 2 - \sqrt{3}$$

19. $\sin \frac{17\pi}{12} = \sin\left(\frac{15\pi}{12} + \frac{2\pi}{12}\right)$

$$= \sin \frac{5\pi}{4} \cdot \cos \frac{\pi}{6} + \cos \frac{5\pi}{4} \cdot \sin \frac{\pi}{6}$$

$$= -\frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \left(-\frac{\sqrt{2}}{2}\right) \cdot \frac{1}{2}$$

$$= -\frac{1}{4}\left(\sqrt{6} + \sqrt{2}\right)$$

20. $\tan \frac{19\pi}{12} = \tan\left(\frac{15\pi}{12} + \frac{4\pi}{12}\right)$

$$= \frac{\tan \frac{5\pi}{4} + \tan \frac{\pi}{3}}{1 - \tan \frac{5\pi}{4} \cdot \tan \frac{\pi}{3}}$$

$$= \frac{1 + \sqrt{3}}{1 - 1 \cdot \sqrt{3}}$$

$$= \frac{1 + \sqrt{3}}{1 - \sqrt{3}} \cdot \frac{1 + \sqrt{3}}{1 + \sqrt{3}}$$

$$= \frac{1 + 2\sqrt{3} + 3}{1 - 3}$$

$$= \frac{4 + 2\sqrt{3}}{-2}$$

$$= -2 - \sqrt{3}$$

21. $\sec\left(-\frac{\pi}{12}\right) = \frac{1}{\cos\left(-\frac{\pi}{12}\right)} = \frac{1}{\cos\left(\frac{3\pi}{12} - \frac{4\pi}{12}\right)}$

$$= \frac{1}{\cos \frac{\pi}{4} \cdot \cos \frac{\pi}{3} + \sin \frac{\pi}{4} \cdot \sin \frac{\pi}{3}}$$

$$= \frac{1}{\frac{\sqrt{2}}{2} \cdot \frac{1}{2} + \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2}}$$

$$= \frac{1}{\frac{\sqrt{2} + \sqrt{6}}{4}}$$

$$= \frac{4}{\sqrt{2} + \sqrt{6}} \cdot \frac{\sqrt{2} - \sqrt{6}}{\sqrt{2} - \sqrt{6}}$$

$$= \frac{4\sqrt{2} - 4\sqrt{6}}{2 - 6}$$

$$= \frac{4\sqrt{2} - 4\sqrt{6}}{-4}$$

$$= \sqrt{6} - \sqrt{2}$$

22. $\cot\left(-\frac{5\pi}{12}\right) = -\cot\frac{5\pi}{12} = \frac{-1}{\tan\frac{5\pi}{12}}$

$$= \frac{-1}{\tan\left(\frac{3\pi}{12} + \frac{2\pi}{12}\right)}$$

$$= \frac{-1}{\frac{\tan\frac{\pi}{4} + \tan\frac{\pi}{6}}{1 - \tan\frac{\pi}{4}\cdot\tan\frac{\pi}{6}}}$$

$$= -\left(\frac{1 - \tan\frac{\pi}{4}\cdot\tan\frac{\pi}{6}}{\tan\frac{\pi}{4} + \tan\frac{\pi}{6}}\right)$$

$$= -\frac{1 - 1\cdot\frac{1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}}\cdot\frac{\sqrt{3}}{\sqrt{3}}$$

$$= -\frac{\sqrt{3} - 1}{\sqrt{3} + 1}\cdot\frac{\sqrt{3} - 1}{\sqrt{3} - 1}$$

$$= -\frac{3 - \sqrt{3} - \sqrt{3} + 1}{3 - 1}$$

$$= -\frac{4 - 2\sqrt{3}}{2}$$

$$= -2 + \sqrt{3}$$

23. $\sin 20^\circ\cdot\cos 10^\circ + \cos 20^\circ\cdot\sin 10^\circ = \sin(20^\circ + 10^\circ)$

$$= \sin 30^\circ$$

$$= \frac{1}{2}$$

24. $\sin 20^\circ\cdot\cos 80^\circ - \cos 20^\circ\cdot\sin 80^\circ = \sin(20^\circ - 80^\circ)$

$$= \sin(-60^\circ)$$

$$= -\sin 60^\circ$$

$$= -\frac{\sqrt{3}}{2}$$

25. $\cos 70^\circ\cdot\cos 20^\circ - \sin 70^\circ\cdot\sin 20^\circ = \cos(70^\circ + 20^\circ)$

$$= \cos 90^\circ$$

$$= 0$$

26. $\cos 40^\circ\cdot\cos 10^\circ + \sin 40^\circ\cdot\sin 10^\circ = \cos(40^\circ - 10^\circ)$

$$= \cos 30^\circ$$

$$= \frac{\sqrt{3}}{2}$$

27. $\frac{\tan 20^\circ + \tan 25^\circ}{1 - \tan 20^\circ\tan 25^\circ} = \tan(20^\circ + 25^\circ)$

$$= \tan 45^\circ$$

$$= 1$$

28. $\frac{\tan 40^\circ - \tan 10^\circ}{1 + \tan 40^\circ\tan 10^\circ} = \tan(40^\circ - 10^\circ)$

$$= \tan 30^\circ$$

$$= \frac{\sqrt{3}}{3}$$

29. $\sin\frac{\pi}{12}\cdot\cos\frac{7\pi}{12} - \cos\frac{\pi}{12}\cdot\sin\frac{7\pi}{12} = \sin\left(\frac{\pi}{12} - \frac{7\pi}{12}\right)$

$$= \sin\left(-\frac{6\pi}{12}\right)$$

$$= \sin\left(-\frac{\pi}{2}\right)$$

$$= -1$$

30. $\cos\frac{5\pi}{12}\cdot\cos\frac{7\pi}{12} - \sin\frac{5\pi}{12}\cdot\sin\frac{7\pi}{12} = \cos\left(\frac{5\pi}{12} + \frac{7\pi}{12}\right)$

$$= \cos\frac{12\pi}{12}$$

$$= \cos\pi$$

$$= -1$$

31. $\cos\frac{\pi}{12}\cdot\cos\frac{5\pi}{12} + \sin\frac{5\pi}{12}\cdot\sin\frac{\pi}{12} = \cos\left(\frac{\pi}{12} - \frac{5\pi}{12}\right)$

$$= \cos\left(-\frac{4\pi}{12}\right)$$

$$= \cos\left(-\frac{\pi}{3}\right)$$

$$= \cos\frac{\pi}{3}$$

$$= \frac{1}{2}$$

32. $\sin\frac{\pi}{18}\cdot\cos\frac{5\pi}{18} + \cos\frac{\pi}{18}\cdot\sin\frac{5\pi}{18} = \sin\left(\frac{\pi}{18} + \frac{5\pi}{18}\right)$

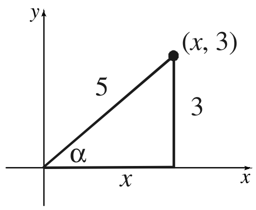
$$= \sin\frac{6\pi}{18}$$

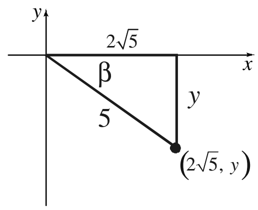
$$= \sin\frac{\pi}{3}$$

$$= \frac{\sqrt{3}}{2}$$

33. $\sin\alpha = \frac{3}{5},\ 0 < \alpha < \frac{\pi}{2}$

$\cos\beta = \frac{2\sqrt{5}}{5},\ -\frac{\pi}{2} < \beta < 0$





$$x^2 + 3^2 = 5^2,\ x > 0$$
$$x^2 = 25 - 9 = 16,\ x > 0$$
$$x = 4$$
$$\cos\alpha = \frac{4}{5}, \qquad \tan\alpha = \frac{3}{4}$$

$$\left(2\sqrt{5}\right)^2 + y^2 = 5^2,\ y < 0$$
$$y^2 = 25 - 20 = 5,\ y < 0$$
$$y = -\sqrt{5}$$
$$\sin\beta = -\frac{\sqrt{5}}{5}, \quad \tan\beta = \frac{-\sqrt{5}}{2\sqrt{5}} = -\frac{1}{2}$$

a. $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$$= \frac{3}{5}\cdot\frac{2\sqrt{5}}{5} + \frac{4}{5}\cdot\left(-\frac{\sqrt{5}}{5}\right)$$
$$= \frac{6\sqrt{5} - 4\sqrt{5}}{25}$$
$$= \frac{2\sqrt{5}}{25}$$

b. $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$$= \frac{4}{5}\cdot\frac{2\sqrt{5}}{5} - \frac{3}{5}\cdot\left(-\frac{\sqrt{5}}{5}\right)$$
$$= \frac{8\sqrt{5} + 3\sqrt{5}}{25}$$
$$= \frac{11\sqrt{5}}{25}$$

c. $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

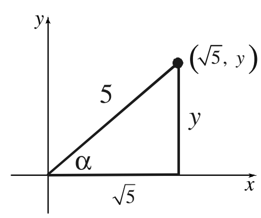
$$= \frac{3}{5}\cdot\frac{2\sqrt{5}}{5} - \frac{4}{5}\cdot\left(-\frac{\sqrt{5}}{5}\right)$$
$$= \frac{6\sqrt{5} + 4\sqrt{5}}{25}$$
$$= \frac{10\sqrt{5}}{25}$$
$$= \frac{2\sqrt{5}}{5}$$

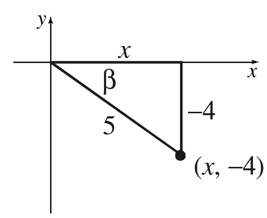
d. $\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\cdot\tan\beta}$

$$= \frac{\frac{3}{4} - \left(-\frac{1}{2}\right)}{1 + \left(\frac{3}{4}\right)\left(-\frac{1}{2}\right)}$$
$$= \frac{\frac{5}{4}}{\frac{5}{8}}$$
$$= 2$$

34. $\cos\alpha = \frac{\sqrt{5}}{5},\ 0 < \alpha < \frac{\pi}{2}$

$\sin\beta = -\frac{4}{5},\ -\frac{\pi}{2} < \beta < 0$





$$\left(\sqrt{5}\right)^2 + y^2 = 5^2,\ y > 0$$
$$y^2 = 25 - 5 = 20,\ y > 0$$
$$y = \sqrt{20} = 2\sqrt{5}$$
$$\sin\alpha = \frac{2\sqrt{5}}{5}, \qquad \tan\alpha = \frac{2\sqrt{5}}{\sqrt{5}} = 2$$

$$x^2 + (-4)^2 = 5^2,\ x > 0$$
$$x^2 = 25 - 16 = 9,\ x > 0$$
$$x = 3$$
$$\cos\beta = \frac{3}{5}, \qquad \tan\beta = \frac{-4}{3} = -\frac{4}{3}$$

a. $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$$= \left(\frac{2\sqrt{5}}{5}\right)\cdot\left(\frac{3}{5}\right) + \left(\frac{\sqrt{5}}{5}\right)\cdot\left(-\frac{4}{5}\right)$$
$$= \frac{6\sqrt{5}-4\sqrt{5}}{25}$$
$$= \frac{2\sqrt{5}}{25}$$

b. $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$$= \left(\frac{\sqrt{5}}{5}\right)\cdot\left(\frac{3}{5}\right) - \left(\frac{2\sqrt{5}}{5}\right)\cdot\left(-\frac{4}{5}\right)$$
$$= \frac{3\sqrt{5}+8\sqrt{5}}{25}$$
$$= \frac{11\sqrt{5}}{25}$$

c. $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

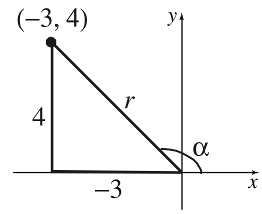
$$= \left(\frac{2\sqrt{5}}{5}\right)\cdot\left(\frac{3}{5}\right) - \left(\frac{\sqrt{5}}{5}\right)\cdot\left(-\frac{4}{5}\right)$$
$$= \frac{6\sqrt{5}+4\sqrt{5}}{25}$$
$$= \frac{10\sqrt{5}}{25}$$
$$= \frac{2\sqrt{5}}{5}$$

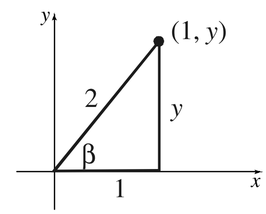
d. $\tan(\alpha-\beta) = \dfrac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$

$$= \frac{2-\left(-\frac{4}{3}\right)}{1+2\cdot\left(-\frac{4}{3}\right)}$$
$$= \frac{\frac{10}{3}}{-\frac{5}{3}}$$
$$= -2$$

35. $\tan\alpha = -\frac{4}{3},\ \frac{\pi}{2} < \alpha < \pi$

$\cos\beta = \frac{1}{2},\ 0 < \beta < \frac{\pi}{2}$





$$r^2 = (-3)^2 + 4^2 = 25$$
$$r = 5$$

$\sin\alpha = \frac{4}{5}$, $\qquad \cos\alpha = \frac{-3}{5} = -\frac{3}{5}$

$$1^2 + y^2 = 2^2,\ \ y > 0$$
$$y^2 = 4 - 1 = 3,\ \ y > 0$$
$$y = \sqrt{3}$$

$\sin\beta = \frac{\sqrt{3}}{2}$, $\qquad \tan\beta = \frac{\sqrt{3}}{1} = \sqrt{3}$

a. $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$$= \left(\frac{4}{5}\right)\cdot\left(\frac{1}{2}\right) + \left(-\frac{3}{5}\right)\cdot\left(\frac{\sqrt{3}}{2}\right)$$
$$= \frac{4-3\sqrt{3}}{10}$$

b. $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$$= \left(-\frac{3}{5}\right)\cdot\left(\frac{1}{2}\right) - \left(\frac{4}{5}\right)\cdot\left(\frac{\sqrt{3}}{2}\right)$$
$$= \frac{-3-4\sqrt{3}}{10}$$

c. $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$$= \left(\frac{4}{5}\right)\cdot\left(\frac{1}{2}\right) - \left(-\frac{3}{5}\right)\cdot\left(\frac{\sqrt{3}}{2}\right)$$
$$= \frac{4+3\sqrt{3}}{10}$$

d. $\tan(\alpha-\beta)=\dfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$

$$=\frac{-\frac{4}{3}-\sqrt{3}}{1+\left(-\frac{4}{3}\right)\cdot\sqrt{3}}$$

$$=\frac{\frac{-4-3\sqrt{3}}{3}}{\frac{3-4\sqrt{3}}{3}}$$

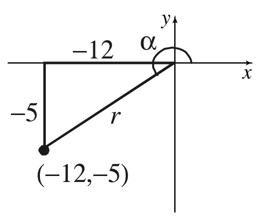
$$=\left(\frac{-4-3\sqrt{3}}{3-4\sqrt{3}}\right)\cdot\left(\frac{3+4\sqrt{3}}{3+4\sqrt{3}}\right)$$

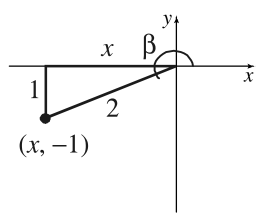
$$=\frac{-48-25\sqrt{3}}{-39}$$

$$=\frac{25\sqrt{3}+48}{39}$$

36. $\tan\alpha=\dfrac{5}{12},\ \pi<\alpha<\dfrac{3\pi}{2}$

$\sin\beta=-\dfrac{1}{2},\ \pi<\beta<\dfrac{3\pi}{2}$





$$r^2=(-12)^2+(-5)^2=169$$
$$r=13$$

$\sin\alpha=\dfrac{-5}{13}=-\dfrac{5}{13},\ \cos\alpha=\dfrac{-12}{13}=-\dfrac{12}{13}$

$x^2+(-1)^2=2^2,\ x<0$

$x^2=4-1=3,\ x<0$

$x=-\sqrt{3}$

$\cos\beta=-\dfrac{\sqrt{3}}{2},\qquad \tan\beta=\dfrac{-1}{-\sqrt{3}}=\dfrac{\sqrt{3}}{3}$

a. $\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$

$$=\left(-\frac{5}{13}\right)\cdot\left(-\frac{\sqrt{3}}{2}\right)+\left(-\frac{12}{13}\right)\cdot\left(-\frac{1}{2}\right)$$

$$=\frac{5\sqrt{3}+12}{26}=\frac{12+5\sqrt{3}}{26}$$

b. $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$

$$=\left(-\frac{12}{13}\right)\cdot\left(-\frac{\sqrt{3}}{2}\right)-\left(-\frac{5}{13}\right)\cdot\left(-\frac{1}{2}\right)$$

$$=\frac{12\sqrt{3}-5}{26}$$

c. $\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$

$$=\left(-\frac{5}{13}\right)\cdot\left(-\frac{\sqrt{3}}{2}\right)-\left(-\frac{12}{13}\right)\cdot\left(-\frac{1}{2}\right)$$

$$=\frac{5\sqrt{3}-12}{26}$$

d. $\tan(\alpha-\beta)=\dfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\cdot\tan\beta}$

$$=\frac{\frac{5}{12}-\frac{\sqrt{3}}{3}}{1+\frac{5}{12}\cdot\frac{\sqrt{3}}{3}}=\frac{\frac{5-4\sqrt{3}}{12}}{\frac{36+5\sqrt{3}}{36}}$$

$$=\left(\frac{15-12\sqrt{3}}{36+5\sqrt{3}}\right)\cdot\left(\frac{36-5\sqrt{3}}{36-5\sqrt{3}}\right)$$

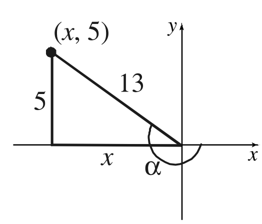
$$=\frac{540-507\sqrt{3}+180}{1296-75}$$

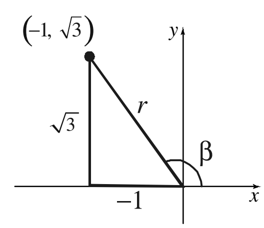
$$=\frac{720-507\sqrt{3}}{1221}$$

$$=\frac{240-169\sqrt{3}}{407}$$

37. $\sin\alpha=\dfrac{5}{13},\ -\dfrac{3\pi}{2}<\alpha<-\pi$

$\tan\beta=-\sqrt{3},\ \dfrac{\pi}{2}<\beta<\pi$





$x^2+5^2=13^2,\ x<0$

$x^2=169-25=144,\ x<0$

$x=-12$

$\cos\alpha=\dfrac{-12}{13}=-\dfrac{12}{13},\qquad \tan\alpha=-\dfrac{5}{12}$

$$r^2 = (-1)^2 + \sqrt{3}^2 = 4$$
$$r = 2$$
$$\sin\beta = \frac{\sqrt{3}}{2}, \quad \cos\beta = \frac{-1}{2} = -\frac{1}{2}$$

a. $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
$$= \left(\frac{5}{13}\right)\cdot\left(-\frac{1}{2}\right) + \left(-\frac{12}{13}\right)\cdot\left(\frac{\sqrt{3}}{2}\right)$$
$$= \frac{-5-12\sqrt{3}}{26} \text{ or } -\frac{5+12\sqrt{3}}{26}$$

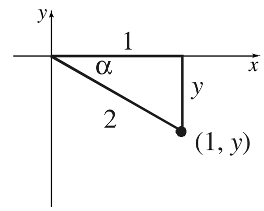
b. $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
$$= \left(-\frac{12}{13}\right)\cdot\left(-\frac{1}{2}\right) - \left(\frac{5}{13}\right)\cdot\left(\frac{\sqrt{3}}{2}\right)$$
$$= \frac{12-5\sqrt{3}}{26}$$

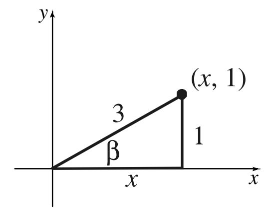
c. $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$
$$= \left(\frac{5}{13}\right)\cdot\left(-\frac{1}{2}\right) - \left(-\frac{12}{13}\right)\cdot\left(\frac{\sqrt{3}}{2}\right)$$
$$= \frac{-5+12\sqrt{3}}{26}$$

d. $\tan(\alpha-\beta) = \dfrac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$
$$= \frac{-\frac{5}{12} - \left(-\sqrt{3}\right)}{1 + \left(-\frac{5}{12}\right)\cdot\left(-\sqrt{3}\right)}$$
$$= \frac{\frac{-5+12\sqrt{3}}{12}}{\frac{12+5\sqrt{3}}{12}}$$
$$= \left(\frac{-5+12\sqrt{3}}{12+5\sqrt{3}}\right)\cdot\left(\frac{12-5\sqrt{3}}{12-5\sqrt{3}}\right)$$
$$= \frac{-240+169\sqrt{3}}{69}$$

38. $\cos\alpha = \dfrac{1}{2}, \ -\dfrac{\pi}{2} < \alpha < 0$

$\sin\beta = \dfrac{1}{3}, \ 0 < \beta < \dfrac{\pi}{2}$





$$1^2 + y^2 = 2^2, \ y < 0$$
$$y^2 = 4 - 1 = 3, \ y < 0$$
$$y = -\sqrt{3}$$
$$\sin\alpha = \frac{-\sqrt{3}}{2} = -\frac{\sqrt{3}}{2}, \quad \tan\alpha = \frac{-\sqrt{3}}{1} = -\sqrt{3}$$
$$x^2 + 1^2 = 3^2, \ x > 0$$
$$x^2 = 9 - 1 = 8. \ x > 0$$
$$x = \sqrt{8} = 2\sqrt{2}$$
$$\cos\beta = \frac{2\sqrt{2}}{3}, \qquad \tan\beta = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$$

a. $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
$$= \left(-\frac{\sqrt{3}}{2}\right)\cdot\left(\frac{2\sqrt{2}}{3}\right) + \left(\frac{1}{2}\right)\cdot\left(\frac{1}{3}\right)$$
$$= \frac{1-2\sqrt{6}}{6}$$

b. $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
$$= \left(\frac{1}{2}\right)\cdot\left(\frac{2\sqrt{2}}{3}\right) - \left(-\frac{\sqrt{3}}{2}\right)\cdot\left(\frac{1}{3}\right)$$
$$= \frac{\sqrt{3}+2\sqrt{2}}{6}$$

c. $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$
$$= \left(-\frac{\sqrt{3}}{2}\right)\cdot\left(\frac{2\sqrt{2}}{3}\right) - \left(\frac{1}{2}\right)\cdot\left(\frac{1}{3}\right)$$
$$= \frac{-1-2\sqrt{6}}{6}$$

d. $\tan(\alpha-\beta)=\dfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$

$=\dfrac{-\sqrt{3}-\frac{\sqrt{2}}{4}}{1+\left(-\sqrt{3}\right)\cdot\frac{\sqrt{2}}{4}}$

$=\dfrac{\frac{-4\sqrt{3}-\sqrt{2}}{4}}{\frac{4-\sqrt{6}}{4}}$

$=\left(\dfrac{-4\sqrt{3}-\sqrt{2}}{4-\sqrt{6}}\right)\cdot\left(\dfrac{4+\sqrt{6}}{4+\sqrt{6}}\right)$

$=\dfrac{-16\sqrt{3}-4\sqrt{2}-4\sqrt{18}-\sqrt{12}}{16-6}$

$=\dfrac{-18\sqrt{3}-16\sqrt{2}}{10}$

$=\dfrac{-9\sqrt{3}-8\sqrt{2}}{5}$

d. $\tan\left(\theta+\dfrac{\pi}{4}\right)=\dfrac{\tan\theta+\tan\frac{\pi}{4}}{1-\tan\theta\cdot\tan\frac{\pi}{4}}$

$=\dfrac{-\frac{1}{2\sqrt{2}}+1}{1-\left(-\frac{1}{2\sqrt{2}}\right)\cdot 1}$

$=\dfrac{\frac{-1+2\sqrt{2}}{2\sqrt{2}}}{\frac{2\sqrt{2}+1}{2\sqrt{2}}}$

$=\left(\dfrac{2\sqrt{2}-1}{2\sqrt{2}+1}\right)\cdot\left(\dfrac{2\sqrt{2}-1}{2\sqrt{2}-1}\right)$

$=\dfrac{8-4\sqrt{2}+1}{8-1}$

$=\dfrac{9-4\sqrt{2}}{7}$

39. $\sin\theta=\dfrac{1}{3}$, θ in quadrant II

a. $\cos\theta=-\sqrt{1-\sin^2\theta}=-\sqrt{1-\left(\dfrac{1}{3}\right)^2}$

$=-\sqrt{1-\dfrac{1}{9}}$

$=-\sqrt{\dfrac{8}{9}}$

$=-\dfrac{2\sqrt{2}}{3}$

b. $\sin\left(\theta+\dfrac{\pi}{6}\right)=\sin\theta\cdot\cos\dfrac{\pi}{6}+\cos\theta\cdot\sin\dfrac{\pi}{6}$

$=\left(\dfrac{1}{3}\right)\left(\dfrac{\sqrt{3}}{2}\right)+\left(-\dfrac{2\sqrt{2}}{3}\right)\left(\dfrac{1}{2}\right)$

$=\dfrac{\sqrt{3}-2\sqrt{2}}{6}=\dfrac{-2\sqrt{2}+\sqrt{3}}{6}$

c. $\cos\left(\theta-\dfrac{\pi}{3}\right)=\cos\theta\cdot\cos\dfrac{\pi}{3}+\sin\theta\cdot\sin\dfrac{\pi}{3}$

$=\left(-\dfrac{2\sqrt{2}}{3}\right)\left(\dfrac{1}{2}\right)+\left(\dfrac{1}{3}\right)\left(\dfrac{\sqrt{3}}{2}\right)$

$=\dfrac{-2\sqrt{2}+\sqrt{3}}{6}$

40. $\cos\theta=\dfrac{1}{4}$, θ in quadrant IV

a. $\sin\theta=-\sqrt{1-\cos^2\theta}$

$=-\sqrt{1-\left(\dfrac{1}{4}\right)^2}$

$=-\sqrt{1-\dfrac{1}{16}}$

$=-\sqrt{\dfrac{15}{16}}$

$=-\dfrac{\sqrt{15}}{4}$

b. $\sin\left(\theta-\dfrac{\pi}{6}\right)=\sin\theta\cdot\cos\dfrac{\pi}{6}-\cos\theta\cdot\sin\dfrac{\pi}{6}$

$=\left(-\dfrac{\sqrt{15}}{4}\right)\cdot\left(\dfrac{\sqrt{3}}{2}\right)-\left(\dfrac{1}{4}\right)\cdot\left(\dfrac{1}{2}\right)$

$=\dfrac{-1-3\sqrt{5}}{8}$

c. $\cos\left(\theta+\dfrac{\pi}{3}\right)=\cos\theta\cdot\cos\dfrac{\pi}{3}-\sin\theta\cdot\sin\dfrac{\pi}{3}$

$=\left(\dfrac{1}{4}\right)\cdot\left(\dfrac{1}{2}\right)-\left(-\dfrac{\sqrt{15}}{4}\right)\cdot\left(\dfrac{\sqrt{3}}{2}\right)$

$=\dfrac{1+3\sqrt{5}}{8}$

d. $\tan\left(\theta-\frac{\pi}{4}\right)=\dfrac{\tan\theta-\tan\frac{\pi}{4}}{1+\tan\theta\cdot\tan\frac{\pi}{4}}$

$$=\frac{-\sqrt{15}-1}{1+\left(-\sqrt{15}\right)\cdot 1}$$
$$=\left(\frac{-1-\sqrt{15}}{1-\sqrt{15}}\right)\cdot\left(\frac{1+\sqrt{15}}{1+\sqrt{15}}\right)$$
$$=\frac{-1-2\sqrt{15}-15}{1-15}$$
$$=\frac{-16-2\sqrt{15}}{-14}$$
$$=\frac{8+\sqrt{15}}{7}$$

41. α lies in quadrant I. Since $x^2+y^2=4$, $r=\sqrt{4}=2$. Now, $(x,1)$ is on the circle, so

$$x^2+1^2=4$$
$$x^2=4-1^2$$
$$x=\sqrt{4-1^2}=\sqrt{3}$$

Thus, $\sin\alpha=\frac{y}{r}=\frac{1}{2}$ and $\cos\alpha=\frac{x}{r}=\frac{\sqrt{3}}{2}$.

β lies in quadrant IV. Since $x^2+y^2=1$, $r=\sqrt{1}=1$. Now, $\left(\frac{1}{3},\ y\right)$ is on the circle, so

$$\left(\frac{1}{3}\right)^2+y^2=1$$
$$y^2=1-\left(\frac{1}{3}\right)^2$$
$$y=-\sqrt{1-\left(\frac{1}{3}\right)^2}=-\sqrt{\frac{8}{9}}=-\frac{2\sqrt{2}}{3}$$

Thus, $\sin\beta=\frac{y}{r}=\frac{\frac{-2\sqrt{2}}{3}}{1}=\frac{-2\sqrt{2}}{3}$ and

$\cos\beta=\frac{x}{r}=\frac{\frac{1}{3}}{1}=\frac{1}{3}$. Thus,

$$f(\alpha+\beta)=\sin(\alpha+\beta)$$
$$=\sin\alpha\cdot\cos\beta+\cos\alpha\cdot\sin\beta$$
$$=\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)+\left(\frac{\sqrt{3}}{2}\right)\left(-\frac{2\sqrt{2}}{3}\right)$$
$$=\frac{1}{6}-\frac{2\sqrt{6}}{6}=\frac{1-2\sqrt{6}}{6}$$

42. From the solution to Problem 41, we have $\sin\alpha=\frac{1}{2}$, $\cos\alpha=\frac{\sqrt{3}}{2}$, $\sin\beta=\frac{-2\sqrt{2}}{3}$, and $\cos\beta=\frac{1}{3}$. Thus,

$$g(\alpha+\beta)=\cos(\alpha+\beta)$$
$$=\cos\alpha\cdot\cos\beta-\sin\alpha\cdot\sin\beta$$
$$=\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{3}\right)-\left(\frac{1}{2}\right)\left(-\frac{2\sqrt{2}}{3}\right)$$
$$=\frac{\sqrt{3}}{6}+\frac{2\sqrt{2}}{6}=\frac{\sqrt{3}+2\sqrt{2}}{6}$$

43. From the solution to Problem 41, we have $\sin\alpha=\frac{1}{2}$, $\cos\alpha=\frac{\sqrt{3}}{2}$, $\sin\beta=\frac{-2\sqrt{2}}{3}$, and $\cos\beta=\frac{1}{3}$. Thus,

$$g(\alpha-\beta)=\cos(\alpha-\beta)$$
$$=\cos\alpha\cdot\cos\beta+\sin\alpha\cdot\sin\beta$$
$$=\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{3}\right)+\left(\frac{1}{2}\right)\left(-\frac{2\sqrt{2}}{3}\right)$$
$$=\frac{\sqrt{3}}{6}-\frac{2\sqrt{2}}{6}=\frac{\sqrt{3}-2\sqrt{2}}{6}$$

44. From the solution to Problem 41, we have $\sin\alpha=\frac{1}{2}$, $\cos\alpha=\frac{\sqrt{3}}{2}$, $\sin\beta=\frac{-2\sqrt{2}}{3}$, and $\cos\beta=\frac{1}{3}$. Thus,

$$f(\alpha-\beta)=\sin(\alpha-\beta)$$
$$=\sin\alpha\cdot\cos\beta-\cos\alpha\cdot\sin\beta$$
$$=\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)-\left(\frac{\sqrt{3}}{2}\right)\left(-\frac{2\sqrt{2}}{3}\right)$$
$$=\frac{1}{6}+\frac{2\sqrt{6}}{6}=\frac{1+2\sqrt{6}}{6}$$

45. From the solution to Problem 41, we have $\sin\alpha=\frac{1}{2}$, $\cos\alpha=\frac{\sqrt{3}}{2}$, $\sin\beta=\frac{-2\sqrt{2}}{3}$, and $\cos\beta=\frac{1}{3}$. Thus,

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \text{ and}$$

$$\tan\beta = \frac{\sin\beta}{\cos\beta} = \frac{-\frac{2\sqrt{2}}{3}}{\frac{1}{3}} = -2\sqrt{2}. \text{ Finally,}$$

$$\begin{aligned}
h(\alpha+\beta) = \tan(\alpha+\beta) &= \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} \\
&= \frac{\frac{\sqrt{3}}{3}+\left(-2\sqrt{2}\right)}{1-\frac{\sqrt{3}}{3}\left(-2\sqrt{2}\right)} \\
&= \frac{\frac{\sqrt{3}}{3}-2\sqrt{2}}{1+\frac{2\sqrt{6}}{3}}\cdot\frac{3}{3} \\
&= \frac{\sqrt{3}-6\sqrt{2}}{3+2\sqrt{6}}\cdot\frac{3-2\sqrt{6}}{3-2\sqrt{6}} \\
&= \frac{3\sqrt{3}-6\sqrt{2}-18\sqrt{2}+24\sqrt{3}}{9-6\sqrt{6}+6\sqrt{6}-24} \\
&= \frac{27\sqrt{3}-24\sqrt{2}}{-15} = \frac{8\sqrt{2}-9\sqrt{3}}{5}
\end{aligned}$$

46. From the solution to Problem 45, we have $\tan\alpha = \frac{\sqrt{3}}{3}$ and $\tan\beta = -2\sqrt{2}$. Thus,

$$\begin{aligned}
h(\alpha-\beta) = \tan(\alpha-\beta) &= \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta} \\
&= \frac{\frac{\sqrt{3}}{3}-\left(-2\sqrt{2}\right)}{1+\frac{\sqrt{3}}{3}\left(-2\sqrt{2}\right)} \\
&= \frac{\frac{\sqrt{3}}{3}+2\sqrt{2}}{1-\frac{2\sqrt{6}}{3}}\cdot\frac{3}{3} \\
&= \frac{\sqrt{3}+6\sqrt{2}}{3-2\sqrt{6}}\cdot\frac{3+2\sqrt{6}}{3+2\sqrt{6}} \\
&= \frac{3\sqrt{3}+6\sqrt{2}+18\sqrt{2}+24\sqrt{3}}{9+6\sqrt{6}-6\sqrt{6}-24} \\
&= \frac{27\sqrt{3}+24\sqrt{2}}{-15} = -\frac{8\sqrt{2}+9\sqrt{3}}{5}
\end{aligned}$$

47. $$\begin{aligned}
\sin\left(\frac{\pi}{2}+\theta\right) &= \sin\frac{\pi}{2}\cdot\cos\theta+\cos\frac{\pi}{2}\cdot\sin\theta \\
&= 1\cdot\cos\theta+0\cdot\sin\theta \\
&= \cos\theta
\end{aligned}$$

48. $$\begin{aligned}
\cos\left(\frac{\pi}{2}+\theta\right) &= \cos\frac{\pi}{2}\cdot\cos\theta-\sin\frac{\pi}{2}\cdot\sin\theta \\
&= 0\cdot\cos\theta-1\cdot\sin\theta \\
&= -\sin\theta
\end{aligned}$$

49. $$\begin{aligned}
\sin(\pi-\theta) &= \sin\pi\cdot\cos\theta-\cos\pi\cdot\sin\theta \\
&= 0\cdot\cos\theta-(-1)\sin\theta \\
&= \sin\theta
\end{aligned}$$

50. $$\begin{aligned}
\cos(\pi-\theta) &= \cos\pi\cdot\cos\theta+\sin\pi\cdot\sin\theta \\
&= -1\cdot\cos\theta+0\cdot\sin\theta \\
&= -\cos\theta
\end{aligned}$$

51. $$\begin{aligned}
\sin(\pi+\theta) &= \sin\pi\cdot\cos\theta+\cos\pi\cdot\sin\theta \\
&= 0\cdot\cos\theta+(-1)\sin\theta \\
&= -\sin\theta
\end{aligned}$$

52. $$\begin{aligned}
\cos(\pi+\theta) &= \cos\pi\cdot\cos\theta-\sin\pi\cdot\sin\theta \\
&= -1\cdot\cos\theta-0\cdot\sin\theta \\
&= -\cos\theta
\end{aligned}$$

53. $$\begin{aligned}
\tan(\pi-\theta) &= \frac{\tan\pi-\tan\theta}{1+\tan\pi\cdot\tan\theta} \\
&= \frac{0-\tan\theta}{1+0\cdot\tan\theta} \\
&= \frac{-\tan\theta}{1} \\
&= -\tan\theta
\end{aligned}$$

54. $$\begin{aligned}
\tan(2\pi-\theta) &= \frac{\tan 2\pi-\tan\theta}{1+\tan 2\pi\cdot\tan\theta} \\
&= \frac{0-\tan\theta}{1+0\cdot\tan\theta} \\
&= \frac{-\tan\theta}{1} \\
&= -\tan\theta
\end{aligned}$$

55. $$\begin{aligned}
\sin\left(\frac{3\pi}{2}+\theta\right) &= \sin\frac{3\pi}{2}\cdot\cos\theta+\cos\frac{3\pi}{2}\cdot\sin\theta \\
&= -1\cdot\cos\theta+0\cdot\sin\theta \\
&= -\cos\theta
\end{aligned}$$

56. $\cos\left(\frac{3\pi}{2}+\theta\right)=\cos\frac{3\pi}{2}\cdot\cos\theta-\sin\frac{3\pi}{2}\cdot\sin\theta$
$=0\cdot\cos\theta-(-1)\cdot\sin\theta$
$=\sin\theta$

57. $\sin(\alpha+\beta)+\sin(\alpha-\beta)$
$=\sin\alpha\cos\beta+\cos\alpha\sin\beta$
$+\sin\alpha\cos\beta-\cos\alpha\sin\beta$
$=2\sin\alpha\cos\beta$

58. $\cos(\alpha+\beta)+\cos(\alpha-\beta)$
$=\cos\alpha\cos\beta-\sin\alpha\sin\beta$
$+\cos\alpha\cos\beta+\sin\alpha\sin\beta$
$=2\cos\alpha\cos\beta$

59. $\frac{\sin(\alpha+\beta)}{\sin\alpha\cos\beta}=\frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\sin\alpha\cos\beta}$
$=\frac{\sin\alpha\cos\beta}{\sin\alpha\cos\beta}+\frac{\cos\alpha\sin\beta}{\sin\alpha\cos\beta}$
$=1+\cot\alpha\tan\beta$

60. $\frac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta}=\frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta}$
$=\frac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}+\frac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}$
$=\tan\alpha+\tan\beta$

61. $\frac{\cos(\alpha+\beta)}{\cos\alpha\cos\beta}=\frac{\cos\alpha\cos\beta-\sin\alpha\sin\beta}{\cos\alpha\cos\beta}$
$=\frac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta}-\frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}$
$=1-\tan\alpha\tan\beta$

62. $\frac{\cos(\alpha-\beta)}{\sin\alpha\cos\beta}=\frac{\cos\alpha\cos\beta+\sin\alpha\sin\beta}{\sin\alpha\cos\beta}$
$=\frac{\cos\alpha\cos\beta}{\sin\alpha\cos\beta}+\frac{\sin\alpha\sin\beta}{\sin\alpha\cos\beta}$
$=\cot\alpha+\tan\beta$

63. $\frac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)}=\frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\sin\alpha\cos\beta-\cos\alpha\sin\beta}$
$=\frac{\frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\frac{\sin\alpha\cos\beta-\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}$
$=\frac{\frac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}+\frac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\frac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}-\frac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}$
$=\frac{\tan\alpha+\tan\beta}{\tan\alpha-\tan\beta}$

64. $\frac{\cos(\alpha+\beta)}{\cos(\alpha-\beta)}=\frac{\cos\alpha\cos\beta-\sin\alpha\sin\beta}{\cos\alpha\cos\beta+\sin\alpha\sin\beta}$
$=\frac{\frac{\cos\alpha\cos\beta-\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}{\frac{\cos\alpha\cos\beta+\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}$
$=\frac{\frac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta}-\frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}{\frac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta}+\frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}$
$=\frac{1-\tan\alpha\tan\beta}{1+\tan\alpha\tan\beta}$

65. $\cot(\alpha+\beta)=\frac{\cos(\alpha+\beta)}{\sin(\alpha+\beta)}$
$=\frac{\cos\alpha\cos\beta-\sin\alpha\sin\beta}{\sin\alpha\cos\beta+\cos\alpha\sin\beta}$
$=\frac{\frac{\cos\alpha\cos\beta-\sin\alpha\sin\beta}{\sin\alpha\sin\beta}}{\frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\sin\alpha\sin\beta}}$
$=\frac{\frac{\cos\alpha\cos\beta}{\sin\alpha\sin\beta}-\frac{\sin\alpha\sin\beta}{\sin\alpha\sin\beta}}{\frac{\sin\alpha\cos\beta}{\sin\alpha\sin\beta}+\frac{\cos\alpha\sin\beta}{\sin\alpha\sin\beta}}$
$=\frac{\cot\alpha\cot\beta-1}{\cot\beta+\cot\alpha}$

66. $\cot(\alpha-\beta)=\dfrac{\cos(\alpha-\beta)}{\sin(\alpha-\beta)}$

$$=\frac{\cos\alpha\cos\beta+\sin\alpha\sin\beta}{\sin\alpha\cos\beta-\cos\alpha\sin\beta}$$

$$=\frac{\dfrac{\cos\alpha\cos\beta+\sin\alpha\sin\beta}{\sin\alpha\sin\beta}}{\dfrac{\sin\alpha\cos\beta-\cos\alpha\sin\beta}{\sin\alpha\sin\beta}}$$

$$=\frac{\dfrac{\cos\alpha\cos\beta}{\sin\alpha\sin\beta}+\dfrac{\sin\alpha\sin\beta}{\sin\alpha\sin\beta}}{\dfrac{\sin\alpha\cos\beta}{\sin\alpha\sin\beta}-\dfrac{\cos\alpha\sin\beta}{\sin\alpha\sin\beta}}$$

$$=\frac{\cot\alpha\cot\beta+1}{\cot\beta-\cot\alpha}$$

67. $\sec(\alpha+\beta)=\dfrac{1}{\cos(\alpha+\beta)}$

$$=\frac{1}{\cos\alpha\cos\beta-\sin\alpha\sin\beta}$$

$$=\frac{\dfrac{1}{\sin\alpha\sin\beta}}{\dfrac{\cos\alpha\cos\beta-\sin\alpha\sin\beta}{\sin\alpha\sin\beta}}$$

$$=\frac{\dfrac{1}{\sin\alpha}\cdot\dfrac{1}{\sin\beta}}{\dfrac{\cos\alpha\cos\beta}{\sin\alpha\sin\beta}-\dfrac{\sin\alpha\sin\beta}{\sin\alpha\sin\beta}}$$

$$=\frac{\csc\alpha\csc\beta}{\cot\alpha\cot\beta-1}$$

68. $\sec(\alpha-\beta)=\dfrac{1}{\cos(\alpha-\beta)}$

$$=\frac{1}{\cos\alpha\cos\beta+\sin\alpha\sin\beta}$$

$$=\frac{\dfrac{1}{\cos\alpha\cos\beta}}{\dfrac{\cos\alpha\cos\beta+\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}$$

$$=\frac{\dfrac{1}{\cos\alpha}\cdot\dfrac{1}{\cos\beta}}{\dfrac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta}+\dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}$$

$$=\frac{\sec\alpha\sec\beta}{1+\tan\alpha\tan\beta}$$

69. $\sin(\alpha-\beta)\sin(\alpha+\beta)$

$$=(\sin\alpha\cos\beta-\cos\alpha\sin\beta)(\sin\alpha\cos\beta+\cos\alpha\sin\beta)$$
$$=\sin^2\alpha\cos^2\beta-\cos^2\alpha\sin^2\beta$$
$$=\sin^2\alpha(1-\sin^2\beta)-(1-\sin^2\alpha)\sin^2\beta$$
$$=\sin^2\alpha-\sin^2\alpha\sin^2\beta-\sin^2\beta+\sin^2\alpha\sin^2\beta$$
$$=\sin^2\alpha-\sin^2\beta$$

70. $\cos(\alpha-\beta)\cos(\alpha+\beta)$

$$=(\cos\alpha\cos\beta+\sin\alpha\sin\beta)(\cos\alpha\cos\beta-\sin\alpha\sin\beta)$$
$$=\cos^2\alpha\cos^2\beta-\sin^2\alpha\sin^2\beta$$
$$=\cos^2\alpha(1-\sin^2\beta)-(1-\cos^2\alpha)\sin^2\beta$$
$$=\cos^2\alpha-\cos^2\alpha\sin^2\beta-\sin^2\beta+\cos^2\alpha\sin^2\beta$$
$$=\cos^2\alpha-\sin^2\beta$$

71. $\sin(\theta+k\pi)=\sin\theta\cdot\cos k\pi+\cos\theta\cdot\sin k\pi$

$$=(\sin\theta)(-1)^k+(\cos\theta)(0)$$
$$=(-1)^k\sin\theta,\quad k \text{ any integer}$$

72. $\cos(\theta+k\pi)=\cos\theta\cdot\cos k\pi-\sin\theta\cdot\sin k\pi$

$$=(\cos\theta)(-1)^k-(\sin\theta)(0)$$
$$=(-1)^k\cos\theta,\quad k \text{ any integer}$$

73. $\sin\left(\sin^{-1}\dfrac{1}{2}+\cos^{-1}0\right)=\sin\left(\dfrac{\pi}{6}+\dfrac{\pi}{2}\right)$

$$=\sin\left(\frac{2\pi}{3}\right)$$
$$=\frac{\sqrt{3}}{2}$$

74. $\sin\left(\sin^{-1}\dfrac{\sqrt{3}}{2}+\cos^{-1}1\right)=\sin\left(\dfrac{\pi}{3}+0\right)$

$$=\sin\frac{\pi}{3}$$
$$=\frac{\sqrt{3}}{2}$$

75. $\sin\left[\sin^{-1}\dfrac{3}{5}-\cos^{-1}\left(-\dfrac{4}{5}\right)\right]$

Let $\alpha=\sin^{-1}\dfrac{3}{5}$ and $\beta=\cos^{-1}\left(-\dfrac{4}{5}\right)$. α is in quadrant I; β is in quadrant II. Then $\sin\alpha=\frac{3}{5}$, $0\le\alpha\le\frac{\pi}{2}$, and $\cos\beta=-\frac{4}{5}$, $\frac{\pi}{2}\le\beta\le\pi$.

$$\cos\alpha=\sqrt{1-\sin^2\alpha}=\sqrt{1-\left(\frac{3}{5}\right)^2}=\sqrt{1-\frac{9}{25}}=\sqrt{\frac{16}{25}}=\frac{4}{5}$$

$$\sin\beta=\sqrt{1-\cos^2\beta}=\sqrt{1-\left(-\frac{4}{5}\right)^2}=\sqrt{1-\frac{16}{25}}=\sqrt{\frac{9}{25}}=\frac{3}{5}$$

$$\begin{aligned}&\sin\left[\sin^{-1}\frac{3}{5}-\cos^{-1}\left(-\frac{4}{5}\right)\right]=\sin(\alpha-\beta)\\&=\sin\alpha\cos\beta-\cos\alpha\sin\beta\\&=\left(\frac{3}{5}\right)\cdot\left(-\frac{4}{5}\right)-\left(\frac{4}{5}\right)\cdot\left(\frac{3}{5}\right)\\&=-\frac{12}{25}-\frac{12}{25}\\&=-\frac{24}{25}\end{aligned}$$

76. $\sin\left[\sin^{-1}\left(-\frac{4}{5}\right)-\tan^{-1}\frac{3}{4}\right]$

Let $\alpha=\sin^{-1}\left(-\frac{4}{5}\right)$ and $\beta=\tan^{-1}\frac{3}{4}$. α is in quadrant IV; β is in quadrant I. Then $\sin\alpha=-\frac{4}{5}$, $-\frac{\pi}{2}\le\alpha\le 0$, and $\tan\beta=\frac{3}{4}$, $0<\beta<\frac{\pi}{2}$.

$$\cos\alpha=\sqrt{1-\sin^2\alpha}=\sqrt{1-\left(-\frac{4}{5}\right)^2}=\sqrt{1-\frac{16}{25}}=\sqrt{\frac{9}{25}}=\frac{3}{5}$$

$$\sec\beta=\sqrt{1+\tan^2\beta}=\sqrt{1+\left(\frac{3}{4}\right)^2}=\sqrt{1-\frac{9}{16}}=\sqrt{\frac{25}{16}}=\frac{5}{4}$$

$$\cos\beta=\frac{4}{5}$$

$$\sin\beta=\sqrt{1-\cos^2\beta}=\sqrt{1-\left(\frac{4}{5}\right)^2}=\sqrt{1-\frac{16}{25}}=\sqrt{\frac{9}{25}}=\frac{3}{5}$$

$$\begin{aligned}&\sin\left[\sin^{-1}\left(-\frac{4}{5}\right)-\tan^{-1}\frac{3}{4}\right]\\&=\sin(\alpha-\beta)\\&=\sin\alpha\cos\beta-\cos\alpha\sin\beta\\&=\left(-\frac{4}{5}\right)\cdot\left(\frac{4}{5}\right)-\left(\frac{3}{5}\right)\cdot\left(\frac{3}{5}\right)\\&=-\frac{16}{25}-\frac{9}{25}=-\frac{25}{25}\\&=-1\end{aligned}$$

77. $\cos\left(\tan^{-1}\frac{4}{3}+\cos^{-1}\frac{5}{13}\right)$

Let $\alpha=\tan^{-1}\frac{4}{3}$ and $\beta=\cos^{-1}\frac{5}{13}$. α is in quadrant I; β is in quadrant I. Then $\tan\alpha=\frac{4}{3}$, $0<\alpha<\frac{\pi}{2}$, and $\cos\beta=\frac{5}{13}$, $0\le\beta\le\frac{\pi}{2}$.

$$\sec\alpha=\sqrt{1+\tan^2\alpha}=\sqrt{1+\left(\frac{4}{3}\right)^2}=\sqrt{1+\frac{16}{9}}=\sqrt{\frac{25}{9}}=\frac{5}{3}$$

$$\cos\alpha=\frac{3}{5}$$

$$\sin\alpha=\sqrt{1-\cos^2\alpha}=\sqrt{1-\left(\frac{3}{5}\right)^2}=\sqrt{1-\frac{9}{25}}=\sqrt{\frac{16}{25}}=\frac{4}{5}$$

$$\sin\beta=\sqrt{1-\cos^2\beta}=\sqrt{1-\left(\frac{5}{13}\right)^2}=\sqrt{1-\frac{25}{169}}=\sqrt{\frac{144}{169}}=\frac{12}{13}$$

$$\cos\left(\tan^{-1}\frac{4}{3}+\cos^{-1}\frac{5}{13}\right)$$
$$=\cos(\alpha+\beta)$$
$$=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$
$$=\left(\frac{3}{5}\right)\cdot\left(\frac{5}{13}\right)-\left(\frac{4}{5}\right)\cdot\left(\frac{12}{13}\right)$$
$$=\frac{15}{65}-\frac{48}{65}=-\frac{33}{65}$$

78. $\cos\left[\tan^{-1}\frac{5}{12}-\sin^{-1}\left(-\frac{3}{5}\right)\right]$

Let $\alpha=\tan^{-1}\frac{5}{12}$ and $\beta=\sin^{-1}\left(-\frac{3}{5}\right)$. α is in quadrant I; β is in quadrant IV. Then $\tan\alpha=\frac{5}{12}$, $0<\alpha<\frac{\pi}{2}$, and $\sin\beta=-\frac{3}{5}$, $-\frac{\pi}{2}<\alpha<0$.

$$\sec\alpha=\sqrt{1+\tan^2\alpha}$$
$$=\sqrt{1+\left(\frac{5}{12}\right)^2}=\sqrt{1+\frac{25}{144}}=\sqrt{\frac{169}{144}}=\frac{13}{12}$$
$$\cos\alpha=\frac{12}{13}$$
$$\sin\alpha=\sqrt{1-\cos^2\alpha}$$
$$=\sqrt{1-\left(\frac{12}{13}\right)^2}=\sqrt{1-\frac{144}{169}}=\sqrt{\frac{25}{169}}=\frac{5}{13}$$
$$\cos\beta=\sqrt{1-\sin^2\beta}$$
$$=\sqrt{1-\left(-\frac{3}{5}\right)^2}=\sqrt{1-\frac{9}{25}}=\sqrt{\frac{16}{25}}=\frac{4}{5}$$
$$\cos\left[\tan^{-1}\frac{5}{12}-\sin^{-1}\left(-\frac{3}{5}\right)\right]$$
$$=\cos(\alpha-\beta)$$
$$=\cos\alpha\cos\beta+\sin\alpha\sin\beta$$
$$=\left(\frac{12}{13}\right)\cdot\left(\frac{4}{5}\right)+\left(\frac{5}{13}\right)\cdot\left(-\frac{3}{5}\right)=\frac{48}{65}-\frac{15}{65}=\frac{33}{65}$$

79. $\cos\left(\sin^{-1}\frac{5}{13}-\tan^{-1}\frac{3}{4}\right)$

Let $\alpha=\sin^{-1}\frac{5}{13}$ and $\beta=\tan^{-1}\frac{3}{4}$. α is in quadrant I; β is in quadrant I. Then $\sin\alpha=\frac{5}{13}$, $0\le\alpha\le\frac{\pi}{2}$, and $\tan\beta=\frac{3}{4}$, $0<\beta<\frac{\pi}{2}$.

$$\cos\alpha=\sqrt{1-\sin^2\alpha}$$
$$=\sqrt{1-\left(\frac{5}{13}\right)^2}=\sqrt{1-\frac{25}{169}}=\sqrt{\frac{144}{169}}=\frac{12}{13}$$
$$\sec\beta=\sqrt{1+\tan^2\beta}$$
$$=\sqrt{1+\left(\frac{3}{4}\right)^2}=\sqrt{1+\frac{9}{16}}=\sqrt{\frac{25}{16}}=\frac{5}{4}$$
$$\cos\beta=\frac{4}{5}$$
$$\sin\beta=\sqrt{1-\cos^2\beta}$$
$$=\sqrt{1-\left(\frac{4}{5}\right)^2}=\sqrt{1-\frac{16}{25}}=\sqrt{\frac{9}{25}}=\frac{3}{5}$$
$$\cos\left[\sin^{-1}\frac{5}{13}-\tan^{-1}\frac{3}{4}\right]$$
$$=\cos(\alpha-\beta)$$
$$=\cos\alpha\cos\beta+\sin\alpha\sin\beta$$
$$=\frac{12}{13}\cdot\frac{4}{5}+\frac{5}{13}\cdot\frac{3}{5}$$
$$=\frac{48}{65}+\frac{15}{65}$$
$$=\frac{63}{65}$$

80. $\cos\left(\tan^{-1}\frac{4}{3}+\cos^{-1}\frac{12}{13}\right)$

Let $\alpha=\tan^{-1}\frac{4}{3}$ and $\beta=\cos^{-1}\frac{12}{13}$. α is in quadrant I; β is in quadrant I. Then $\tan\alpha=\frac{4}{3}$, $0<\alpha<\frac{\pi}{2}$, and $\cos\beta=\frac{12}{13}$, $0\le\beta\le\frac{\pi}{2}$.

$$\sec\alpha=\sqrt{1+\tan^2\alpha}$$
$$=\sqrt{1+\left(\frac{4}{3}\right)^2}=\sqrt{1+\frac{16}{9}}=\sqrt{\frac{25}{9}}=\frac{5}{3}$$
$$\cos\alpha=\frac{3}{5}$$

$$\sin\alpha = \sqrt{1-\cos^2\alpha} = \sqrt{1-\left(\frac{3}{5}\right)^2} = \sqrt{1-\frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$$

$$\sin\beta = \sqrt{1-\cos^2\beta} = \sqrt{1-\left(\frac{12}{13}\right)^2} = \sqrt{1-\frac{144}{169}} = \sqrt{\frac{25}{169}} = \frac{5}{13}$$

$$\begin{aligned}&\cos\left(\tan^{-1}\frac{4}{3}+\cos^{-1}\frac{12}{13}\right)\\ &= \cos(\alpha+\beta)\\ &= \cos\alpha\cos\beta - \sin\alpha\sin\beta\\ &= \left(\frac{3}{5}\right)\cdot\left(\frac{12}{13}\right)-\left(\frac{4}{5}\right)\cdot\left(\frac{5}{13}\right)\\ &= \frac{36}{65}-\frac{20}{65}\\ &= \frac{16}{65}\end{aligned}$$

81. $\tan\left(\sin^{-1}\frac{3}{5}+\frac{\pi}{6}\right)$

Let $\alpha = \sin^{-1}\frac{3}{5}$. α is in quadrant I. Then

$\sin\alpha = \frac{3}{5},\ 0 \le \alpha \le \frac{\pi}{2}$.

$$\cos\alpha = \sqrt{1-\sin^2\alpha} = \sqrt{1-\left(\frac{3}{5}\right)^2} = \sqrt{1-\frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$$

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{\frac{3}{5}}{\frac{4}{5}} = \frac{3}{5}\cdot\frac{5}{4} = \frac{3}{4}$$

$$\begin{aligned}\tan\left(\sin^{-1}\frac{3}{5}+\frac{\pi}{6}\right) &= \frac{\tan\left(\sin^{-1}\frac{3}{5}\right)+\tan\frac{\pi}{6}}{1-\tan\left(\sin^{-1}\frac{3}{5}\right)\cdot\tan\frac{\pi}{6}}\\ &= \frac{\frac{3}{4}+\frac{\sqrt{3}}{3}}{1-\frac{3}{4}\cdot\frac{\sqrt{3}}{3}}\\ &= \frac{\frac{9+\sqrt{3}}{12}}{\frac{12-3\sqrt{3}}{12}}\\ &= \frac{9+\sqrt{3}}{12-3\sqrt{3}}\cdot\frac{12+3\sqrt{3}}{12+3\sqrt{3}}\\ &= \frac{108+75\sqrt{3}+36}{144-27}\\ &= \frac{144+75\sqrt{3}}{117}\\ &= \frac{48+25\sqrt{3}}{39}\end{aligned}$$

82. $\tan\left(\frac{\pi}{4}-\cos^{-1}\frac{3}{5}\right)$

Let $\alpha = \cos^{-1}\frac{3}{5}$. α is in quadrant I. Then

$\cos\alpha = \frac{3}{5},\ 0 \le \alpha \le \frac{\pi}{2}$.

$$\sin\alpha = \sqrt{1-\cos^2\alpha} = \sqrt{1-\left(\frac{3}{5}\right)^2} = \sqrt{1-\frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$$

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{\frac{4}{5}}{\frac{3}{5}} = \frac{4}{5}\cdot\frac{5}{3} = \frac{4}{3}$$

$$\begin{aligned}\tan\left(\frac{\pi}{4}-\cos^{-1}\frac{3}{5}\right) &= \frac{\tan\frac{\pi}{4}-\tan\left(\cos^{-1}\frac{3}{5}\right)}{1+\tan\frac{\pi}{4}\cdot\tan\left(\cos^{-1}\frac{3}{5}\right)}\\ &= \frac{1-\frac{4}{3}}{1+1\cdot\frac{4}{3}} = \frac{-\frac{1}{3}}{\frac{7}{3}} = -\frac{1}{3}\cdot\frac{3}{7} = -\frac{1}{7}\end{aligned}$$

83. $\tan\left(\sin^{-1}\frac{4}{5}+\cos^{-1}1\right)$

Let $\alpha=\sin^{-1}\frac{4}{5}$ and $\beta=\cos^{-1}1$; α is in quadrant I. Then $\sin\alpha=\frac{4}{5}$, $0\le\alpha\le\frac{\pi}{2}$, and $\cos\beta=1$, $0\le\beta\le\pi$. So, $\beta=\cos^{-1}1=0$.

$$\cos\alpha=\sqrt{1-\sin^2\alpha}=\sqrt{1-\left(\frac{4}{5}\right)^2}=\sqrt{1-\frac{16}{25}}=\sqrt{\frac{9}{25}}=\frac{3}{5}$$

$$\tan\alpha=\frac{\sin\alpha}{\cos\alpha}=\frac{\frac{4}{5}}{\frac{3}{5}}=\frac{4}{5}\cdot\frac{5}{3}=\frac{4}{3}$$

$$\tan\left(\sin^{-1}\frac{4}{5}-\cos^{-1}1\right)=\frac{\tan\left(\sin^{-1}\frac{4}{5}\right)+\tan\left(\cos^{-1}1\right)}{1-\tan\left(\sin^{-1}\frac{4}{5}\right)\cdot\tan\left(\cos^{-1}1\right)}=\frac{\frac{4}{3}+0}{1-\frac{4}{3}\cdot 0}=\frac{\frac{4}{3}}{1}=\frac{4}{3}$$

84. $\tan\left(\cos^{-1}\frac{4}{5}+\sin^{-1}1\right)$

Let $\alpha=\cos^{-1}\frac{4}{5}$ and $\beta=\sin^{-1}1$; α is in quadrant I. Then $\cos\alpha=\frac{4}{5}$, $0\le\alpha\le\frac{\pi}{2}$, and $\sin\beta=1$, $-\frac{\pi}{2}\le\beta\le\frac{\pi}{2}$. So, $\beta=\sin^{-1}1=\frac{\pi}{2}$.

$$\sin\alpha=\sqrt{1-\cos^2\alpha}=\sqrt{1-\left(\frac{4}{5}\right)^2}=\sqrt{1-\frac{16}{25}}=\sqrt{\frac{9}{25}}=\frac{3}{5}$$

$\tan\alpha=\dfrac{\sin\alpha}{\cos\alpha}=\dfrac{\frac{3}{5}}{\frac{4}{5}}=\dfrac{3}{5}\cdot\dfrac{5}{4}=\dfrac{3}{4}$, but $\tan\frac{\pi}{2}$ is undefined. Therefore, we cannot use the sum formula for tangent. Rewriting using sine and cosine, we obtain:

$$\tan\left(\cos^{-1}\frac{4}{5}+\sin^{-1}1\right)=\frac{\sin\left(\cos^{-1}\frac{4}{5}+\sin^{-1}1\right)}{\tan\left(\cos^{-1}\frac{4}{5}+\sin^{-1}1\right)}=\frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}=\frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta}=\frac{\left(\frac{3}{5}\right)(0)+\left(\frac{4}{5}\right)(1)}{\left(\frac{4}{5}\right)(0)-\left(\frac{3}{5}\right)(1)}=\frac{\frac{4}{5}}{-\frac{3}{5}}=-\frac{4}{3}$$

85. $\cos\left(\cos^{-1}u+\sin^{-1}v\right)$

Let $\alpha=\cos^{-1}u$ and $\beta=\sin^{-1}v$.
Then $\cos\alpha=u,\ 0\le\alpha\le\pi$, and
$\sin\beta=v,\ -\frac{\pi}{2}\le\beta\le\frac{\pi}{2}$
$-1\le u\le 1,\ -1\le v\le 1$
$\sin\alpha=\sqrt{1-\cos^2\alpha}=\sqrt{1-u^2}$
$\cos\beta=\sqrt{1-\sin^2\beta}=\sqrt{1-v^2}$

$$\cos\left(\cos^{-1}u+\sin^{-1}v\right)=\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta=u\sqrt{1-v^2}-v\sqrt{1-u^2}$$

86. $\sin\left(\sin^{-1}u-\cos^{-1}v\right)$

Let $\alpha=\sin^{-1}u$ and $\beta=\cos^{-1}v$. Then
$\sin\alpha=u,\ -\frac{\pi}{2}\le\alpha\le\frac{\pi}{2}$, and
$\cos\beta=v,\ 0\le\beta\le\pi$.
$-1\le u\le 1,\ -1\le v\le 1$
$\cos\alpha=\sqrt{1-\sin^2\alpha}=\sqrt{1-u^2}$
$\sin\beta=\sqrt{1-\cos^2\beta}=\sqrt{1-v^2}$

$$\sin\left(\sin^{-1}u-\cos^{-1}v\right)=\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta=uv-\sqrt{1-u^2}\sqrt{1-v^2}$$

87. $\sin\left(\tan^{-1} u - \sin^{-1} v\right)$

Let $\alpha = \tan^{-1} u$ and $\beta = \sin^{-1} v$. Then

$\tan\alpha = u,\ -\frac{\pi}{2} < \alpha < \frac{\pi}{2}$, and

$\sin\beta = v, -\frac{\pi}{2} \le \beta \le \frac{\pi}{2}$.

$-\infty < u < \infty,\ -1 \le v \le 1$

$\sec\alpha = \sqrt{\tan^2\alpha + 1} = \sqrt{u^2+1}$

$\cos\alpha = \frac{1}{\sqrt{u^2+1}}$

$\cos\beta = \sqrt{1-\sin^2\beta} = \sqrt{1-v^2}$

$$\begin{aligned}\sin\alpha &= \sqrt{1-\cos^2\alpha}\\ &= \sqrt{1-\frac{1}{u^2+1}}\\ &= \sqrt{\frac{u^2+1-1}{u^2+1}}\\ &= \sqrt{\frac{u^2}{u^2+1}}\\ &= \frac{u}{\sqrt{u^2+1}}\end{aligned}$$

$$\begin{aligned}&\sin\left(\tan^{-1} u - \sin^{-1} v\right)\\ &= \sin(\alpha-\beta)\\ &= \sin\alpha\cos\beta - \cos\alpha\sin\beta\\ &= \frac{u}{\sqrt{u^2+1}}\cdot\sqrt{1-v^2} - \frac{1}{\sqrt{u^2+1}}\cdot v\\ &= \frac{u\sqrt{1-v^2}-v}{\sqrt{u^2+1}}\end{aligned}$$

88. $\cos\left(\tan^{-1} u + \tan^{-1} v\right)$

Let $\alpha = \tan^{-1} u$ and $\beta = \tan^{-1} v$. Then

$\tan\alpha = u,\ -\frac{\pi}{2} < \alpha < \frac{\pi}{2}$, and

$\tan\beta = v, -\frac{\pi}{2} < \beta < \frac{\pi}{2}$.

$-\infty < u < \infty,\ -\infty < v < \infty$

$\sec\alpha = \sqrt{\tan^2\alpha + 1} = \sqrt{u^2+1}$

$\cos\alpha = \frac{1}{\sqrt{u^2+1}}$

$$\begin{aligned}\sin\alpha &= \sqrt{1-\cos^2\alpha}\\ &= \sqrt{1-\frac{1}{u^2+1}}\\ &= \sqrt{\frac{u^2+1-1}{u^2+1}}\\ &= \sqrt{\frac{u^2}{u^2+1}}\\ &= \frac{u}{\sqrt{u^2+1}}\end{aligned}$$

$\sec\beta = \sqrt{\tan^2\beta + 1} = \sqrt{v^2+1}$

$\cos\beta = \frac{1}{\sqrt{v^2+1}}$

$$\begin{aligned}\sin\beta &= \sqrt{1-\cos^2\beta}\\ &= \sqrt{1-\frac{1}{v^2+1}}\\ &= \sqrt{\frac{v^2+1-1}{v^2+1}}\\ &= \sqrt{\frac{v^2}{v^2+1}}\\ &= \frac{v}{\sqrt{v^2+1}}\end{aligned}$$

$$\begin{aligned}&\cos\left(\tan^{-1} u + \tan^{-1} v\right)\\ &= \cos(\alpha+\beta)\\ &= \cos\alpha\cos\beta - \sin\alpha\sin\beta\\ &= \frac{1}{\sqrt{u^2+1}}\cdot\frac{1}{\sqrt{v^2+1}} - \frac{u}{\sqrt{u^2+1}}\cdot\frac{v}{\sqrt{v^2+1}}\\ &= \frac{1-uv}{\sqrt{u^2+1}\cdot\sqrt{v^2+1}}\end{aligned}$$

89. $\tan\left(\sin^{-1} u - \cos^{-1} v\right)$

Let $\alpha = \sin^{-1} u$ and $\beta = \cos^{-1} v$. Then

$\sin\alpha = u,\ -\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$, and

$\cos\beta = v, 0 \le \beta \le \pi$.

$-1 \le u \le 1,\ -1 \le v \le 1$

$\cos\alpha = \sqrt{1-\sin^2\alpha} = \sqrt{1-u^2}$

$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{u}{\sqrt{1-u^2}}$

$\sin\beta = \sqrt{1-\cos^2\beta} = \sqrt{1-v^2}$

$\tan\beta = \frac{\sin\beta}{\cos\beta} = \frac{\sqrt{1-v^2}}{v}$

$$\begin{aligned}\tan\left(\sin^{-1}u - \cos^{-1}v\right) &= \tan(\alpha-\beta)\\ &= \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}\\ &= \frac{\frac{u}{\sqrt{1-u^2}} - \frac{\sqrt{1-v^2}}{v}}{1+\frac{u}{\sqrt{1-u^2}}\cdot\frac{\sqrt{1-v^2}}{v}}\\ &= \frac{\frac{uv-\sqrt{1-u^2}\sqrt{1-v^2}}{v\sqrt{1-u^2}}}{\frac{v\sqrt{1-u^2}+u\sqrt{1-v^2}}{v\sqrt{1-u^2}}}\\ &= \frac{uv-\sqrt{1-u^2}\sqrt{1-v^2}}{v\sqrt{1-u^2}+u\sqrt{1-v^2}}\end{aligned}$$

90. $\sec\left(\tan^{-1}u + \cos^{-1}v\right)$

Let $\alpha = \tan^{-1}u$ and $\beta = \cos^{-1}v$. Then

$\tan\alpha = u,\ -\frac{\pi}{2} < \alpha < \frac{\pi}{2}$, and

$\cos\beta = v, 0 \le \beta \le \pi$.

$-\infty < u < \infty,\ -1 \le v \le 1$

$\sec\alpha = \sqrt{\tan^2\alpha + 1} = \sqrt{u^2+1}$

$\cos\alpha = \frac{1}{\sqrt{u^2+1}}$

$$\begin{aligned}\sin\alpha &= \sqrt{1-\cos^2\alpha}\\ &= \sqrt{1-\frac{1}{u^2+1}}\\ &= \sqrt{\frac{u^2+1-1}{u^2+1}}\\ &= \sqrt{\frac{u^2}{u^2+1}}\\ &= \frac{u}{\sqrt{u^2+1}}\end{aligned}$$

$\sin\beta = \sqrt{1-\cos^2\beta} = \sqrt{1-v^2}$

$$\begin{aligned}&\sec\left(\tan^{-1}u + \cos^{-1}v\right)\\ &= \sec(\alpha+\beta)\\ &= \frac{1}{\cos(\alpha+\beta)}\\ &= \frac{1}{\cos\alpha\cos\beta - \sin\alpha\sin\beta}\\ &= \frac{1}{\frac{1}{\sqrt{u^2+1}}\cdot v - \frac{u}{\sqrt{u^2+1}}\cdot\sqrt{1-v^2}}\\ &= \frac{1}{\frac{v}{\sqrt{u^2+1}} - \frac{u\sqrt{1-v^2}}{\sqrt{u^2+1}}}\\ &= \frac{1}{\frac{v-u\sqrt{1-v^2}}{\sqrt{u^2+1}}}\\ &= \frac{\sqrt{u^2+1}}{v-u\sqrt{1-v^2}}\end{aligned}$$

91. $\sin\theta - \sqrt{3}\cos\theta = 1$

Divide each side by 2:

$\frac{1}{2}\sin\theta - \frac{\sqrt{3}}{2}\cos\theta = \frac{1}{2}$

Rewrite in the difference of two angles form using $\cos\phi = \frac{1}{2}$, $\sin\phi = \frac{\sqrt{3}}{2}$, and $\phi = \frac{\pi}{3}$:

$$\begin{aligned}\sin\theta\cos\phi - \cos\theta\sin\phi &= \frac{1}{2}\\ \sin(\theta-\phi) &= \frac{1}{2}\end{aligned}$$

$$\begin{aligned}\theta-\phi &= \frac{\pi}{6} \quad\text{or}\quad & \theta-\phi &= \frac{5\pi}{6}\\ \theta-\frac{\pi}{3} &= \frac{\pi}{6} & \theta-\frac{\pi}{3} &= \frac{5\pi}{6}\\ \theta &= \frac{\pi}{2} & \theta &= \frac{7\pi}{6}\end{aligned}$$

The solution set is $\left\{\frac{\pi}{2}, \frac{7\pi}{6}\right\}$.

92. $\sqrt{3}\sin\theta+\cos\theta=1$

Divide each side by 2:

$$\frac{\sqrt{3}}{2}\sin\theta+\frac{1}{2}\cos\theta=\frac{1}{2}$$

Rewrite in the sum of two angles form using

$\cos\phi=\frac{\sqrt{3}}{2}$, $\sin\phi=\frac{1}{2}$, and $\phi=\frac{\pi}{6}$:

$$\sin\theta\cos\phi+\cos\theta\sin\phi=\frac{1}{2}$$

$$\sin(\theta+\phi)=\frac{1}{2}$$

$$\theta+\phi=\frac{\pi}{6} \quad\text{or}\quad \theta+\phi=\frac{5\pi}{6}$$

$$\theta+\frac{\pi}{6}=\frac{\pi}{6} \quad\text{or}\quad \theta+\frac{\pi}{6}=\frac{5\pi}{6}$$

$$\theta=0 \quad\text{or}\quad \theta=\frac{2\pi}{3}$$

The solution set is $\left\{0,\ \frac{2\pi}{3}\right\}$.

93. $\sin\theta+\cos\theta=\sqrt{2}$

Divide each side by $\sqrt{2}$:

$$\frac{1}{\sqrt{2}}\sin\theta+\frac{1}{\sqrt{2}}\cos\theta=1$$

Rewrite in the sum of two angles form using

$\cos\phi=\frac{1}{\sqrt{2}}$, $\sin\phi=\frac{1}{\sqrt{2}}$, and $\phi=\frac{\pi}{4}$:

$$\sin\theta\cos\phi+\cos\theta\sin\phi=1$$

$$\sin(\theta+\phi)=1$$

$$\theta+\phi=\frac{\pi}{2}$$

$$\theta+\frac{\pi}{4}=\frac{\pi}{2}$$

$$\theta=\frac{\pi}{4}$$

The solution set is $\left\{\frac{\pi}{4}\right\}$.

94. $\sin\theta-\cos\theta=-\sqrt{2}$

Divide each side by $\sqrt{2}$:

$$\frac{1}{\sqrt{2}}\sin\theta-\frac{1}{\sqrt{2}}\cos\theta=-1$$

Rewrite in the sum of two angles form using

$\cos\phi=\frac{1}{\sqrt{2}}$, $\sin\phi=\frac{1}{\sqrt{2}}$, and $\phi=\frac{\pi}{4}$:

$$\sin\theta\cos\phi-\sin\phi\cos\theta=-1$$

$$\sin(\theta-\phi)=-1$$

$$\theta-\phi=\frac{3\pi}{2}$$

$$\theta-\frac{\pi}{4}=\frac{3\pi}{2}$$

$$\theta=\frac{7\pi}{4}$$

The solution set is $\left\{\frac{7\pi}{4}\right\}$.

95. $\tan\theta+\sqrt{3}=\sec\theta$

$$\frac{\sin\theta}{\cos\theta}+\sqrt{3}=\frac{1}{\cos\theta}$$

$$\sin\theta+\sqrt{3}\cos\theta=1$$

$$\sin\theta+\sqrt{3}\cos\theta=1$$

Divide each side by 2:

$$\frac{1}{2}\sin\theta+\frac{\sqrt{3}}{2}\cos\theta=\frac{1}{2}$$

Rewrite in the difference of two angles form

using $\cos\phi=\frac{1}{2}$, $\sin\phi=\frac{\sqrt{3}}{2}$, and $\phi=\frac{\pi}{3}$:

$$\sin\theta\cos\phi+\cos\theta\sin\phi=\frac{1}{2}$$

$$\sin(\theta+\phi)=\frac{1}{2}$$

$$\theta+\phi=\frac{\pi}{6} \quad\text{or}\quad \theta+\phi=\frac{5\pi}{6}$$

$$\theta+\frac{\pi}{3}=\frac{\pi}{6} \qquad \theta+\frac{\pi}{3}=\frac{5\pi}{6}$$

$$\theta=-\frac{\pi}{6}=\frac{11\pi}{6} \qquad \theta=\frac{\pi}{2}$$

But since $\frac{\pi}{2}$ is not in the domain of the tangent function then the solution set is $\left\{\frac{11\pi}{6}\right\}$.

96. $\cot\theta + \csc\theta = -\sqrt{3}$

$$\frac{\cos\theta}{\sin\theta} + \frac{1}{\sin\theta} = -\sqrt{3}$$

$$\cos\theta + 1 = -\sqrt{3}\sin\theta$$

$$\sqrt{3}\sin\theta + \cos\theta = -1$$

Divide each side by 2:

$$\frac{\sqrt{3}}{2}\sin\theta + \frac{1}{2}\cos\theta = -\frac{1}{2}$$

Rewrite in the sum of two angles form using $\cos\phi = \frac{\sqrt{3}}{2}$, $\sin\phi = \frac{1}{2}$, and $\phi = \frac{\pi}{6}$:

$$\sin\theta\cos\phi + \cos\theta\sin\phi = -\frac{1}{2}$$

$$\sin(\theta + \phi) = -\frac{1}{2}$$

$$\theta + \phi = \frac{7\pi}{6} \quad \text{or} \quad \theta + \phi = \frac{11\pi}{6}$$

$$\theta + \frac{\pi}{6} = \frac{7\pi}{6} \quad \text{or} \quad \theta + \frac{\pi}{6} = \frac{11\pi}{6}$$

$$\theta = \pi \quad \text{or} \quad \theta = \frac{5\pi}{3}$$

But since π is not in the domain of the cotangent function then the solution set is $\left\{\frac{5\pi}{3}\right\}$.

97. Let $\alpha = \sin^{-1} v$ and $\beta = \cos^{-1} v$. Then $\sin\alpha = v = \cos\beta$, and since $\sin\alpha = \cos\left(\frac{\pi}{2} - \alpha\right)$, $\cos\left(\frac{\pi}{2} - \alpha\right) = \cos\beta$. If $v \geq 0$, then $0 \leq \alpha \leq \frac{\pi}{2}$, so that $\left(\frac{\pi}{2} - \alpha\right)$ and β both lie in the interval $\left[0, \frac{\pi}{2}\right]$. If $v < 0$, then $-\frac{\pi}{2} \leq \alpha < 0$, so that $\left(\frac{\pi}{2} - \alpha\right)$ and β both lie in the interval $\left(\frac{\pi}{2}, \pi\right]$. Either way, $\cos\left(\frac{\pi}{2} - \alpha\right) = \cos\beta$ implies $\frac{\pi}{2} - \alpha = \beta$, or $\alpha + \beta = \frac{\pi}{2}$. Thus, $\sin^{-1} v + \cos^{-1} v = \frac{\pi}{2}$.

98. Let $\alpha = \tan^{-1} v$ and $\beta = \cot^{-1} v$. Then $\tan\alpha = v = \cot\beta$, and since $\tan\alpha = \cot\left(\frac{\pi}{2} - \alpha\right)$, $\cot\left(\frac{\pi}{2} - \alpha\right) = \cot\beta$. If $v \geq 0$, then $0 \leq \alpha < \frac{\pi}{2}$, so that $\left(\frac{\pi}{2} - \alpha\right)$ and β both lie in the interval $\left(0, \frac{\pi}{2}\right]$. If $v < 0$, then $-\frac{\pi}{2} < \alpha < 0$, so that $\left(\frac{\pi}{2} - \alpha\right)$ and β both lie in the interval $\left(\frac{\pi}{2}, \pi\right)$. Either way, $\cot\left(\frac{\pi}{2} - \alpha\right) = \cot\beta$ implies $\frac{\pi}{2} - \alpha = \beta$, or $\alpha + \beta = \frac{\pi}{2}$. Thus, $\tan^{-1} v + \cot^{-1} v = \frac{\pi}{2}$. Note that $v \neq 0$ since $\cot^{-1} 0$ is undefined.

99. Let $\alpha = \tan^{-1}\left(\frac{1}{v}\right)$ and $\beta = \tan^{-1} v$. Because $\frac{1}{v}$ must be defined, $v \neq 0$ and so $\alpha, \beta \neq 0$. Then $\tan\alpha = \frac{1}{v} = \frac{1}{\tan\beta} = \cot\beta$, and since $\tan\alpha = \cot\left(\frac{\pi}{2} - \alpha\right)$, $\cot\left(\frac{\pi}{2} - \alpha\right) = \cot\beta$. Because $v > 0$, $0 < \alpha < \frac{\pi}{2}$ and so $\left(\frac{\pi}{2} - \alpha\right)$ and β both lie in the interval $\left(0, \frac{\pi}{2}\right)$. Then $\cot\left(\frac{\pi}{2} - \alpha\right) = \cot\beta$ implies $\frac{\pi}{2} - \alpha = \beta$ or $\alpha = \frac{\pi}{2} - \beta$. Thus,

$$\tan^{-1}\left(\frac{1}{v}\right) = \frac{\pi}{2} - \tan^{-1} v, \text{ if } v > 0.$$

100. Let $\theta = \tan^{-1} e^{-v}$. Then $\tan\theta = e^{-v}$, so $\cot\theta = \frac{1}{e^{-v}} = e^{v}$. Because $0 < \theta < \frac{\pi}{2}$, we know that $e^{-v} > 0$, which means $\cot^{-1} e^{v} = \cot^{-1}(\cot\theta) = \theta = \tan^{-1} e^{-v}$.

101. $\sin\left(\sin^{-1} v + \cos^{-1} v\right)$

$= \sin\left(\sin^{-1} v\right)\cos\left(\cos^{-1} v\right) + \cos\left(\sin^{-1} v\right)\sin\left(\cos^{-1} v\right)$

$= v \cdot v + \sqrt{1-v^2}\sqrt{1-v^2}$

$= v^2 + 1 - v^2$

$= 1$

102. $\cos\left(\sin^{-1} v + \cos^{-1} v\right)$

$= \cos\left(\sin^{-1} v\right)\cos\left(\cos^{-1} v\right) - \sin\left(\sin^{-1} v\right)\sin\left(\cos^{-1} v\right)$

$= \sqrt{1-v^2} \cdot v - v \cdot \sqrt{1-v^2}$

$= 0$

103. $\dfrac{f(x+h)-f(x)}{h}$

$= \dfrac{\sin(x+h)-\sin x}{h}$

$= \dfrac{\sin x\cos h + \cos x\sin h - \sin x}{h}$

$= \dfrac{\cos x\sin h - \sin x + \sin x\cos h}{h}$

$= \dfrac{\cos x\sin h - \sin x(1-\cos h)}{h}$

$= \cos x \cdot \dfrac{\sin h}{h} - \sin x \cdot \dfrac{1-\cos h}{h}$

104. $\dfrac{f(x+h)-f(x)}{h}$

$\dfrac{\cos(x+h)-\cos x}{h}$

$= \dfrac{\cos x\cos h - \sin x\sin h - \cos x}{h}$

$= \dfrac{-\sin x\sin h + \cos x\cos h - \cos x}{h}$

$= \dfrac{-\sin x\sin h - \cos x(1-\cos h)}{h}$

$= -\sin x \cdot \dfrac{\sin h}{h} - \cos x \cdot \dfrac{1-\cos h}{h}$

105. a. $\tan\left(\tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3\right) = \tan\left(\left(\tan^{-1}1 + \tan^{-1}2\right) + \tan^{-1}3\right) = \dfrac{\tan\left(\tan^{-1}1 + \tan^{-1}2\right) + \tan\left(\tan^{-1}3\right)}{1 - \tan\left(\tan^{-1}1 + \tan^{-1}2\right)\tan\left(\tan^{-1}3\right)}$

$$= \frac{\dfrac{\tan\left(\tan^{-1}1\right) + \tan\left(\tan^{-1}2\right)}{1-\tan\left(\tan^{-1}1\right)\tan\left(\tan^{-1}2\right)} + 3}{1 - \dfrac{\tan\left(\tan^{-1}1\right) + \tan\left(\tan^{-1}2\right)}{1-\tan\left(\tan^{-1}1\right)\tan\left(\tan^{-1}2\right)} \cdot 3} = \frac{\dfrac{1+2}{1-1\cdot 2} + 3}{1 - \dfrac{1+2}{1-1\cdot 2}\cdot 3} = \frac{\dfrac{3}{-1} + 3}{1 - \dfrac{3}{-1}\cdot 3} = \frac{-3+3}{1+9} = \frac{0}{10} = 0$$

b. From the definition of the inverse tangent function we know $0 < \tan^{-1}1 < \dfrac{\pi}{2}$, $0 < \tan^{-1}2 < \dfrac{\pi}{2}$, and $0 < \tan^{-1}3 < \dfrac{\pi}{2}$. Thus, $0 < \tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3 < \dfrac{3\pi}{2}$. On the interval $\left(0, \dfrac{3\pi}{2}\right)$, $\tan\theta = 0$ if and only if $\theta = \pi$. Therefore, from part (a), $\tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3 = \pi$.

106. $$\begin{aligned}\cos\phi\sin^2(\omega t)-\sin\phi\sin(\omega t)\cos(\omega t)&=\sin(\omega t)\big(\cos\phi\sin(\omega t)-\sin\phi\cos(\omega t)\big)\\&=\sin(\omega t)\big(\sin(\omega t)\cos\phi-\cos(\omega t)\sin\phi\big)\\&=\sin(\omega t)\sin(\omega t-\phi)\end{aligned}$$

107. Note that $\theta=\theta_2-\theta_1$.

Then $\tan\theta=\tan(\theta_2-\theta_1)=\dfrac{\tan\theta_2-\tan\theta_1}{1+\tan\theta_2\tan\theta_1}=\dfrac{m_2-m_1}{1+m_2m_1}$

108. $$\begin{aligned}&\sin(\alpha-\theta)\sin(\beta-\theta)\sin(\gamma-\theta)\\&=(\sin\alpha\cos\theta-\cos\alpha\sin\theta)(\sin\beta\cos\theta-\cos\beta\sin\theta)(\sin\gamma\cos\theta-\cos\gamma\sin\theta)\\&=\sin\theta\left(\sin\alpha\left(\frac{\cos\theta}{\sin\theta}\right)-\cos\alpha\right)\sin\theta\left(\sin\beta\left(\frac{\cos\theta}{\sin\theta}\right)-\cos\beta\right)\sin\theta\left(\sin\gamma\left(\frac{\cos\theta}{\sin\theta}\right)-\cos\gamma\right)\\&=\sin^3\theta\left(\sin\alpha\left(\frac{\cos\theta}{\sin\theta}-\frac{\cos\alpha}{\sin\alpha}\right)\right)\left(\sin\beta\left(\frac{\cos\theta}{\sin\theta}-\frac{\cos\beta}{\sin\beta}\right)\right)\left(\sin\gamma\left(\frac{\cos\theta}{\sin\theta}-\frac{\cos\gamma}{\sin\gamma}\right)\right)\\&=\sin^3\theta\big(\sin\alpha(\cot\theta-\cot\alpha)\big)\big(\sin\beta(\cot\theta-\cot\beta)\big)\big(\sin\gamma(\cot\theta-\cot\gamma)\big)\\&=\sin^3\theta\sin\alpha\sin\beta\sin\gamma(\cot\beta+\cot\gamma)(\cot\alpha+\cot\gamma)(\cot\alpha+\cot\beta)\\&=\sin^3\theta\sin\alpha\sin\beta\sin\gamma\left(\frac{\cos\beta}{\sin\beta}+\frac{\cos\gamma}{\sin\gamma}\right)\left(\frac{\cos\alpha}{\sin\alpha}+\frac{\cos\gamma}{\sin\gamma}\right)\left(\frac{\cos\alpha}{\sin\alpha}+\frac{\cos\beta}{\sin\beta}\right)\\&=\sin^3\theta\sin\alpha\sin\beta\sin\gamma\left(\frac{\sin(\gamma+\beta)}{\sin\beta\sin\gamma}\right)\left(\frac{\sin(\gamma+\alpha)}{\sin\alpha\sin\gamma}\right)\left(\frac{\sin(\beta+\alpha)}{\sin\alpha\sin\beta}\right)\\&=\sin^3\theta\sin\alpha\sin\beta\sin\gamma\left(\frac{\sin(180^\circ-\alpha)}{\sin\beta\sin\gamma}\right)\left(\frac{\sin(180^\circ-\beta)}{\sin\alpha\sin\gamma}\right)\left(\frac{\sin(180^\circ-\gamma)}{\sin\alpha\sin\beta}\right)\\&=\sin^3\theta\sin\alpha\sin\beta\sin\gamma\left(\frac{\sin\alpha}{\sin\beta\sin\gamma}\right)\left(\frac{\sin\beta}{\sin\alpha\sin\gamma}\right)\left(\frac{\sin\gamma}{\sin\alpha\sin\beta}\right)\\&=\sin^3\theta\end{aligned}$$

109. If $\tan\alpha=x+1$ and $\tan\beta=x-1$, then

$$\begin{aligned}2\cot(\alpha-\beta)&=2\cdot\frac{1}{\tan(\alpha-\beta)}\\&=\frac{2}{\dfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}}\\&=\frac{2(1+\tan\alpha\tan\beta)}{\tan\alpha-\tan\beta}\\&=\frac{2\big(1+(x+1)(x-1)\big)}{x+1-(x-1)}\\&=\frac{2\big(1+(x^2-1)\big)}{x+1-x+1}\\&=\frac{2x^2}{2}\\&=x^2\end{aligned}$$

110. The first step in the derivation,

$\tan\left(\theta+\dfrac{\pi}{2}\right)=\dfrac{\tan\theta+\tan\frac{\pi}{2}}{1-\tan\theta\cdot\tan\frac{\pi}{2}}$, is impossible

because $\tan\dfrac{\pi}{2}$ is undefined.

111. If formula (7) is used, we obtain

$\tan\left(\dfrac{\pi}{2}-\theta\right)=\dfrac{\tan\frac{\pi}{2}-\tan\theta}{1+\tan\frac{\pi}{2}\cdot\tan\theta}$. However, this is

impossible because $\tan\dfrac{\pi}{2}$ is undefined. Using formulas (3a) and (3b), we obtain

$$\tan\left(\frac{\pi}{2}-\theta\right)=\frac{\sin\left(\frac{\pi}{2}-\theta\right)}{\cos\left(\frac{\pi}{2}-\theta\right)}.$$
$$=\frac{\cos\theta}{\sin\theta}$$
$$=\cot\theta$$

Section 7.6

1. $\sin^2\theta$, $2\cos^2\theta$, $2\sin^2\theta$

2. $1-\cos\theta$

3. $\sin\theta$

4. True

5. False, only the first one is equivalent.

6. False, you cannot add the arguments or tan.

7. $\sin\theta=\frac{3}{5}$, $0<\theta<\frac{\pi}{2}$. Thus, $0<\frac{\theta}{2}<\frac{\pi}{4}$, which means $\frac{\theta}{2}$ lies in quadrant I.

$y=3,\ r=5$

$x^2+3^2=5^2,\ x>0$

$x^2=25-9=16,\ x>0$

$x=4$

So, $\cos\theta=\frac{4}{5}$ and $\tan\theta=\frac{3}{4}$.

a. $\sin(2\theta)=2\sin\theta\cos\theta=2\cdot\frac{3}{5}\cdot\frac{4}{5}=\frac{24}{25}$

b. $\cos(2\theta)=\cos^2\theta-\sin^2\theta$

$$=\left(\frac{4}{5}\right)^2-\left(\frac{3}{5}\right)^2=\frac{16}{25}-\frac{9}{25}=\frac{7}{25}$$

c. $\sin\frac{\theta}{2}=\sqrt{\frac{1-\cos\theta}{2}}$

$$=\sqrt{\frac{1-\frac{4}{5}}{2}}=\sqrt{\frac{\frac{1}{5}}{2}}=\sqrt{\frac{1}{10}}=\frac{1}{\sqrt{10}}\frac{\sqrt{10}}{\sqrt{10}}=\frac{\sqrt{10}}{10}$$

d. $\cos\frac{\theta}{2}=\sqrt{\frac{1+\cos\theta}{2}}$

$$=\sqrt{\frac{1+\frac{4}{5}}{2}}=\sqrt{\frac{\frac{9}{5}}{2}}=\sqrt{\frac{9}{10}}=\frac{3}{\sqrt{10}}\frac{\sqrt{10}}{\sqrt{10}}=\frac{3\sqrt{10}}{10}$$

e. $\tan(2\theta)=\frac{2\tan\theta}{1-\tan^2\theta}$

$$=\frac{2\left(\frac{3}{4}\right)}{1-\left(\frac{3}{4}\right)^2}=\frac{\frac{3}{2}}{1-\frac{9}{16}}=\frac{\frac{3}{2}}{\frac{7}{16}}=\frac{24}{7}$$

f. The angle is in QI so

$$\tan\left(\frac{\theta}{2}\right)=+\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}=\sqrt{\frac{1-\frac{4}{5}}{1+\frac{4}{5}}}$$
$$=\sqrt{\frac{\frac{1}{5}}{\frac{9}{5}}}=\sqrt{\frac{1}{9}}=\frac{1}{3}$$

8. $\cos\theta=\frac{3}{5}$, $0<\theta<\frac{\pi}{2}$. Thus, $0<\frac{\theta}{2}<\frac{\pi}{4}$, which means $\frac{\theta}{2}$ lies in quadrant I.

$x=3,\ r=5$

$3^2+y^2=5^2,\ y>0$

$y^2=25-9=16,\ y>0$

$y=4$

So, $\sin\theta=\frac{4}{5}$ and $\tan\theta=\frac{4}{3}$.

a. $\sin(2\theta)=2\sin\theta\cos\theta=2\cdot\frac{4}{5}\cdot\frac{3}{5}=\frac{24}{25}$

b. $\cos(2\theta)=\cos^2\theta-\sin^2\theta$

$$=\left(\frac{3}{5}\right)^2-\left(\frac{4}{5}\right)^2=\frac{9}{25}-\frac{16}{25}=-\frac{7}{25}$$

c. $\sin\frac{\theta}{2}=\sqrt{\frac{1-\cos\theta}{2}}$

$$=\sqrt{\frac{1-\frac{3}{5}}{2}}=\sqrt{\frac{\frac{2}{5}}{2}}=\sqrt{\frac{1}{5}}=\frac{1}{\sqrt{5}}\frac{\sqrt{5}}{\sqrt{5}}=\frac{\sqrt{5}}{5}$$

d. $\cos\frac{\theta}{2}=\sqrt{\frac{1+\cos\theta}{2}}$

$=\sqrt{\frac{1+\frac{3}{5}}{2}}=\sqrt{\frac{\frac{8}{5}}{2}}=\sqrt{\frac{4}{5}}=\frac{2}{\sqrt{5}}\frac{\sqrt{5}}{\sqrt{5}}=\frac{2\sqrt{5}}{5}$

e. $\tan(2\theta)=\frac{2\tan\theta}{1-\tan^2\theta}$

$=\frac{2\left(\frac{4}{3}\right)}{1-\left(\frac{4}{3}\right)^2}=\frac{\frac{8}{3}}{1-\frac{16}{9}}=\frac{\frac{8}{3}}{-\frac{7}{9}}=-\frac{24}{7}$

f. The angle is in QI so

$\tan\left(\frac{\theta}{2}\right)=+\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}=\sqrt{\frac{1-\frac{3}{5}}{1+\frac{3}{5}}}$

$=\sqrt{\frac{\frac{2}{5}}{\frac{8}{5}}}=\sqrt{\frac{1}{4}}=\frac{1}{2}$

9. $\tan\theta=\frac{4}{3},\quad \pi<\theta<\frac{3\pi}{2}$. Thus, $\frac{\pi}{2}<\frac{\theta}{2}<\frac{3\pi}{4}$,

which means $\frac{\theta}{2}$ lies in quadrant II.

$x=-3,\ \ y=-4$

$r^2=(-3)^2+(-4)^2=9+16=25$

$r=5$

$\sin\theta=-\frac{4}{5},\quad \cos\theta=-\frac{3}{5},\quad \tan\theta=\frac{4}{3}$

a. $\sin(2\theta)=2\sin\theta\cos\theta$

$=2\cdot\left(-\frac{4}{5}\right)\cdot\left(-\frac{3}{5}\right)=\frac{24}{25}$

b. $\cos(2\theta)=\cos^2\theta-\sin^2\theta$

$=\left(-\frac{3}{5}\right)^2-\left(-\frac{4}{5}\right)^2=\frac{9}{25}-\frac{16}{25}=-\frac{7}{25}$

c. $\sin\frac{\theta}{2}=\sqrt{\frac{1-\cos\theta}{2}}$

$=\sqrt{\frac{1-\left(-\frac{3}{5}\right)}{2}}$

$=\sqrt{\frac{\frac{8}{5}}{2}}=\sqrt{\frac{4}{5}}=\frac{2}{\sqrt{5}}\frac{\sqrt{5}}{\sqrt{5}}=\frac{2\sqrt{5}}{5}$

d. $\cos\frac{\theta}{2}=-\sqrt{\frac{1+\cos\theta}{2}}$

$=-\sqrt{\frac{1+\left(-\frac{3}{5}\right)}{2}}$

$=-\sqrt{\frac{\frac{2}{5}}{2}}=-\sqrt{\frac{1}{5}}=-\frac{1}{\sqrt{5}}\frac{\sqrt{5}}{\sqrt{5}}=-\frac{\sqrt{5}}{5}$

e. $\tan(2\theta)=\frac{2\tan\theta}{1-\tan^2\theta}$

$=\frac{2\left(\frac{4}{3}\right)}{1-\left(\frac{4}{3}\right)^2}=\frac{\frac{8}{3}}{1-\frac{16}{9}}=\frac{\frac{8}{3}}{-\frac{7}{9}}=-\frac{24}{7}$

f. The angle is in QII so

$\tan\left(\frac{\theta}{2}\right)=-\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}=-\sqrt{\frac{1-\left(-\frac{3}{5}\right)}{1+\left(-\frac{3}{5}\right)}}$

$=-\sqrt{\frac{\frac{8}{5}}{\frac{2}{5}}}=-\sqrt{4}=-2$

10. $\tan\theta=\frac{1}{2},\quad \pi<\theta<\frac{3\pi}{2}$. Thus, $\frac{\pi}{2}<\frac{\theta}{2}<\frac{3\pi}{4}$,

which means $\frac{\theta}{2}$ lie in quadrant II.

$x=-2,\ \ y=-1$

$r^2=(-2)^2+(-1)^2=4+1=5$

$r=\sqrt{5}$

$\sin\theta=-\frac{1}{\sqrt{5}}=-\frac{\sqrt{5}}{5},\ \ \cos\theta=-\frac{2}{\sqrt{5}}=-\frac{2\sqrt{5}}{5}$

$\tan\theta=\frac{1}{2}$

a. $\sin(2\theta)=2\sin\theta\cos\theta$

$=2\cdot\left(-\frac{\sqrt{5}}{5}\right)\cdot\left(-\frac{2\sqrt{5}}{5}\right)=\frac{4}{5}$

b. $\cos(2\theta)=\cos^2\theta-\sin^2\theta$

$=\left(-\frac{2\sqrt{5}}{5}\right)^2-\left(-\frac{\sqrt{5}}{5}\right)^2$

$=\frac{20}{25}-\frac{5}{25}=\frac{15}{25}=\frac{3}{5}$

c. $\sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}} = \sqrt{\frac{1-\left(\frac{-2\sqrt{5}}{5}\right)}{2}} = \sqrt{\frac{\frac{5+2\sqrt{5}}{5}}{2}} = \sqrt{\frac{5+2\sqrt{5}}{10}}$

d. $\cos\frac{\theta}{2} = -\sqrt{\frac{1+\cos\theta}{2}} = -\sqrt{\frac{1+\left(\frac{-2\sqrt{5}}{5}\right)}{2}} = -\sqrt{\frac{\frac{5-2\sqrt{5}}{5}}{2}} = -\sqrt{\frac{5-2\sqrt{5}}{10}}$

e. $\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2\left(\frac{1}{2}\right)}{1-\left(\frac{1}{2}\right)^2} = \frac{1}{1-\frac{1}{4}} = \frac{1}{\frac{3}{4}} = \frac{4}{3}$

f. The angle is in QII so

$$\tan\left(\frac{\theta}{2}\right) = -\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = -\sqrt{\frac{1-\left(-\frac{2}{\sqrt{5}}\right)}{1+\left(-\frac{2}{\sqrt{5}}\right)}}$$

$$= -\sqrt{\frac{\frac{\sqrt{5}+2}{\sqrt{5}}}{\frac{\sqrt{5}-2}{\sqrt{5}}}} = -\sqrt{\frac{5+2\sqrt{5}}{5-2\sqrt{5}}}$$

$$= -\sqrt{\frac{\left(5+2\sqrt{5}\right)}{\left(5-2\sqrt{5}\right)}\cdot\frac{\left(5+2\sqrt{5}\right)}{\left(5+2\sqrt{5}\right)}}$$

$$= -\sqrt{\frac{25+20\sqrt{5}+20}{25-20}} = -\sqrt{\frac{45+20\sqrt{5}}{5}}$$

$$= -\sqrt{9+4\sqrt{5}}$$

11. $\cos\theta = -\frac{\sqrt{6}}{3}$, $\frac{\pi}{2} < \theta < \pi$. Thus, $\frac{\pi}{4} < \frac{\theta}{2} < \frac{\pi}{2}$, which means $\frac{\theta}{2}$ lies in quadrant I.

$x = -\sqrt{6},\ r = 3$

$\left(-\sqrt{6}\right)^2 + y^2 = 3^2$

$y^2 = 9 - 6 = 3$

$y = \sqrt{3}$

$\sin\theta = \frac{\sqrt{3}}{3}$ and $\tan\theta = -\frac{\sqrt{2}}{2}$

a. $\sin(2\theta) = 2\sin\theta\cos\theta = 2\cdot\left(\frac{\sqrt{3}}{3}\right)\cdot\left(-\frac{\sqrt{6}}{3}\right) = -\frac{2\sqrt{18}}{9} = -\frac{6\sqrt{2}}{9} = -\frac{2\sqrt{2}}{3}$

b. $\cos(2\theta) = \cos^2\theta - \sin^2\theta = \left(-\frac{\sqrt{6}}{3}\right)^2 - \left(\frac{\sqrt{3}}{3}\right)^2 = \frac{6}{9} - \frac{3}{9} = \frac{3}{9} = \frac{1}{3}$

c. $\sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}} = \sqrt{\frac{1-\left(-\frac{\sqrt{6}}{3}\right)}{2}} = \sqrt{\frac{\frac{3+\sqrt{6}}{3}}{2}} = \sqrt{\frac{3+\sqrt{6}}{6}}$

d. $\cos\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}} = \sqrt{\frac{1+\left(-\frac{\sqrt{6}}{3}\right)}{2}} = \sqrt{\frac{\frac{3-\sqrt{6}}{3}}{2}} = \sqrt{\frac{3-\sqrt{6}}{6}}$

e. $\tan(2\theta) = \dfrac{2\tan\theta}{1-\tan^2\theta}$

$= \dfrac{2\left(-\frac{\sqrt{2}}{2}\right)}{1-\left(\frac{\sqrt{2}}{2}\right)^2} = \dfrac{-\sqrt{2}}{1-\frac{1}{2}} = -\dfrac{\sqrt{2}}{\frac{1}{2}} = -2\sqrt{2}$

f. The angle is in QI so

$\tan\left(\dfrac{\theta}{2}\right) = \sqrt{\dfrac{1-\cos\theta}{1+\cos\theta}} = \sqrt{\dfrac{1-\left(-\frac{\sqrt{6}}{3}\right)}{1+\left(-\frac{\sqrt{6}}{3}\right)}}$

$= \sqrt{\dfrac{\frac{3+\sqrt{6}}{3}}{\frac{3-\sqrt{6}}{3}}} = \sqrt{\dfrac{3+\sqrt{6}}{3-\sqrt{6}}}$

$= \sqrt{\dfrac{(3+\sqrt{6})}{(3-\sqrt{6})}\cdot\dfrac{(3+\sqrt{6})}{(3+\sqrt{6})}} = \sqrt{\dfrac{9+6\sqrt{6}+6}{9-6}}$

$= \sqrt{\dfrac{15+6\sqrt{6}}{3}} = \sqrt{5+2\sqrt{6}}$

12. $\sin\theta = -\dfrac{\sqrt{3}}{3}$, $\dfrac{3\pi}{2} < \theta < 2\pi$. Thus, $\dfrac{3\pi}{4} < \dfrac{\theta}{2} < \pi$, which means $\dfrac{\theta}{2}$ lies in quadrant II.

$y = -\sqrt{3},\ r = 3$

$x^2 + \left(-\sqrt{3}\right)^2 = 3$

$x^2 = 9-3 = 6$

$x = \sqrt{6}$

$\cos\theta = \dfrac{\sqrt{6}}{3}$ and $\tan\theta = -\dfrac{\sqrt{2}}{2}$

a. $\sin(2\theta) = 2\sin\theta\cos\theta$

$= 2\cdot\left(-\dfrac{\sqrt{3}}{3}\right)\cdot\left(\dfrac{\sqrt{6}}{3}\right)$

$= -\dfrac{2\sqrt{18}}{9} = -\dfrac{6\sqrt{2}}{9} = -\dfrac{2\sqrt{2}}{3}$

b. $\cos(2\theta) = \cos^2\theta - \sin^2\theta$

$= \left(\dfrac{\sqrt{6}}{3}\right)^2 - \left(-\dfrac{\sqrt{3}}{3}\right)^2$

$= \dfrac{6}{9} - \dfrac{3}{9} = \dfrac{3}{9} = \dfrac{1}{3}$

c. $\sin\dfrac{\theta}{2} = \sqrt{\dfrac{1-\cos\theta}{2}} = \sqrt{\dfrac{1-\frac{\sqrt{6}}{3}}{2}}$

$= \sqrt{\dfrac{\frac{3-\sqrt{6}}{3}}{2}}$

$= \sqrt{\dfrac{3-\sqrt{6}}{6}}$

d. $\cos\dfrac{\theta}{2} = -\sqrt{\dfrac{1+\cos\theta}{2}} = -\sqrt{\dfrac{1+\frac{\sqrt{6}}{3}}{2}}$

$= -\sqrt{\dfrac{\frac{3+\sqrt{6}}{3}}{2}}$

$= -\sqrt{\dfrac{3+\sqrt{6}}{6}}$

e. $\tan(2\theta) = \dfrac{2\tan\theta}{1-\tan^2\theta}$

$= \dfrac{2\left(-\frac{\sqrt{2}}{2}\right)}{1-\left(\frac{\sqrt{2}}{2}\right)^2} = \dfrac{-\sqrt{2}}{1-\frac{1}{2}} = -\dfrac{\sqrt{2}}{\frac{1}{2}} = -2\sqrt{2}$

f. The angle is in QII so

$\tan\left(\dfrac{\theta}{2}\right) = -\sqrt{\dfrac{1-\cos\theta}{1+\cos\theta}} = -\sqrt{\dfrac{1-\left(\frac{\sqrt{6}}{3}\right)}{1+\left(\frac{\sqrt{6}}{3}\right)}}$

$= -\sqrt{\dfrac{\frac{3-\sqrt{6}}{3}}{\frac{3+\sqrt{6}}{3}}} = -\sqrt{\dfrac{3-\sqrt{6}}{3+\sqrt{6}}}$

$= -\sqrt{\dfrac{(3-\sqrt{6})}{(3+\sqrt{6})}\cdot\dfrac{(3-\sqrt{6})}{(3-\sqrt{6})}}$

$= -\sqrt{\dfrac{9-6\sqrt{6}+6}{9-6}} = -\sqrt{\dfrac{15-6\sqrt{6}}{3}}$

$= -\sqrt{5-2\sqrt{6}}$

13. $\sec\theta = 3$, $\sin\theta > 0$, so $0 < \theta < \dfrac{\pi}{2}$. Thus, $0 < \dfrac{\theta}{2} < \dfrac{\pi}{4}$, which means $\dfrac{\theta}{2}$ lies in quadrant I.

$\cos\theta = \frac{1}{3}$, $x = 1$, $r = 3$.

$$1^2 + y^2 = 3^2$$
$$y^2 = 9 - 1 = 8$$
$$y = \sqrt{8} = 2\sqrt{2}$$

$\sin\theta = \frac{2\sqrt{2}}{3}$ and $\tan\theta = 2\sqrt{2}$

a. $\sin(2\theta) = 2\sin\theta\cos\theta = 2\cdot\frac{2\sqrt{2}}{3}\cdot\frac{1}{3} = \frac{4\sqrt{2}}{9}$

b. $\cos(2\theta) = \cos^2\theta - \sin^2\theta$

$$= \left(\frac{1}{3}\right)^2 - \left(\frac{2\sqrt{2}}{3}\right)^2 = \frac{1}{9} - \frac{8}{9} = -\frac{7}{9}$$

c. $\sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}}$

$$= \sqrt{\frac{1-\frac{1}{3}}{2}} = \sqrt{\frac{\frac{2}{3}}{2}} = \sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}}\frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

d. $\cos\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}}$

$$= \sqrt{\frac{1+\frac{1}{3}}{2}} = \sqrt{\frac{\frac{4}{3}}{2}} = \sqrt{\frac{2}{3}} = \frac{\sqrt{2}}{\sqrt{3}}\frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{6}}{3}$$

e. $\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$

$$= \frac{2\left(2\sqrt{2}\right)}{1-\left(2\sqrt{2}\right)^2} = \frac{4\sqrt{2}}{1-8} = -\frac{4\sqrt{2}}{7}$$

f. The angle is in QI so

$$\tan\left(\frac{\theta}{2}\right) = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \sqrt{\frac{1-\left(\frac{1}{3}\right)}{1+\left(\frac{1}{3}\right)}}$$
$$= \sqrt{\frac{\frac{2}{3}}{\frac{4}{3}}} = \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2}$$

14. $\csc\theta = -\sqrt{5}$, $\cos\theta < 0$, so $\pi < \theta < \frac{3\pi}{2}$. Thus, $\frac{\pi}{2} < \frac{\theta}{2} < \frac{3\pi}{4}$, which means $\frac{\theta}{2}$ lies in quadrant II.

$\sin\theta = \frac{-1}{\sqrt{5}} = -\frac{\sqrt{5}}{5}$, $r = \sqrt{5}$, $y = -1$

$$x^2 + (-1)^2 = \left(\sqrt{5}\right)^2$$
$$x^2 = 5 - 1 = 4$$
$$x = -2$$

$\cos\theta = \frac{-2}{\sqrt{5}} = -\frac{2\sqrt{5}}{5}$ and $\tan\theta = \frac{1}{2}$

a. $\sin(2\theta) = 2\sin\theta\cos\theta$

$$= 2\cdot\left(-\frac{\sqrt{5}}{5}\right)\cdot\left(-\frac{2\sqrt{5}}{5}\right) = \frac{4}{5}$$

b. $\cos(2\theta) = \cos^2\theta - \sin^2\theta$

$$= \left(-\frac{2\sqrt{5}}{5}\right)^2 - \left(-\frac{\sqrt{5}}{5}\right)^2$$
$$= \frac{20}{25} - \frac{5}{25} = \frac{15}{25} = \frac{3}{5}$$

c. $\sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}} = \sqrt{\frac{1-\left(-\frac{2\sqrt{5}}{5}\right)}{2}}$

$$= \sqrt{\frac{\frac{5+2\sqrt{5}}{5}}{2}}$$
$$= \sqrt{\frac{5+2\sqrt{5}}{10}}$$

d. $\cos\frac{\theta}{2} = -\sqrt{\frac{1+\cos\theta}{2}} = -\sqrt{\frac{1+\left(-\frac{2\sqrt{5}}{5}\right)}{2}}$

$$= -\sqrt{\frac{\frac{5-2\sqrt{5}}{5}}{2}}$$
$$= -\sqrt{\frac{5-2\sqrt{5}}{10}}$$

e. $\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$

$$= \frac{2\left(\frac{1}{2}\right)}{1-\left(\frac{1}{2}\right)^2} = \frac{1}{1-\frac{1}{4}} = \frac{1}{\frac{3}{4}} = \frac{4}{3}$$

f. The angle is in QII so

$$\tan\left(\frac{\theta}{2}\right) = -\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = -\sqrt{\frac{1-\left(-\frac{2}{\sqrt{5}}\right)}{1+\left(-\frac{2}{\sqrt{5}}\right)}}$$

$$= -\sqrt{\frac{\frac{\sqrt{5}+2}{\sqrt{5}}}{\frac{\sqrt{5}-2}{\sqrt{5}}}} = -\sqrt{\frac{5+2\sqrt{5}}{5-2\sqrt{5}}} = -\sqrt{9+4\sqrt{5}}$$

15. $\cot\theta = -2,\ \sec\theta < 0$, so $\frac{\pi}{2} < \theta < \pi$. Thus,

$\frac{\pi}{4} < \frac{\theta}{2} < \frac{\pi}{2}$, which means $\frac{\theta}{2}$ lies in quadrant I.

$x = -2,\ \ y = 1$

$r^2 = (-2)^2 + 1^2 = 4 + 1 = 5$

$r = \sqrt{5}$

$\sin\theta = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$,

$\cos\theta = \frac{-2}{\sqrt{5}} = -\frac{2\sqrt{5}}{5}$, $\tan\theta = -\frac{1}{2}$

a. $\sin(2\theta) = 2\sin\theta\cos\theta$

$$= 2\cdot\left(\frac{\sqrt{5}}{5}\right)\cdot\left(-\frac{2\sqrt{5}}{5}\right) = -\frac{20}{25} = -\frac{4}{5}$$

b. $\cos(2\theta) = \cos^2\theta - \sin^2\theta$

$$= \left(-\frac{2\sqrt{5}}{5}\right)^2 - \left(\frac{\sqrt{5}}{5}\right)^2$$

$$= \frac{20}{25} - \frac{5}{25} = \frac{15}{25} = \frac{3}{5}$$

c. $\sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}} = \sqrt{\frac{1-\left(-\frac{2\sqrt{5}}{5}\right)}{2}}$

$$= \sqrt{\frac{\frac{5+2\sqrt{5}}{5}}{2}}$$

$$= \sqrt{\frac{5+2\sqrt{5}}{10}}$$

d. $\cos\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}} = \sqrt{\frac{1+\left(-\frac{2\sqrt{5}}{5}\right)}{2}}$

$$= \sqrt{\frac{\frac{5-2\sqrt{5}}{5}}{2}}$$

$$= \sqrt{\frac{5-2\sqrt{5}}{10}}$$

e. $\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$

$$= \frac{2\left(-\frac{1}{2}\right)}{1-\left(-\frac{1}{2}\right)^2} = \frac{-1}{1-\frac{1}{4}} = -\frac{1}{\frac{3}{4}} = -\frac{4}{3}$$

f. The angle is in QI so

$$\tan\left(\frac{\theta}{2}\right) = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \sqrt{\frac{1-\left(-\frac{2}{\sqrt{5}}\right)}{1+\left(-\frac{2}{\sqrt{5}}\right)}}$$

$$= \sqrt{\frac{\frac{\sqrt{5}+2}{\sqrt{5}}}{\frac{\sqrt{5}-2}{\sqrt{5}}}} = \sqrt{\frac{5+2\sqrt{5}}{5-2\sqrt{5}}}$$

$$= \sqrt{\frac{\left(5+2\sqrt{5}\right)}{\left(5-2\sqrt{5}\right)}\cdot\frac{\left(5+2\sqrt{5}\right)}{\left(5+2\sqrt{5}\right)}}$$

$$= \sqrt{\frac{25+40\sqrt{5}+20}{25-20}} = \sqrt{\frac{45+40\sqrt{5}}{5}}$$

$$= \sqrt{9+4\sqrt{5}}$$

16. $\sec\theta = 2,\ \csc\theta < 0$, so $\frac{3\pi}{2} < \theta < 2\pi$. Thus, $\frac{3\pi}{4} < \frac{\theta}{2} < \pi$, which means $\frac{\theta}{2}$ lies in quadrant II.

$\cos\theta = \frac{1}{2},\ x = 1,\ \ r = 2$

$1^2 + y^2 = 2^2$

$y^2 = 4 - 1 = 3$

$y = \sqrt{3}$

$\sin\theta = -\frac{\sqrt{3}}{2}$ and $\tan\theta = -\sqrt{3}$

a. $\sin(2\theta) = 2\sin\theta\cos\theta$

$$= 2\cdot\left(-\frac{\sqrt{3}}{2}\right)\cdot\left(\frac{1}{2}\right) = -\frac{\sqrt{3}}{2}$$

b. $\cos(2\theta) = \cos^2\theta - \sin^2\theta$

$$= \left(\frac{1}{2}\right)^2 - \left(-\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4} - \frac{3}{4} = -\frac{1}{2}$$

c. $\sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}}$

$$= \sqrt{\frac{1-\frac{1}{2}}{2}} = \sqrt{\frac{\frac{1}{2}}{2}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$$

d. $\cos\frac{\theta}{2} = -\sqrt{\frac{1+\cos\theta}{2}}$

$$= -\sqrt{\frac{1+\frac{1}{2}}{2}} = -\sqrt{\frac{\frac{3}{2}}{2}} = -\sqrt{\frac{3}{4}} = -\frac{\sqrt{3}}{2}$$

e. $\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$

$$= \frac{2\left(-\sqrt{3}\right)}{1-\left(-\sqrt{3}\right)^2} = \frac{-2\sqrt{3}}{1-3} = \sqrt{3}$$

f. The angle is in QII so

$$\tan\left(\frac{\theta}{2}\right) = -\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = -\sqrt{\frac{1-\left(\frac{1}{2}\right)}{1+\left(\frac{1}{2}\right)}}$$

$$= -\sqrt{\frac{\frac{1}{2}}{\frac{3}{2}}} = -\sqrt{\frac{1}{3}} = -\frac{\sqrt{3}}{3}$$

17. $\tan\theta = -3,\ \sin\theta < 0$, so $\frac{3\pi}{2} < \theta < 2\pi$. Thus, $\frac{3\pi}{4} < \frac{\theta}{2} < \pi$, which means $\frac{\theta}{2}$ lies in quadrant II.

$x = 1,\ \ y = -3$

$r^2 = 1^2 + (-3)^2 = 1 + 9 = 10$

$r = \sqrt{10}$

$\sin\theta = \frac{-3}{\sqrt{10}} = -\frac{3\sqrt{10}}{10},\ \ \cos\theta = \frac{1}{\sqrt{10}} = \frac{\sqrt{10}}{10}$,

$\tan\theta = -3$

a. $\sin(2\theta) = 2\sin\theta\cos\theta$

$$= 2\cdot\left(-\frac{3\sqrt{10}}{10}\right)\cdot\left(\frac{\sqrt{10}}{10}\right)$$

$$= -\frac{6}{10} = -\frac{3}{5}$$

b. $\cos(2\theta) = \cos^2\theta - \sin^2\theta$

$$= \left(\frac{\sqrt{10}}{10}\right)^2 - \left(-\frac{3\sqrt{10}}{10}\right)^2$$

$$= \frac{10}{100} - \frac{90}{100} = -\frac{80}{100} = -\frac{4}{5}$$

c. $\sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}} = \sqrt{\frac{1-\frac{\sqrt{10}}{10}}{2}}$

$$= \sqrt{\frac{\frac{10-\sqrt{10}}{10}}{2}}$$

$$= \sqrt{\frac{10-\sqrt{10}}{20}}$$

$$= \frac{1}{2}\sqrt{\frac{10-\sqrt{10}}{5}}$$

d. $\cos\frac{\theta}{2} = -\sqrt{\frac{1+\cos\theta}{2}} = -\sqrt{\frac{1+\frac{\sqrt{10}}{10}}{2}}$

$= -\sqrt{\frac{\frac{10+\sqrt{10}}{10}}{2}}$

$= -\sqrt{\frac{10+\sqrt{10}}{20}}$

$= -\frac{1}{2}\sqrt{\frac{10+\sqrt{10}}{5}}$

e. $\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$

$= \frac{2(-3)}{1-(-3)^2} = \frac{-6}{1-9} = -\frac{6}{-8} = \frac{3}{4}$

f. The angle is in QII so

$\tan\left(\frac{\theta}{2}\right) = -\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = -\sqrt{\frac{1-\left(\frac{1}{\sqrt{10}}\right)}{1+\left(\frac{1}{\sqrt{10}}\right)}}$

$= -\sqrt{\frac{\sqrt{10}-1}{\sqrt{10}+1}} = -\sqrt{\frac{10-\sqrt{10}}{10+\sqrt{10}}}$

$= -\sqrt{\frac{\left(10-\sqrt{10}\right)}{\left(10+\sqrt{10}\right)}\cdot\frac{\left(10-\sqrt{10}\right)}{\left(10-\sqrt{10}\right)}}$

$= -\sqrt{\frac{100-20\sqrt{10}+10}{100-10}} = -\sqrt{\frac{110-20\sqrt{10}}{90}}$

$= -\frac{\sqrt{11-2\sqrt{10}}}{3}$

18. $\cot\theta = 3$, $\cos\theta < 0$, so $\pi < \theta < \frac{3\pi}{2}$. Thus,

$\frac{\pi}{2} < \frac{\theta}{2} < \frac{3\pi}{4}$ which means $\frac{\theta}{2}$ is in quadrant II.

$x = -3, \quad y = -1$

$r^2 = (-3)^2 + (-1)^2 = 9 + 1 = 10$

$r = \sqrt{10}$

$\sin\theta = -\frac{1}{\sqrt{10}} = -\frac{\sqrt{10}}{10}$,

$\cos\theta = -\frac{3}{\sqrt{10}} = -\frac{3\sqrt{10}}{10}$ and $\tan\theta = \frac{1}{3}$

a. $\sin(2\theta) = 2\sin\theta\cos\theta$

$= 2\cdot\left(-\frac{\sqrt{10}}{10}\right)\cdot\left(-\frac{3\sqrt{10}}{10}\right) = \frac{6}{10} = \frac{3}{5}$

b. $\cos(2\theta) = \cos^2\theta - \sin^2\theta$

$= \left(-\frac{3\sqrt{10}}{10}\right)^2 - \left(-\frac{\sqrt{10}}{10}\right)^2$

$= \frac{90}{100} - \frac{10}{100} = \frac{80}{100} = \frac{4}{5}$

c. $\sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}} = \sqrt{\frac{1-\left(-\frac{3\sqrt{10}}{10}\right)}{2}}$

$= \sqrt{\frac{\frac{10+3\sqrt{10}}{10}}{2}}$

$= \sqrt{\frac{10+3\sqrt{10}}{20}}$

$= \frac{1}{2}\sqrt{\frac{10+3\sqrt{10}}{5}}$

d. $\cos\frac{\theta}{2} = -\sqrt{\frac{1+\cos\theta}{2}} = -\sqrt{\frac{1+\left(-\frac{3\sqrt{10}}{10}\right)}{2}}$

$= -\sqrt{\frac{\frac{10-3\sqrt{10}}{10}}{2}}$

$= -\sqrt{\frac{10-3\sqrt{10}}{20}}$

$= -\frac{1}{2}\sqrt{\frac{10-3\sqrt{10}}{5}}$

e. $\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta}$

$= \frac{2\left(\frac{1}{3}\right)}{1-\left(\frac{1}{3}\right)^2} = \frac{\frac{2}{3}}{\frac{8}{9}} = \frac{3}{4}$

f. The angle is in QII so

$$\tan\left(\frac{\theta}{2}\right)=-\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}=-\sqrt{\frac{1-\left(-\frac{3}{\sqrt{10}}\right)}{1+\left(-\frac{3}{\sqrt{10}}\right)}}$$

$$=-\sqrt{\frac{\sqrt{10}+3}{\sqrt{10}-3}}=-\sqrt{\frac{10+3\sqrt{10}}{10-3\sqrt{10}}}$$

$$=-\sqrt{\frac{\left(10+3\sqrt{10}\right)}{\left(10-3\sqrt{10}\right)}\cdot\frac{\left(10+3\sqrt{10}\right)}{\left(10+3\sqrt{10}\right)}}$$

$$=-\sqrt{\frac{100+60\sqrt{10}+90}{100-90}}=-\sqrt{\frac{190+60\sqrt{10}}{10}}$$

$$=-\sqrt{19+6\sqrt{10}}$$

19. $$\sin 22.5°=\sin\left(\frac{45°}{2}\right)$$

$$=\sqrt{\frac{1-\cos 45°}{2}}$$

$$=\sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}}=\sqrt{\frac{2-\sqrt{2}}{4}}=\frac{\sqrt{2-\sqrt{2}}}{2}$$

20. $$\cos 22.5°=\cos\left(\frac{45°}{2}\right)$$

$$=\sqrt{\frac{1+\cos 45°}{2}}$$

$$=\sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}}=\sqrt{\frac{2+\sqrt{2}}{4}}=\frac{\sqrt{2+\sqrt{2}}}{2}$$

21. $$\tan\frac{7\pi}{8}=\tan\left(\frac{\frac{7\pi}{4}}{2}\right)=-\sqrt{\frac{1-\cos\frac{7\pi}{4}}{1+\cos\frac{7\pi}{4}}}$$

$$=-\sqrt{\frac{1-\frac{\sqrt{2}}{2}}{1+\frac{\sqrt{2}}{2}}\cdot\frac{2}{2}}$$

$$=-\sqrt{\left(\frac{2-\sqrt{2}}{2+\sqrt{2}}\right)\cdot\left(\frac{2-\sqrt{2}}{2-\sqrt{2}}\right)}$$

$$=-\sqrt{\frac{\left(2-\sqrt{2}\right)^2}{2}}$$

$$=-\left(\frac{2-\sqrt{2}}{\sqrt{2}}\right)$$

$$=-\left(\sqrt{2}-1\right)$$

$$=1-\sqrt{2}$$

22. $$\tan\frac{9\pi}{8}=\tan\left(\frac{\frac{9\pi}{4}}{2}\right)=\sqrt{\frac{1-\cos\frac{9\pi}{4}}{1+\cos\frac{9\pi}{4}}}$$

$$=\sqrt{\frac{1-\frac{\sqrt{2}}{2}}{1+\frac{\sqrt{2}}{2}}\cdot\frac{2}{2}}$$

$$=\sqrt{\left(\frac{2-\sqrt{2}}{2+\sqrt{2}}\right)\cdot\left(\frac{2-\sqrt{2}}{2-\sqrt{2}}\right)}$$

$$=\sqrt{\frac{\left(2-\sqrt{2}\right)^2}{2}}$$

$$=\frac{2-\sqrt{2}}{\sqrt{2}}$$

$$=\sqrt{2}-1$$

$$=-1+\sqrt{2}$$

23. $$\cos 165°=\cos\left(\frac{330°}{2}\right)$$

$$=-\sqrt{\frac{1+\cos 330°}{2}}$$

$$=-\sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}}=-\sqrt{\frac{2+\sqrt{3}}{4}}=-\frac{\sqrt{2+\sqrt{3}}}{2}$$

24. $\sin 195° = \sin\left(\frac{390°}{2}\right) = -\sqrt{\frac{1-\cos 390°}{2}}$

$= -\sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}}$

$= -\sqrt{\frac{2-\sqrt{3}}{4}}$

$= -\frac{\sqrt{2-\sqrt{3}}}{2}$

25. $\sec\frac{15\pi}{8} = \frac{1}{\cos\frac{15\pi}{8}} = \frac{1}{\cos\left(\frac{\frac{15\pi}{4}}{2}\right)}$

$= \frac{1}{\sqrt{\frac{1+\cos\frac{15\pi}{4}}{2}}}$

$= \frac{1}{\sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}}}$

$= \frac{1}{\sqrt{\frac{2+\sqrt{2}}{4}}}$

$= \frac{2}{\sqrt{2+\sqrt{2}}}$

$= \left(\frac{2}{\sqrt{2+\sqrt{2}}}\right)\cdot\left(\frac{\sqrt{2+\sqrt{2}}}{\sqrt{2+\sqrt{2}}}\right)$

$= \left(\frac{2\sqrt{2+\sqrt{2}}}{2+\sqrt{2}}\right)\cdot\left(\frac{2-\sqrt{2}}{2-\sqrt{2}}\right)$

$= \frac{2\left(2-\sqrt{2}\right)\sqrt{2+\sqrt{2}}}{2}$

$= \left(2-\sqrt{2}\right)\sqrt{2+\sqrt{2}}$

26. $\csc\frac{7\pi}{8} = \frac{1}{\sin\frac{7\pi}{8}} = \frac{1}{\sin\left(\frac{\frac{7\pi}{4}}{2}\right)}$

$= \frac{1}{\sqrt{\frac{1-\cos\frac{7\pi}{4}}{2}}}$

$= \frac{1}{\sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}}}$

$= \frac{1}{\sqrt{\frac{2-\sqrt{2}}{4}}}$

$= \frac{2}{\sqrt{2-\sqrt{2}}}$

$= \left(\frac{2}{\sqrt{2-\sqrt{2}}}\right)\cdot\left(\frac{\sqrt{2-\sqrt{2}}}{\sqrt{2-\sqrt{2}}}\right)$

$= \left(\frac{2\sqrt{2-\sqrt{2}}}{2-\sqrt{2}}\right)\cdot\left(\frac{2+\sqrt{2}}{2+\sqrt{2}}\right)$

$= \frac{2\left(2+\sqrt{2}\right)\sqrt{2-\sqrt{2}}}{2}$

$= \left(2+\sqrt{2}\right)\sqrt{2-\sqrt{2}}$

27. $\sin\left(-\frac{\pi}{8}\right) = \sin\left(\frac{-\frac{\pi}{4}}{2}\right)$

$= -\sqrt{\frac{1-\cos\left(-\frac{\pi}{4}\right)}{2}}$

$= -\sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}} = -\sqrt{\frac{2-\sqrt{2}}{4}} = -\frac{\sqrt{2-\sqrt{2}}}{2}$

28. $\cos\left(-\frac{3\pi}{8}\right) = \cos\left(\frac{-\frac{3\pi}{4}}{2}\right)$

$= \sqrt{\frac{1+\cos\left(-\frac{3\pi}{4}\right)}{2}}$

$= \sqrt{\frac{1+\left(-\frac{\sqrt{2}}{2}\right)}{2}} = \sqrt{\frac{2-\sqrt{2}}{4}} = \frac{\sqrt{2-\sqrt{2}}}{2}$

29. θ lies in quadrant II. Since $x^2+y^2=5$, $r=\sqrt{5}$. Now, the point $(a, 2)$ is on the circle, so

$$a^2+2^2=5$$
$$a^2=5-2^2$$
$$a=-\sqrt{5-2^2}=-\sqrt{1}=-1$$

(a is negative because θ lies in quadrant II.)

Thus, $\sin\theta=\frac{b}{r}=\frac{2}{\sqrt{5}}=\frac{2\sqrt{5}}{5}$ and

$\cos\theta=\frac{a}{r}=\frac{-1}{\sqrt{5}}=-\frac{\sqrt{5}}{5}$. Thus,

$$f(2\theta)=\sin(2\theta)=2\sin\theta\cos\theta$$
$$=2\cdot\left(\frac{2\sqrt{5}}{5}\right)\cdot\left(-\frac{\sqrt{5}}{5}\right)=-\frac{20}{25}=-\frac{4}{5}$$

30. From the solution to Problem 29, we have $\sin\theta=\frac{2\sqrt{5}}{5}$ and $\cos\theta=-\frac{\sqrt{5}}{5}$.

Thus, $g(2\theta)=\cos(2\theta)=\cos^2\theta-\sin^2\theta$

$$=\left(-\frac{\sqrt{5}}{5}\right)^2-\left(\frac{2\sqrt{5}}{5}\right)^2$$
$$=\frac{5}{25}-\frac{20}{25}=-\frac{15}{25}=-\frac{3}{5}$$

31. Note: Since θ lies in quadrant II, $\frac{\theta}{2}$ must lie in quadrant I. Therefore, $\cos\frac{\theta}{2}$ is positive. From the solution to Problem 29, we have $\cos\theta=-\frac{\sqrt{5}}{5}$.

Thus, $g\left(\frac{\theta}{2}\right)=\cos\frac{\theta}{2}=\sqrt{\frac{1+\cos\theta}{2}}$

$$=\sqrt{\frac{1+\left(\frac{-\sqrt{5}}{5}\right)}{2}}$$
$$=\sqrt{\frac{\frac{5-\sqrt{5}}{5}}{2}}$$
$$=\sqrt{\frac{5-\sqrt{5}}{10}}$$
$$=\frac{\sqrt{10\left(5-\sqrt{5}\right)}}{10}$$

32. Note: Since θ lies in quadrant II, $\frac{\theta}{2}$ must lie in quadrant I. Therefore, $\sin\frac{\theta}{2}$ is positive. From the solution to Problem 29, we have $\cos\theta=-\frac{\sqrt{5}}{5}$.

Thus, $f\left(\frac{\theta}{2}\right)=\sin\frac{\theta}{2}=\sqrt{\frac{1-\cos\theta}{2}}$

$$=\sqrt{\frac{1-\left(\frac{-\sqrt{5}}{5}\right)}{2}}$$
$$=\sqrt{\frac{\frac{5+\sqrt{5}}{5}}{2}}$$
$$=\sqrt{\frac{5+\sqrt{5}}{10}}$$
$$=\frac{\sqrt{10\left(5+\sqrt{5}\right)}}{10}$$

33. θ lies in quadrant II. Since $x^2+y^2=5$, $r=\sqrt{5}$. Now, the point $(a, 2)$ is on the circle, so

$$a^2+2^2=5$$
$$a^2=5-2^2$$
$$a=-\sqrt{5-2^2}=-\sqrt{1}=-1$$

(a is negative because θ lies in quadrant II.)

Thus, $\tan\theta=\frac{b}{a}=\frac{2}{-1}=-2$.

$$h(2\theta)=\tan(2\theta)$$
$$=\frac{2\tan\theta}{1-\tan^2\theta}$$
$$=\frac{2(-2)}{1-(-2)^2}=\frac{-4}{1-4}=\frac{-4}{-3}=\frac{4}{3}$$

34. From the solution to Problem 29, we have $\sin\theta = \frac{2\sqrt{5}}{5}$ and $\cos\theta = -\frac{\sqrt{5}}{5}$. Thus,

$$h\left(\frac{\theta}{2}\right) = \tan\frac{\theta}{2} = \frac{1-\cos\theta}{\sin\theta} = \frac{1-\left(\frac{-\sqrt{5}}{5}\right)}{\frac{2\sqrt{5}}{5}} = \frac{\frac{5+\sqrt{5}}{5}}{\frac{2\sqrt{5}}{5}} = \frac{5+\sqrt{5}}{2\sqrt{5}} = \frac{5+\sqrt{5}}{2\sqrt{5}}\cdot\frac{\sqrt{5}}{\sqrt{5}} = \frac{5\sqrt{5}+5}{10} = \frac{\sqrt{5}+1}{2} = \frac{1+\sqrt{5}}{2}$$

35. α lies in quadrant III. Since $x^2 + y^2 = 1$, $r = \sqrt{1} = 1$. Now, the point $\left(-\frac{1}{4}, b\right)$ is on the circle, so

$$\left(-\frac{1}{4}\right)^2 + b^2 = 1$$

$$b^2 = 1-\left(-\frac{1}{4}\right)^2$$

$$b = -\sqrt{1-\left(-\frac{1}{4}\right)^2} = -\sqrt{\frac{15}{16}} = -\frac{\sqrt{15}}{4}$$

(*b* is negative because α lies in quadrant III.)

Thus, $\cos\alpha = \frac{a}{r} = \frac{-\frac{1}{4}}{1} = -\frac{1}{4}$ and $\sin\alpha = \frac{b}{r} = \frac{-\frac{\sqrt{15}}{4}}{1} = -\frac{\sqrt{15}}{4}$. Thus,

$$g(2\alpha) = \cos(2\alpha) = \cos^2\alpha - \sin^2\alpha = \left(-\frac{1}{4}\right)^2 - \left(\frac{\sqrt{15}}{4}\right)^2 = \frac{1}{16} - \frac{15}{16} = -\frac{14}{16} = -\frac{7}{8}$$

36. From the solution to Problem 35, we have $\sin\alpha = -\frac{\sqrt{15}}{4}$ and $\cos\alpha = -\frac{1}{4}$. Thus,

$$f(2\alpha) = \sin(2\alpha) = 2\sin\alpha\cos\alpha = 2\cdot\left(-\frac{\sqrt{15}}{4}\right)\cdot\left(-\frac{1}{4}\right) = \frac{\sqrt{15}}{8}$$

37. Note: Since α lies in quadrant III, $\frac{\alpha}{2}$ must lie in quadrant II. Therefore, $\sin\frac{\alpha}{2}$ is positive. From the solution to Problem 35, we have $\cos\alpha = -\frac{1}{4}$.

Thus, $f\left(\frac{\alpha}{2}\right) = \sin\frac{\alpha}{2} = \sqrt{\frac{1-\cos\alpha}{2}} = \sqrt{\frac{1-\left(-\frac{1}{4}\right)}{2}} = \sqrt{\frac{\frac{5}{4}}{2}} = \sqrt{\frac{5}{8}} = \sqrt{\frac{5}{8}\cdot\frac{2}{2}} = \sqrt{\frac{10}{16}} = \frac{\sqrt{10}}{4}$

38. Note: Since α lies in quadrant III, $\frac{\alpha}{2}$ must lie in quadrant II. Therefore, $\cos\frac{\alpha}{2}$ is negative. From the solution to Problem 35, we have $\cos\alpha = -\frac{1}{4}$.

Thus,

$$g\left(\frac{\alpha}{2}\right) = \cos\frac{\alpha}{2} = -\sqrt{\frac{1+\cos\alpha}{2}} = -\sqrt{\frac{1+\left(-\frac{1}{4}\right)}{2}} = -\sqrt{\frac{\frac{3}{4}}{2}} = -\sqrt{\frac{3}{8}} = -\sqrt{\frac{3}{8}\cdot\frac{2}{2}} = -\sqrt{\frac{6}{16}} = -\frac{\sqrt{6}}{4}$$

39. From the solution to Problem 35, we have $\sin\alpha = -\frac{\sqrt{15}}{4}$ and $\cos\alpha = -\frac{1}{4}$. Thus,

$$h\left(\frac{\alpha}{2}\right) = \tan\frac{\alpha}{2} = \frac{1-\cos\alpha}{\sin\alpha} = \frac{1-\left(-\frac{1}{4}\right)}{-\frac{\sqrt{15}}{4}} = \frac{\frac{5}{4}}{-\frac{\sqrt{15}}{4}} = -\frac{5}{\sqrt{15}} = -\frac{5}{\sqrt{15}}\cdot\frac{\sqrt{15}}{\sqrt{15}} = -\frac{5\sqrt{15}}{15} = -\frac{\sqrt{15}}{3}$$

40. α lies in quadrant III. Since $x^2 + y^2 = 1$, $r = \sqrt{1} = 1$. Now, the point $\left(-\frac{1}{4}, b\right)$ is on the circle, so

$$\left(-\frac{1}{4}\right)^2 + b^2 = 1$$

$$b^2 = 1 - \left(-\frac{1}{4}\right)^2$$

$$b = -\sqrt{1-\left(-\frac{1}{4}\right)^2} = -\sqrt{\frac{15}{16}} = -\frac{\sqrt{15}}{4}$$

(b is negative because α lies in quadrant III.)

Thus, $\tan\theta = \frac{b}{a} = \frac{-\frac{\sqrt{15}}{4}}{-\frac{1}{4}} = \sqrt{15}$.

$$h(2\alpha) = \tan(2\alpha) = \frac{2\tan\alpha}{1-\tan^2\alpha} = \frac{2\left(\sqrt{15}\right)}{1-\left(\sqrt{15}\right)^2} = \frac{2\sqrt{15}}{1-15} = \frac{2\sqrt{15}}{-14} = -\frac{\sqrt{15}}{7}$$

41. $$\sin^4\theta = \left(\sin^2\theta\right)^2 = \left(\frac{1-\cos(2\theta)}{2}\right)^2 = \frac{1}{4}\left[1-2\cos(2\theta)+\cos^2(2\theta)\right] = \frac{1}{4}-\frac{1}{2}\cos(2\theta)+\frac{1}{4}\cos^2(2\theta) = \frac{1}{4}-\frac{1}{2}\cos(2\theta)+\frac{1}{4}\left(\frac{1+\cos(4\theta)}{2}\right) = \frac{1}{4}-\frac{1}{2}\cos(2\theta)+\frac{1}{8}+\frac{1}{8}\cos(4\theta) = \frac{3}{8}-\frac{1}{2}\cos(2\theta)+\frac{1}{8}\cos(4\theta)$$

42. $$\sin(4\theta) = \sin(2\cdot 2\theta) = 2\sin(2\theta)\cos(2\theta) = 2(2\sin\theta\cos\theta)\left(1-2\sin^2\theta\right) = 4\sin\theta\cos\theta\left(1-2\sin^2\theta\right) = (\cos\theta)\left[4\sin\theta\left(1-2\sin^2\theta\right)\right] = (\cos\theta)\left(4\sin\theta-8\sin^3\theta\right)$$

43. $$\cos(3\theta) = \cos(2\theta+\theta) = \cos(2\theta)\cos\theta-\sin(2\theta)\sin\theta = \left(2\cos^2\theta-1\right)\cos\theta-2\sin\theta\cos\theta\sin\theta = 2\cos^3\theta-\cos\theta-2\sin^2\theta\cos\theta = 2\cos^3\theta-\cos\theta-2\left(1-\cos^2\theta\right)\cos\theta = 2\cos^3\theta-\cos\theta-2\cos\theta+2\cos^3\theta = 4\cos^3\theta-3\cos\theta$$

44. $$\cos(4\theta) = \cos(2\cdot 2\theta) = 2\cos^2(2\theta)-1 = 2\left(2\cos^2\theta-1\right)^2-1 = 2\left(4\cos^4\theta-4\cos^2\theta+1\right)-1 = 8\cos^4\theta-8\cos^2\theta+2-1 = 8\cos^4\theta-8\cos^2\theta+1$$

45. We use the result of problem 42 to help solve this problem:

$$\begin{aligned}
\sin(5\theta) &= \sin(4\theta+\theta)\\
&= \sin(4\theta)\cos\theta + \cos(4\theta)\sin\theta\\
&= \cos\theta\left(4\sin\theta - 8\sin^3\theta\right)\cos\theta + \cos(2(2\theta))\sin\theta\\
&= \cos^2\theta\left(4\sin\theta - 8\sin^3\theta\right) + \left(1-2\sin^2(2\theta)\right)\sin\theta\\
&= \left(1-\sin^2\theta\right)\left(4\sin\theta - 8\sin^3\theta\right)\\
&\qquad + \sin\theta\left(1-2(2\sin\theta\cos\theta)^2\right)\\
&= 4\sin\theta - 12\sin^3\theta + 8\sin^5\theta\\
&\qquad + \sin\theta\left(1-8\sin^2\theta\cos^2\theta\right)\\
&= 4\sin\theta - 12\sin^3\theta + 8\sin^5\theta\\
&\qquad + \sin\theta - 8\sin^3\theta\left(1-\sin^2\theta\right)\\
&= 5\sin\theta - 12\sin^3\theta + 8\sin^5\theta - 8\sin^3\theta + 8\sin^5\theta\\
&= 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta
\end{aligned}$$

46. We use the results from problems 42 and 44 to help solve this problem:

$$\begin{aligned}
\cos(5\theta) &= \cos(4\theta+\theta)\\
&= \cos(4\theta)\cos\theta - \sin(4\theta)\sin\theta\\
&= \left(8\cos^4\theta - 8\cos^2\theta + 1\right)\cos\theta\\
&\qquad - \left(\cos\theta\left(4\sin\theta - 8\sin^3\theta\right)\right)\sin\theta\\
&= 8\cos^5\theta - 8\cos^3\theta + \cos\theta\\
&\qquad - 4\cos\theta\sin^2\theta + 8\cos\theta\sin^4\theta\\
&= 8\cos^5\theta - 8\cos^3\theta + \cos\theta\\
&\qquad - 4\cos\theta(1-\cos^2\theta) + 8\cos\theta(1-\cos^2\theta)^2\\
&= 8\cos^5\theta - 8\cos^3\theta + \cos\theta - 4\cos\theta\\
&\qquad + 4\cos^3\theta + 8\cos\theta(1-2\cos^2\theta+\cos^4\theta)\\
&= 8\cos^5\theta - 4\cos^3\theta - 3\cos\theta\\
&\qquad + 8\cos\theta - 16\cos^3\theta + 8\cos^5\theta\\
&= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta
\end{aligned}$$

47.
$$\begin{aligned}
\cos^4\theta - \sin^4\theta &= \left(\cos^2\theta + \sin^2\theta\right)\left(\cos^2\theta - \sin^2\theta\right)\\
&= 1\cdot\cos(2\theta)\\
&= \cos(2\theta)
\end{aligned}$$

48.
$$\begin{aligned}
\frac{\cot\theta - \tan\theta}{\cot\theta + \tan\theta} &= \frac{\dfrac{\cos\theta}{\sin\theta} - \dfrac{\sin\theta}{\cos\theta}}{\dfrac{\cos\theta}{\sin\theta} + \dfrac{\sin\theta}{\cos\theta}}\\
&= \frac{\dfrac{\cos^2\theta - \sin^2\theta}{\sin\theta\cos\theta}}{\dfrac{\cos^2\theta + \sin^2\theta}{\sin\theta\cos\theta}}\\
&= \frac{\cos^2\theta - \sin^2\theta}{\sin\theta\cos\theta}\cdot\frac{\sin\theta\cos\theta}{\cos^2\theta + \sin^2\theta}\\
&= \frac{\cos^2\theta - \sin^2\theta}{1}\\
&= \cos(2\theta)
\end{aligned}$$

49.
$$\begin{aligned}
\cot(2\theta) = \frac{1}{\tan(2\theta)} &= \frac{1}{\dfrac{2\tan\theta}{1-\tan^2\theta}}\\
&= \frac{1-\tan^2\theta}{2\tan\theta}\\
&= \frac{1-\dfrac{1}{\cot^2\theta}}{\dfrac{2}{\cot\theta}}\\
&= \frac{\dfrac{\cot^2\theta - 1}{\cot^2\theta}}{\dfrac{2}{\cot\theta}}\\
&= \frac{\cot^2\theta - 1}{\cot^2\theta}\cdot\frac{\cot\theta}{2}\\
&= \frac{\cot^2\theta - 1}{2\cot\theta}
\end{aligned}$$

50.
$$\begin{aligned}
\cot(2\theta) = \frac{1}{\tan(2\theta)} &= \frac{1}{\dfrac{2\tan\theta}{1-\tan^2\theta}}\\
&= \frac{1-\tan^2\theta}{2\tan\theta}\\
&= \frac{1}{2}\left(\frac{1}{\tan\theta} - \frac{\tan^2\theta}{\tan\theta}\right)\\
&= \frac{1}{2}(\cot\theta - \tan\theta)
\end{aligned}$$

51. $\sec(2\theta) = \dfrac{1}{\cos(2\theta)} = \dfrac{1}{2\cos^2\theta - 1}$

$= \dfrac{1}{\dfrac{2}{\sec^2\theta} - 1}$

$= \dfrac{1}{\dfrac{2 - \sec^2\theta}{\sec^2\theta}}$

$= \dfrac{\sec^2\theta}{2 - \sec^2\theta}$

52. $\csc(2\theta) = \dfrac{1}{\sin(2\theta)} = \dfrac{1}{2\sin\theta\cos\theta}$

$= \dfrac{1}{2} \cdot \dfrac{1}{\cos\theta} \cdot \dfrac{1}{\sin\theta}$

$= \dfrac{1}{2}\sec\theta\csc\theta$

53. $\cos^2(2u) - \sin^2(2u) = \cos[2(2u)] = \cos(4u)$

54. $(4\sin u\cos u)(1 - 2\sin^2 u)$

$= 2(2\sin u\cos u)(1 - 2\sin^2 u)$

$= 2\sin 2u\cos 2u$

$= \sin(2 \cdot 2u)$

$= \sin(4u)$

55. $\dfrac{\cos(2\theta)}{1 + \sin(2\theta)} = \dfrac{\cos^2\theta - \sin^2\theta}{1 + 2\sin\theta\cos\theta}$

$= \dfrac{(\cos\theta - \sin\theta)(\cos\theta + \sin\theta)}{\cos^2\theta + \sin^2\theta + 2\sin\theta\cos\theta}$

$= \dfrac{(\cos\theta - \sin\theta)(\cos\theta + \sin\theta)}{(\cos\theta + \sin\theta)(\cos\theta + \sin\theta)}$

$= \dfrac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta}$

$= \dfrac{\dfrac{\cos\theta - \sin\theta}{\sin\theta}}{\dfrac{\cos\theta + \sin\theta}{\sin\theta}}$

$= \dfrac{\dfrac{\cos\theta}{\sin\theta} - \dfrac{\sin\theta}{\sin\theta}}{\dfrac{\cos\theta}{\sin\theta} + \dfrac{\sin\theta}{\sin\theta}}$

$= \dfrac{\cot\theta - 1}{\cot\theta + 1}$

56. $\sin^2\theta\cos^2\theta = \dfrac{1}{4}\left(4\sin^2\theta\cos^2\theta\right)$

$= \dfrac{1}{4}(2\sin\theta\cos\theta)^2$

$= \dfrac{1}{4}[\sin(2\theta)]^2$

$= \dfrac{1}{4} \cdot \left[\dfrac{1 - \cos(4\theta)}{2}\right]$

$= \dfrac{1}{8}[1 - \cos(4\theta)]$

57. $\sec^2\left(\dfrac{\theta}{2}\right) = \dfrac{1}{\cos^2\left(\dfrac{\theta}{2}\right)} = \dfrac{1}{\dfrac{1 + \cos\theta}{2}} = \dfrac{2}{1 + \cos\theta}$

58. $\csc^2\left(\dfrac{\theta}{2}\right) = \dfrac{1}{\sin^2\left(\dfrac{\theta}{2}\right)} = \dfrac{1}{\dfrac{1 - \cos\theta}{2}} = \dfrac{2}{1 - \cos\theta}$

59. $\cot^2\left(\dfrac{v}{2}\right) = \dfrac{1}{\tan^2\left(\dfrac{v}{2}\right)} = \dfrac{1}{\dfrac{1 - \cos v}{1 + \cos v}}$

$= \dfrac{1 + \cos v}{1 - \cos v}$

$= \dfrac{1 + \dfrac{1}{\sec v}}{1 - \dfrac{1}{\sec v}}$

$= \dfrac{\dfrac{\sec v + 1}{\sec v}}{\dfrac{\sec v - 1}{\sec v}}$

$= \dfrac{\sec v + 1}{\sec v} \cdot \dfrac{\sec v}{\sec v - 1}$

$= \dfrac{\sec v + 1}{\sec v - 1}$

60. $\tan\frac{v}{2}=\frac{1-\cos v}{\sin v}=\frac{1}{\sin v}-\frac{\cos v}{\sin v}=\csc v-\cot v$

61.
$$\frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}=\frac{1-\frac{1-\cos\theta}{1+\cos\theta}}{1+\frac{1-\cos\theta}{1+\cos\theta}}$$
$$=\frac{\frac{1+\cos\theta-(1-\cos\theta)}{1+\cos\theta}}{\frac{1+\cos\theta+1-\cos\theta}{1+\cos\theta}}$$
$$=\frac{\frac{2\cos\theta}{1+\cos\theta}}{\frac{2}{1+\cos\theta}}$$
$$=\frac{2\cos\theta}{1+\cos\theta}\cdot\frac{1+\cos\theta}{2}$$
$$=\cos\theta$$

62.
$$\frac{\sin^3\theta+\cos^3\theta}{\sin\theta+\cos\theta}$$
$$=\frac{(\sin\theta+\cos\theta)\left(\sin^2\theta-\sin\theta\cos\theta+\cos^2\theta\right)}{\sin\theta+\cos\theta}$$
$$=\sin^2\theta-\sin\theta\cos\theta+\cos^2\theta$$
$$=\left(\sin^2\theta+\cos^2\theta\right)-\frac{1}{2}(2\sin\theta\cos\theta)$$
$$=1-\frac{1}{2}\sin(2\theta)$$

63.
$$\frac{\sin(3\theta)}{\sin\theta}-\frac{\cos(3\theta)}{\cos\theta}=\frac{\sin(3\theta)\cos\theta-\cos(3\theta)\sin\theta}{\sin\theta\cos\theta}$$
$$=\frac{\sin(3\theta-\theta)}{\sin\theta\cos\theta}$$
$$=\frac{\sin 2\theta}{\sin\theta\cos\theta}$$
$$=\frac{2\sin\theta\cos\theta}{\sin\theta\cos\theta}$$
$$=2$$

64.
$$\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}-\frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}=\frac{(\cos\theta+\sin\theta)^2-(\cos\theta-\sin\theta)^2}{(\cos\theta-\sin\theta)(\cos\theta+\sin\theta)}$$
$$=\frac{\cos^2\theta+2\cos\theta\sin\theta+\sin^2\theta-\left(\cos^2\theta-2\cos\theta\sin\theta+\sin^2\theta\right)}{\cos^2\theta-\sin^2\theta}$$
$$=\frac{\cos^2\theta+2\cos\theta\sin\theta+\sin^2\theta-\cos^2\theta+2\cos\theta\sin\theta-\sin^2\theta}{\cos^2\theta-\sin^2\theta}$$
$$=\frac{4\cos\theta\sin\theta}{\cos(2\theta)}$$
$$=\frac{2(2\sin\theta\cos\theta)}{\cos(2\theta)}$$
$$=\frac{2\sin(2\theta)}{\cos(2\theta)}$$
$$=2\tan(2\theta)$$

65. $\tan(3\theta) = \tan(2\theta + \theta)$

$$= \frac{\tan(2\theta) + \tan\theta}{1 - \tan(2\theta)\tan\theta} = \frac{\frac{2\tan\theta}{1-\tan^2\theta} + \tan\theta}{1 - \frac{2\tan\theta}{1-\tan^2\theta}\cdot\tan\theta} = \frac{\frac{2\tan\theta + \tan\theta - \tan^3\theta}{1-\tan^2\theta}}{\frac{1-\tan^2\theta - 2\tan^2\theta}{1-\tan^2\theta}} = \frac{3\tan\theta - \tan^3\theta}{1-\tan^2\theta}\cdot\frac{1-\tan^2\theta}{1-3\tan^2\theta} = \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$$

66. $\tan\theta + \tan(\theta + 120°) + \tan(\theta + 240°)$

$$= \tan\theta + \frac{\tan\theta + \tan 120°}{1 - \tan\theta\tan 120°} + \frac{\tan\theta + \tan 240°}{1 - \tan\theta\tan 240°}$$

$$= \tan\theta + \frac{\tan\theta - \sqrt{3}}{1 - \tan\theta\left(-\sqrt{3}\right)} + \frac{\tan\theta + \sqrt{3}}{1 - \tan\theta\left(\sqrt{3}\right)}$$

$$= \tan\theta + \frac{\tan\theta - \sqrt{3}}{1 + \sqrt{3}\tan\theta} + \frac{\tan\theta + \sqrt{3}}{1 - \sqrt{3}\tan\theta}$$

$$= \frac{\tan\theta\left(1 - 3\tan^2\theta\right) + \left(\tan\theta - \sqrt{3}\right)\left(1 - \sqrt{3}\tan\theta\right) + \left(\tan\theta + \sqrt{3}\right)\left(1 + \sqrt{3}\tan\theta\right)}{1 - 3\tan^2\theta}$$

$$= \frac{\tan\theta - 3\tan^3\theta + \tan\theta - \sqrt{3}\tan^2\theta - \sqrt{3} + 3\tan\theta + \tan\theta + \sqrt{3}\tan^2\theta + \sqrt{3} + 3\tan\theta}{1 - 3\tan^2\theta}$$

$$= \frac{-3\tan^3\theta + 9\tan\theta}{1 - 3\tan^2\theta}$$

$$= \frac{3\left(3\tan\theta - \tan^3\theta\right)}{1 - 3\tan^2\theta}$$

$= 3\tan(3\theta)$ (from Problem 65)

67. $\frac{1}{2}\cdot\left(\ln\left|1 - \cos(2\theta)\right| - \ln 2\right)$

$$= \frac{1}{2}\cdot\ln\left|\frac{1 - \cos 2\theta}{2}\right|$$

$$= \ln\left(\left|\frac{1 - \cos(2\theta)}{2}\right|^{1/2}\right)$$

$$= \ln\left(\left|\sin^2\theta\right|^{1/2}\right)$$

$$= \ln\left|\sin\theta\right|$$

68. $\frac{1}{2}\cdot\left(\ln\left|1 + \cos(2\theta)\right| - \ln 2\right)$

$$= \frac{1}{2}\cdot\ln\left|\frac{1 + \cos 2\theta}{2}\right|$$

$$= \ln\left(\left|\frac{1 + \cos(2\theta)}{2}\right|^{1/2}\right)$$

$$= \ln\left(\left|\cos^2\theta\right|^{1/2}\right)$$

$$= \ln\left|\cos\theta\right|$$

69.
$$\cos(2\theta)+6\sin^2\theta=4$$
$$1-2\sin^2\theta+6\sin^2\theta=4$$
$$4\sin^2\theta=3$$
$$\sin^2\theta=\frac{3}{4}$$
$$\sin\theta=\pm\frac{\sqrt{3}}{2}$$
$$\theta=\frac{\pi}{3},\frac{2\pi}{3},\frac{4\pi}{3},\frac{5\pi}{3}$$
The solution set is $\left\{\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}\right\}$.

70.
$$\cos(2\theta)=2-2\sin^2\theta$$
$$1-2\sin^2\theta=2-2\sin^2\theta$$
$$1=2 \quad \text{(not possible)}$$
The equation has no real solution.

71.
$$\cos(2\theta)=\cos\theta$$
$$2\cos^2\theta-1=\cos\theta$$
$$2\cos^2\theta-\cos\theta-1=0$$
$$(2\cos\theta+1)(\cos\theta-1)=0$$
$$2\cos\theta+1=0 \quad \text{or} \quad \cos\theta-1=0$$
$$\cos\theta=-\frac{1}{2} \qquad \cos\theta=1$$
$$\theta=\frac{2\pi}{3},\frac{4\pi}{3} \qquad \theta=0$$
The solution set is $\left\{0, \frac{2\pi}{3}, \frac{4\pi}{3}\right\}$.

72.
$$\sin(2\theta)=\cos\theta$$
$$2\sin\theta\cos\theta=\cos\theta$$
$$2\sin\theta\cos\theta-\cos\theta=0$$
$$(\cos\theta)(2\sin\theta-1)=0$$
$$\cos\theta=0 \quad \text{or} \quad 2\sin\theta=1$$
$$\cos\theta=0 \qquad \sin\theta=\frac{1}{2}$$
$$\theta=\frac{\pi}{2},\frac{3\pi}{2} \qquad \theta=\frac{\pi}{6},\frac{5\pi}{6}$$
The solution set is $\left\{\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{3\pi}{2}\right\}$.

73.
$$\sin(2\theta)+\sin(4\theta)=0$$
$$\sin(2\theta)+2\sin(2\theta)\cos(2\theta)=0$$
$$\sin(2\theta)\left(1+2\cos(2\theta)\right)=0$$
$$\sin(2\theta)=0 \quad \text{or} \quad 1+2\cos(2\theta)=0$$
$$\cos(2\theta)=-\frac{1}{2}$$
$$2\theta=0+2k\pi \quad \text{or} \quad 2\theta=\pi+2k\pi \quad \text{or}$$
$$\theta=k\pi \qquad \theta=\frac{\pi}{2}+k\pi$$
$$2\theta=\frac{2\pi}{3}+2k\pi \quad \text{or} \quad 2\theta=\frac{4\pi}{3}+2k\pi$$
$$\theta=\frac{\pi}{3}+k\pi \qquad \theta=\frac{2\pi}{3}+k\pi$$
On the interval $0\le\theta<2\pi$, the solution set is $\left\{0, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{3\pi}{2}, \frac{5\pi}{3}\right\}$.

74.
$$\cos(2\theta)+\cos(4\theta)=0$$
$$\left(2\cos^2\theta-1\right)+\left(2\cos^2(2\theta)-1\right)=0$$
$$2\cos^2\theta-1+2\left[\cos(2\theta)\cos(2\theta)\right]-1=0$$
$$2\cos^2\theta+2\left(2\cos^2(\theta)-1\right)\left(2\cos^2(\theta)-1\right)-2=0$$
$$\left(2\cos^2\theta-1\right)+2\left[4\cos^4\theta-4\cos^2\theta+1\right]-1=0$$
$$2\cos^2\theta-1+8\cos^4\theta-8\cos^2\theta+2-1=0$$
$$8\cos^4\theta-6\cos^2\theta=0$$
$$4\cos^4\theta-3\cos^2\theta=0$$
$$\cos^2\theta\left(4\cos^2\theta-3\right)=0$$
$$\cos^2(\theta)=0 \quad \text{or} \quad 4\cos^2\theta-3=0$$
$$\cos\theta=0 \quad \text{or} \quad \cos^2\theta=\frac{3}{4}$$
$$\cos\theta=\pm\frac{\sqrt{3}}{2}$$
$$\theta=\frac{\pi}{2},\frac{3\pi}{2} \quad \text{or} \quad \theta=\frac{\pi}{6},\frac{5\pi}{6},\frac{7\pi}{6},\frac{11\pi}{6}$$
On the interval $0\le\theta<2\pi$, the solution set is $\left\{\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}\right\}$.

75.

$$3-\sin\theta=\cos(2\theta)$$
$$3-\sin\theta=1-2\sin^2\theta$$
$$2\sin^2\theta-\sin\theta+2=0$$

This equation is quadratic in $\sin\theta$.

The discriminant is $b^2-4ac=1-16=-15<0$.

The equation has no real solutions.

76.

$$\cos(2\theta)+5\cos\theta+3=0$$
$$2\cos^2\theta-1+5\cos\theta+3=0$$
$$2\cos^2\theta+5\cos\theta+2=0$$
$$(2\cos\theta+1)(\cos\theta+2)=0$$

$2\cos\theta=-1$ or $\cos\theta=-2$ (not possible)

$$\cos\theta=-\frac{1}{2}$$
$$\theta=\frac{2\pi}{3},\frac{4\pi}{3}$$

The solution set is $\left\{\frac{2\pi}{3},\ \frac{4\pi}{3}\right\}$.

77.

$$\tan(2\theta)+2\sin\theta=0$$
$$\frac{\sin(2\theta)}{\cos(2\theta)}+2\sin\theta=0$$
$$\frac{\sin 2\theta+2\sin\theta\cos 2\theta}{\cos 2\theta}=0$$
$$2\sin\theta\cos\theta+2\sin\theta(2\cos^2\theta-1)=0$$
$$2\sin\theta\left(\cos\theta+2\cos^2\theta-1\right)=0$$
$$2\sin\theta\left(2\cos^2\theta+\cos\theta-1\right)=0$$
$$2\sin\theta(2\cos\theta-1)(\cos\theta+1)=0$$

$2\cos\theta-1=0$ or $2\sin\theta=0$ or

$$\cos\theta=\frac{1}{2}\qquad \sin\theta=0$$
$$\theta=\frac{\pi}{3},\frac{5\pi}{3}\qquad \theta=0,\pi$$

$$\cos\theta+1=0$$
$$\cos\theta=-1$$
$$\theta=\pi$$

The solution set is $\left\{0,\ \frac{\pi}{3},\ \pi,\ \frac{5\pi}{3}\right\}$.

78.

$$\tan(2\theta)+2\cos\theta=0$$
$$\frac{\sin(2\theta)}{\cos(2\theta)}+2\cos\theta=0$$
$$\frac{\sin(2\theta)+2\cos\theta\cos 2\theta}{\cos(2\theta)}=0$$
$$2\sin\theta\cos\theta+2\cos\theta(1-2\sin^2\theta)=0$$
$$2\cos\theta\left(\sin\theta+1-2\sin^2\theta\right)=0$$
$$-2\cos\theta\left(2\sin^2\theta-\sin\theta-1\right)=0$$
$$-2\cos\theta(2\sin\theta+1)(\sin\theta-1)=0$$

$-2\cos\theta=0$ or $2\sin\theta+1=0$ or

$$\cos\theta=0\qquad \sin\theta=-\frac{1}{2}$$
$$\theta=\frac{\pi}{2},\frac{3\pi}{2}\qquad \theta=\frac{7\pi}{6},\frac{11\pi}{6}$$

$$\sin\theta-1=0$$
$$\sin\theta=1$$
$$\theta=\frac{\pi}{2}$$

The solution set is $\left\{\frac{\pi}{2},\ \frac{7\pi}{6},\ \frac{3\pi}{2},\ \frac{11\pi}{6}\right\}$.

79. $\sin\left(2\sin^{-1}\frac{1}{2}\right)=\sin\left(2\cdot\frac{\pi}{6}\right)=\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}$

80. $\sin\left[2\sin^{-1}\frac{\sqrt{3}}{2}\right]=\sin\left(2\cdot\frac{\pi}{3}\right)=\sin\frac{2\pi}{3}=\frac{\sqrt{3}}{2}$

81.

$$\cos\left(2\sin^{-1}\frac{3}{5}\right)=1-2\sin^2\left(\sin^{-1}\frac{3}{5}\right)$$
$$=1-2\left(\frac{3}{5}\right)^2$$
$$=1-\frac{18}{25}$$
$$=\frac{7}{25}$$

82. $\cos\left(2\cos^{-1}\frac{4}{5}\right) = 2\cos^2\left(\cos^{-1}\frac{4}{5}\right) - 1$

$$= 2\left(\frac{4}{5}\right)^2 - 1$$

$$= \frac{32}{25} - 1$$

$$= \frac{7}{25}$$

83. $\tan\left[2\cos^{-1}\left(-\frac{3}{5}\right)\right]$

Let $\alpha = \cos^{-1}\left(-\frac{3}{5}\right)$. α lies in quadrant II.

Then $\cos\alpha = -\frac{3}{5}$, $\frac{\pi}{2} \le \alpha \le \pi$.

$\sec\alpha = -\frac{5}{3}$

$\tan\alpha = -\sqrt{\sec^2\alpha - 1}$

$$= -\sqrt{\left(-\frac{5}{3}\right)^2 - 1} = -\sqrt{\frac{25}{9} - 1} = -\sqrt{\frac{16}{9}} = -\frac{4}{3}$$

$$\tan\left[2\cos^{-1}\left(-\frac{3}{5}\right)\right] = \tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$$

$$= \frac{2\left(-\frac{4}{3}\right)}{1-\left(-\frac{4}{3}\right)^2}$$

$$= \frac{-\frac{8}{3}}{1-\frac{16}{9}} \cdot \frac{9}{9}$$

$$= \frac{-24}{9-16}$$

$$= \frac{-24}{-7}$$

$$= \frac{24}{7}$$

84. $\tan\left(2\tan^{-1}\frac{3}{4}\right) = \dfrac{2\tan\left(\tan^{-1}\frac{3}{4}\right)}{1-\tan^2\left(\tan^{-1}\frac{3}{4}\right)}$

$$= \frac{2\cdot\left(\frac{3}{4}\right)}{1-\left(\frac{3}{4}\right)^2}$$

$$= \frac{\frac{3}{2}}{1-\frac{9}{16}} \cdot \frac{16}{16}$$

$$= \frac{24}{16-9}$$

$$= \frac{24}{7}$$

85. $\sin\left(2\cos^{-1}\frac{4}{5}\right)$

Let $\alpha = \cos^{-1}\frac{4}{5}$. α is in quadrant I.

Then $\cos\alpha = \frac{4}{5}$, $0 \le \alpha \le \frac{\pi}{2}$.

$\sin\alpha = \sqrt{1-\cos^2\alpha}$

$$= \sqrt{1-\left(\frac{4}{5}\right)^2} = \sqrt{1-\frac{16}{25}} = \sqrt{\frac{9}{25}} = \frac{3}{5}$$

$$\sin\left(2\cos^{-1}\frac{4}{5}\right) = \sin 2\alpha$$

$$= 2\sin\alpha\cos\alpha = 2\cdot\frac{3}{5}\cdot\frac{4}{5} = \frac{24}{25}$$

86. $\cos\left[2\tan^{-1}\left(-\frac{4}{3}\right)\right]$

Let $\alpha = \tan^{-1}\left(-\frac{4}{3}\right)$. α is in quadrant IV.

Then $\tan\alpha = -\frac{4}{3}$, $-\frac{\pi}{2} < \alpha < 0$.

$\sec\alpha = \sqrt{\tan^2\alpha + 1}$

$$= \sqrt{\left(-\frac{4}{3}\right)^2 + 1} = \sqrt{\frac{16}{9}+1} = \sqrt{\frac{25}{9}} = \frac{5}{3}$$

$$\cos\alpha = \frac{3}{5}$$

$$\cos\left[2\tan^{-1}\left(-\frac{4}{3}\right)\right] = \cos 2\alpha = 2\cos^2\alpha - 1 = 2\left(\frac{3}{5}\right)^2 - 1 = \frac{18}{25} - 1 = -\frac{7}{25}$$

87. $\sin^2\left(\frac{1}{2}\cos^{-1}\frac{3}{5}\right) = \dfrac{1-\cos\left(\cos^{-1}\frac{3}{5}\right)}{2} = \dfrac{1-\frac{3}{5}}{2} = \dfrac{\frac{2}{5}}{2} = \dfrac{1}{5}$

88. $\cos^2\left(\frac{1}{2}\sin^{-1}\frac{3}{5}\right)$

Let $\alpha = \sin^{-1}\frac{3}{5}$. α is in quadrant I. Then

$\sin\alpha = \frac{3}{5},\ 0 < \alpha < \frac{\pi}{2}$.

$$\cos\alpha = \sqrt{1-\sin^2\alpha} = \sqrt{1-\left(\frac{3}{5}\right)^2} = \sqrt{1-\frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$$

$$\cos^2\left(\frac{1}{2}\sin^{-1}\frac{3}{5}\right) = \cos^2\left(\frac{1}{2}\cdot\alpha\right) = \frac{1+\cos\alpha}{2} = \frac{1+\frac{4}{5}}{2} = \frac{\frac{9}{5}}{2} = \frac{9}{10}$$

89. $\sec\left(2\tan^{-1}\frac{3}{4}\right)$

Let $\alpha = \tan^{-1}\left(\frac{3}{4}\right)$. α is in quadrant I.

Then $\tan\alpha = \frac{3}{4},\ 0 < \alpha < \frac{\pi}{2}$.

$$\sec\alpha = \sqrt{\tan^2\alpha + 1} = \sqrt{\left(\frac{3}{4}\right)^2 + 1} = \sqrt{\frac{9}{16}+1} = \sqrt{\frac{25}{16}} = \frac{5}{4}$$

$$\cos\alpha = \frac{4}{5}$$

$$\sec\left(2\tan^{-1}\frac{3}{4}\right) = \sec(2\alpha) = \frac{1}{\cos(2\alpha)} = \frac{1}{2\cos^2\alpha - 1} = \frac{1}{2\left(\frac{4}{5}\right)^2 - 1} = \frac{1}{\frac{32}{25} - 1} = \frac{1}{\frac{7}{25}} = \frac{25}{7}$$

90. $\csc\left[2\sin^{-1}\left(-\frac{3}{5}\right)\right]$

Let $\alpha = \sin^{-1}\left(-\frac{3}{5}\right)$. α is in quadrant IV.

Then $\sin\alpha = -\frac{3}{5},\ -\frac{\pi}{2} \le \alpha \le 0$.

$$\cos\alpha = \sqrt{1-\sin^2\alpha} = \sqrt{1-\left(-\frac{3}{5}\right)^2} = \sqrt{1-\frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$$

$$\csc\left[2\sin^{-1}\left(-\frac{3}{5}\right)\right] = \csc(2\alpha) = \frac{1}{\sin(2\alpha)} = \frac{1}{2\sin\alpha\cos\alpha} = \frac{1}{2\left(-\frac{3}{5}\right)\left(\frac{4}{5}\right)} = \frac{1}{-\frac{24}{25}} = -\frac{25}{24}$$

91.
$$f(x) = 0$$
$$\sin(2x) - \sin x = 0$$
$$2\sin x \cos x - \sin x = 0$$
$$\sin x(2\cos x - 1) = 0$$
$$\sin x = 0 \quad \text{or} \quad 2\cos x - 1 = 0$$
$$x = 0, \pi \qquad \cos x = \frac{1}{2}$$
$$x = \frac{\pi}{3}, \frac{5\pi}{3}$$

The zeros on $0 \le x < 2\pi$ are $0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}$.

92.
$$f(x) = 0$$
$$\cos(2x) + \cos x = 0$$
$$2\cos^2 x - 1 + \cos x = 0$$
$$2\cos^2 x + \cos x - 1 = 0$$
$$(2\cos x - 1)(\cos x + 1) = 0$$
$$2\cos x - 1 = 0 \quad \text{or} \quad \cos x + 1 = 0$$
$$\cos x = \frac{1}{2} \qquad \cos x = -1$$
$$x = \frac{\pi}{3}, \frac{5\pi}{3} \qquad x = \pi$$

The zeros on $0 \le x < 2\pi$ are $\frac{\pi}{3}, \pi, \frac{5\pi}{3}$.

93.
$$f(x) = 0$$
$$\cos(2x) + \sin^2 x = 0$$
$$\cos^2 x - \sin^2 x + \sin^2 x = 0$$
$$\cos^2 x = 0$$
$$\cos x = 0$$
$$x = \frac{\pi}{2}, \frac{3\pi}{2}$$

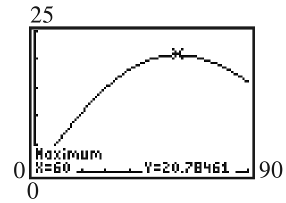
The zeros on $0 \le x < 2\pi$ are $\frac{\pi}{2}, \frac{3\pi}{2}$.

94. a. $\cos(2\theta) + \cos\theta = 0$, $0° < \theta < 90°$
$$2\cos^2\theta - 1 + \cos\theta = 0$$
$$2\cos^2\theta + \cos\theta - 1 = 0$$
$$(2\cos\theta - 1)(\cos\theta + 1) = 0$$
$$2\cos\theta - 1 = 0 \quad \text{or} \quad \cos\theta + 1 = 0$$
$$\cos\theta = \frac{1}{2} \qquad \cos\theta = -1$$
$$\theta = 60°, 300° \qquad \theta = 180°$$

On the interval $0° < \theta < 90°$, the solution is 60°.

b.
$$A(60°) = 16\sin(60°)\left[\cos(60°) + 1\right]$$
$$= 16 \cdot \frac{\sqrt{3}}{2}\left(\frac{1}{2} + 1\right)$$
$$= 12\sqrt{3} \text{ in}^2 \approx 20.78 \text{ in}^2$$

c. Graph $Y_1 = 16\sin x(\cos x + 1)$ and use the MAXIMUM feature:



The maximum area is approximately 20.78 in.^2 when the angle is 60°.

95. a.
$$D = \frac{\frac{1}{2}W}{\csc\theta - \cot\theta}$$
$$W = 2D(\csc\theta - \cot\theta)$$
$$\csc\theta - \cot\theta = \frac{1}{\sin\theta} - \frac{\cos\theta}{\sin\theta} = \frac{1 - \cos\theta}{\sin\theta}$$
$$= \tan\frac{\theta}{2}$$

Therefore, $W = 2D\tan\frac{\theta}{2}$.

b. Here we have $D = 15$ and $W = 6.5$.
$$6.5 = 2(15)\tan\frac{\theta}{2}$$
$$\tan\frac{\theta}{2} = \frac{13}{60}$$
$$\frac{\theta}{2} = \tan^{-1}\frac{13}{60}$$
$$\theta = 2\tan^{-1}\frac{13}{60} \approx 24.45°$$

96.
$$I_x \sin\theta\cos\theta - I_y \sin\theta\cos\theta + I_{xy}\left(\cos^2\theta - \sin^2\theta\right)$$
$$= \left(I_x - I_y\right)(\sin\theta\cos\theta) + I_{xy}\left(\cos^2\theta - \sin^2\theta\right)$$
$$= \left(I_x - I_y\right)\frac{1}{2}\sin 2\theta + I_{xy}\cos 2\theta$$
$$= \frac{I_x - I_y}{2}\sin 2\theta + I_{xy}\cos 2\theta$$

97. a. $R(\theta)=\dfrac{v_0^2\sqrt{2}}{16}\cos\theta(\sin\theta-\cos\theta)$

$$=\frac{v_0^2\sqrt{2}}{16}(\cos\theta\sin\theta-\cos^2\theta)$$
$$=\frac{v_0^2\sqrt{2}}{16}\cdot\frac{1}{2}(2\cos\theta\sin\theta-2\cos^2\theta)$$
$$=\frac{v_0^2\sqrt{2}}{32}\left[\sin 2\theta-2\left(\frac{1+\cos 2\theta}{2}\right)\right]$$
$$=\frac{v_0^2\sqrt{2}}{32}\left[\sin(2\theta)-1-\cos(2\theta)\right]$$
$$=\frac{v_0^2\sqrt{2}}{32}\left[\sin(2\theta)-\cos(2\theta)-1\right]$$

b. $\sin(2\theta)+\cos(2\theta)=0$

Divide each side by $\sqrt{2}$:

$$\frac{1}{\sqrt{2}}\sin(2\theta)+\frac{1}{\sqrt{2}}\cos(2\theta)=0$$

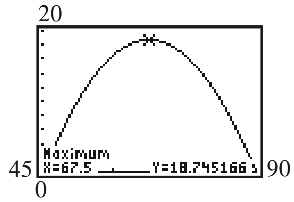
Rewrite in the sum of two angles form using $\cos\phi=\dfrac{1}{\sqrt{2}}$ and $\sin\phi=\dfrac{1}{\sqrt{2}}$ and $\phi=\dfrac{\pi}{4}$:

$$\sin(2\theta)\cos\phi+\cos(2\theta)\sin\phi=0$$
$$\sin(2\theta+\phi)=0$$
$$2\theta+\phi=0+k\pi$$
$$2\theta+\frac{\pi}{4}=0+k\pi$$
$$2\theta=-\frac{\pi}{4}+k\pi$$
$$\theta=-\frac{\pi}{8}+\frac{k\pi}{2}$$
$$\theta=\frac{3\pi}{8}=67.5°$$

c. $R=\dfrac{32^2\sqrt{2}}{32}\left(\sin(2\cdot 67.5°)-\cos(2\cdot 67.5°)-1\right)$

$$=32\sqrt{2}\left(\sin(135°)-\cos(135°)-1\right)$$
$$=32\sqrt{2}\left(\frac{\sqrt{2}}{2}-\left(-\frac{\sqrt{2}}{2}\right)-1\right)$$
$$=32\sqrt{2}\left(\sqrt{2}-1\right)$$
$$=32\left(2-\sqrt{2}\right)\text{ feet}\approx 18.75\text{ feet}$$

d. Graph $Y_1=\dfrac{32^2\sqrt{2}}{32}\left(\sin(2x)-\cos(2x)-1\right)$ and use the MAXIMUM feature:



The angle that maximizes the distance is 67.5°, and the maximum distance is 18.75 feet.

98. $y=\dfrac{1}{2}\sin(2\pi x)+\dfrac{1}{4}\sin(4\pi x)$

$$=\frac{1}{2}\sin(2\pi x)+\frac{1}{4}\sin(2\cdot 2\pi x)$$
$$=\frac{1}{2}\sin(2\pi x)+\frac{1}{4}\left[2\sin(2\pi x)\cos(2\pi x)\right]$$
$$=\frac{1}{2}\sin(2\pi x)+\frac{1}{2}\left[\sin(2\pi x)\cos(2\pi x)\right]$$
$$=\frac{1}{2}\sin(2\pi x)+\frac{1}{2}\left[\sin(2\pi x)\cdot\left(2\cos^2(\pi x)-1\right)\right]$$
$$=\frac{1}{2}\sin(2\pi x)+\sin(2\pi x)\cos^2(\pi x)-\frac{1}{2}\sin(2\pi x)$$
$$=\sin(2\pi x)\cos^2(\pi x)$$

99. Let b represent the base of the triangle.

$\cos\dfrac{\theta}{2}=\dfrac{h}{s}$ $\quad$ $\sin\dfrac{\theta}{2}=\dfrac{b/2}{s}$

$h=s\cos\dfrac{\theta}{2}$ $\quad$ $b=2s\sin\dfrac{\theta}{2}$

$$A=\frac{1}{2}b\cdot h$$
$$=\frac{1}{2}\cdot\left(2s\sin\frac{\theta}{2}\right)\left(s\cos\frac{\theta}{2}\right)$$
$$=s^2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$
$$=\frac{1}{2}s^2\sin\theta$$

100. $\sin\theta=\dfrac{y}{1}=y;\ \cos\theta=\dfrac{x}{1}=x$

a. $A=2xy=2\cos\theta\sin\theta=2\sin\theta\cos\theta$

b. $2\sin\theta\cos\theta=\sin(2\theta)$

c. The largest value of the sine function is 1. Solve:

$$\sin 2\theta = 1$$
$$2\theta = \frac{\pi}{2}$$
$$\theta = \frac{\pi}{4} = 45°$$

d. $x = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$ $\quad y = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

The dimensions of the largest rectangle are $\sqrt{2}$ by $\frac{\sqrt{2}}{2}$.

101. $$\sin(2\theta) = 2\sin\theta\cos\theta = \frac{2\sin\theta}{\cos\theta}\cdot\frac{\cos^2\theta}{1} = \frac{2\cdot\frac{\sin\theta}{\cos\theta}}{\frac{1}{\cos^2\theta}} = \frac{2\tan\theta}{\sec^2\theta} = \frac{2\tan\theta}{1+\tan^2\theta}\cdot\frac{4}{4} = \frac{4(2\tan\theta)}{4+(2\tan\theta)^2} = \frac{4x}{4+x^2}$$

102. $$\cos(2\theta) = \cos^2\theta - \sin^2\theta = \frac{\cos^2\theta-\sin^2\theta}{\cos^2\theta+\sin^2\theta} = \frac{\frac{\cos^2\theta-\sin^2\theta}{\cos^2\theta}}{\frac{\cos^2\theta+\sin^2\theta}{\cos^2\theta}} = \frac{1-\tan^2\theta}{1+\tan^2\theta}\cdot\frac{4}{4} = \frac{4-4\tan^2\theta}{4+4\tan^2\theta} = \frac{4-(2\tan\theta)^2}{4+(2\tan\theta)^2} = \frac{4-x^2}{4+x^2}$$

103. $$\frac{1}{2}\cdot\sin^2 x + C = -\frac{1}{4}\cdot\cos(2x)$$
$$C = -\frac{1}{4}\cdot\cos(2x) - \frac{1}{2}\cdot\sin^2 x = -\frac{1}{4}\cdot\left(\cos(2x)+2\sin^2 x\right) = -\frac{1}{4}\cdot\left(1-2\sin^2 x+2\sin^2 x\right) = -\frac{1}{4}\cdot(1) = -\frac{1}{4}$$

104. $$\frac{1}{2}\cdot\cos^2 x + C = \frac{1}{4}\cdot\cos(2x)$$
$$C = \frac{1}{4}\cdot\cos(2x) - \frac{1}{2}\cdot\cos^2 x = \frac{1}{4}\cdot\left(2\cos^2 x-1\right) - \frac{1}{2}\cos^2 x = \frac{1}{2}\cos^2 x - \frac{1}{4} - \frac{1}{2}\cos^2 x = -\frac{1}{4}$$

105. If $z = \tan\left(\frac{\alpha}{2}\right)$, then

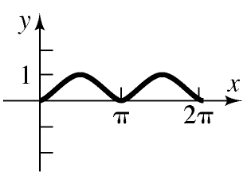
$$\frac{2z}{1+z^2} = \frac{2\tan\left(\frac{\alpha}{2}\right)}{1+\tan^2\left(\frac{\alpha}{2}\right)} = \frac{2\tan\left(\frac{\alpha}{2}\right)}{\sec^2\left(\frac{\alpha}{2}\right)} = 2\tan\left(\frac{\alpha}{2}\right)\cos^2\left(\frac{\alpha}{2}\right) = \frac{2\sin\left(\frac{\alpha}{2}\right)}{\cos\left(\frac{\alpha}{2}\right)}\cdot\cos^2\left(\frac{\alpha}{2}\right) = 2\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right) = \sin\left[2\left(\frac{\alpha}{2}\right)\right] = \sin\alpha$$

106. If $z=\tan\left(\frac{\alpha}{2}\right)$, then

$$\frac{1-z^2}{1+z^2}=\frac{1-\tan^2\left(\frac{\alpha}{2}\right)}{1+\tan^2\left(\frac{\alpha}{2}\right)}=\frac{1-\frac{1-\cos\alpha}{1+\cos\alpha}}{1+\frac{1-\cos\alpha}{1+\cos\alpha}}=\frac{\frac{1+\cos\alpha-(1-\cos\alpha)}{1+\cos\alpha}}{\frac{1+\cos\alpha+1-\cos\alpha}{1+\cos\alpha}}=\frac{1+\cos\alpha-(1-\cos\alpha)}{1+\cos\alpha+1-\cos\alpha}=\frac{2\cos\alpha}{2}=\cos\alpha$$

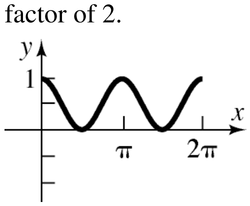
107. $f(x)=\sin^2 x=\frac{1-\cos(2x)}{2}$

Starting with the graph of $y=\cos x$, compress horizontally by a factor of 2, reflect across the x-axis, shift 1 unit up, and shrink vertically by a factor of 2.



108. $g(x)=\cos^2 x=\frac{1+\cos(2x)}{2}$

Starting with the graph of $y=\cos x$, compress horizontally by a factor of 2, reflect across the x-axis, shift 1 unit up, and shrink vertically by a factor of 2.



109. $$\sin\frac{\pi}{24}=\sin\left(\frac{\frac{\pi}{12}}{2}\right)=\sqrt{\frac{1-\cos\frac{\pi}{12}}{2}}=\sqrt{\frac{1-\left(\frac{1}{4}\left(\sqrt{6}+\sqrt{2}\right)\right)}{2}}=\sqrt{\frac{1}{2}-\frac{1}{8}\left(\sqrt{6}+\sqrt{2}\right)}=\sqrt{\frac{8-2\left(\sqrt{6}+\sqrt{2}\right)}{16}}=\frac{\sqrt{8-2\left(\sqrt{6}+\sqrt{2}\right)}}{4}=\frac{\sqrt{2\left(4-\left(\sqrt{6}+\sqrt{2}\right)\right)}}{4}=\frac{\sqrt{2}}{4}\sqrt{4-\sqrt{6}-\sqrt{2}}$$

$$\cos\frac{\pi}{24}=\cos\left(\frac{\frac{\pi}{12}}{2}\right)=\sqrt{\frac{1+\cos\frac{\pi}{12}}{2}}=\sqrt{\frac{1+\left(\frac{1}{4}\left(\sqrt{6}+\sqrt{2}\right)\right)}{2}}=\sqrt{\frac{1}{2}+\frac{1}{8}\left(\sqrt{6}+\sqrt{2}\right)}=\sqrt{\frac{8+2\left(\sqrt{6}+\sqrt{2}\right)}{16}}=\frac{\sqrt{8+2\left(\sqrt{6}+\sqrt{2}\right)}}{4}=\frac{\sqrt{2\left(4+\sqrt{6}+\sqrt{2}\right)}}{4}=\frac{\sqrt{2}}{4}\sqrt{4+\sqrt{6}+\sqrt{2}}$$

110. $$\cos\frac{\pi}{8}=\cos\left(\frac{\frac{\pi}{4}}{2}\right)=\sqrt{\frac{1+\cos\frac{\pi}{4}}{2}}=\sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}}=\sqrt{\frac{2+\sqrt{2}}{4}}=\frac{\sqrt{2+\sqrt{2}}}{2}$$

$$\sin\frac{\pi}{16}=\sin\left(\frac{\frac{\pi}{8}}{2}\right)=\sqrt{\frac{1-\cos\frac{\pi}{8}}{2}}=\sqrt{\frac{1-\frac{\sqrt{2+\sqrt{2}}}{2}}{2}}=\sqrt{\frac{2-\sqrt{2+\sqrt{2}}}{4}}=\frac{\sqrt{2-\sqrt{2+\sqrt{2}}}}{2}$$

$$\cos\frac{\pi}{16} = \cos\left(\frac{\frac{\pi}{8}}{2}\right) = \sqrt{\frac{1+\cos\frac{\pi}{8}}{2}}$$
$$= \sqrt{\frac{1+\frac{\sqrt{2+\sqrt{2}}}{2}}{2}} = \sqrt{\frac{2+\sqrt{2+\sqrt{2}}}{4}}$$
$$= \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}$$

111. $\sin^3\theta + \sin^3(\theta+120°) + \sin^3(\theta+240°)$

$$= \sin^3\theta + \left(\sin\theta\cos(120°) + \cos\theta\sin(120°)\right)^3 + \left(\sin\theta\cos(240°) + \cos\theta\sin(240°)\right)^3$$
$$= \sin^3\theta + \left(-\frac{1}{2}\cdot\sin\theta + \frac{\sqrt{3}}{2}\cdot\cos\theta\right)^3 + \left(-\frac{1}{2}\cdot\sin\theta - \frac{\sqrt{3}}{2}\cdot\cos\theta\right)^3$$
$$= \sin^3\theta + \frac{1}{8}\cdot\left(-\sin^3\theta + 3\sqrt{3}\sin^2\theta\cos\theta - 9\sin\theta\cos^2\theta + 3\sqrt{3}\cos^3\theta\right)$$
$$-\frac{1}{8}\left(\sin^3\theta + 3\sqrt{3}\sin^2\theta\cos\theta + 9\sin\theta\cos^2\theta + 3\sqrt{3}\cos^3\theta\right)$$
$$= \sin^3\theta - \frac{1}{8}\cdot\sin^3\theta + \frac{3\sqrt{3}}{8}\cdot\sin^2\theta\cos\theta - \frac{9}{8}\cdot\sin\theta\cos^2\theta + \frac{3\sqrt{3}}{8}\cdot\cos^3\theta$$
$$-\frac{1}{8}\cdot\sin^3\theta - \frac{3\sqrt{3}}{8}\cdot\sin^2\theta\cos\theta - \frac{9}{8}\cdot\sin\theta\cos^2\theta - \frac{3\sqrt{3}}{8}\cdot\cos^3\theta$$
$$= \frac{3}{4}\cdot\sin^3\theta - \frac{9}{4}\cdot\sin\theta\cos^2\theta = \frac{3}{4}\cdot\left[\sin^3\theta - 3\sin\theta\left(1-\sin^2\theta\right)\right] = \frac{3}{4}\cdot\left(\sin^3\theta - 3\sin\theta + 3\sin^3\theta\right)$$
$$= \frac{3}{4}\cdot\left(4\sin^3\theta - 3\sin\theta\right) = -\frac{3}{4}\cdot\sin(3\theta) \qquad \text{(from Example 2)}$$

112. $\tan\theta = \tan\left(3\cdot\frac{\theta}{3}\right)$

$$= \frac{3\tan\frac{\theta}{3} - \tan^3\frac{\theta}{3}}{1 - 3\tan^2\frac{\theta}{3}} \quad \text{(from problem 65)}$$

$$a\tan\frac{\theta}{3} = \frac{\tan\frac{\theta}{3}\left(3 - \tan^2\frac{\theta}{3}\right)}{1 - 3\tan^2\frac{\theta}{3}}$$

$$3\tan\frac{\theta}{3} - \tan^3\frac{\theta}{3} = a\tan\frac{\theta}{3}\left(1 - 3\tan^2\frac{\theta}{3}\right)$$

$$3 - \tan^2\frac{\theta}{3} = a\left(1 - 3\tan^2\frac{\theta}{3}\right)$$

$$3 - \tan^2\frac{\theta}{3} = a - 3a\tan^2\frac{\theta}{3}$$

$$3a\tan^2\frac{\theta}{3} - \tan^2\frac{\theta}{3} = a - 3$$

$$(3a - 1)\tan^2\frac{\theta}{3} = a - 3$$

$$\tan^2\frac{\theta}{3} = \frac{a - 3}{3a - 1}$$

$$\tan\frac{\theta}{3} = \pm\sqrt{\frac{a - 3}{3a - 1}}$$

113. Answers will vary.

Section 7.7

1. $\sin(195°)\cos(75°) = \sin(150° + 45°)\cos(30° + 45°)$

$\sin(150° + 45°)\cos(30° + 45°) =$

$= \left(\sin 150°\cos 45° + \cos 150°\sin 45°\right)\left(\cos 30°\cos 45° - \sin 30°\sin 45°\right)$

$= \left[\left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)\right]\left[\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)\right]$

$= \left(\frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4}\right)\left(\frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}\right) = \frac{\sqrt{12}}{16} - \frac{\sqrt{4}}{16} - \frac{\sqrt{36}}{16} + \frac{\sqrt{12}}{16}$

$= \frac{2\sqrt{3}}{16} - \frac{2}{16} - \frac{6}{16} + \frac{2\sqrt{3}}{16} = \frac{\sqrt{3}}{8} - \frac{1}{8} - \frac{3}{8} + \frac{\sqrt{3}}{8} = \frac{2\sqrt{3}}{8} - \frac{4}{8} = \frac{\sqrt{3}}{4} - \frac{1}{2} = \frac{1}{2}\left(\frac{\sqrt{3}}{2} - 1\right)$

2. $\cos(285°)\cos(195°) = \cos(240° + 45°)\cos(240° - 45°)$

$\cos(240° + 45°)\cos(240° - 45°) =$

$= \left(\cos 240°\cos 45° - \sin 240°\sin 45°\right)\left(\cos 240°\cos 45° + \sin 240°\sin 45°\right)$

$= \left(\cos 240°\right)^2\left(\cos 45°\right)^2 - \left(\sin 240°\right)^2\left(\sin 45°\right)^2$

$= \left(-\frac{1}{2}\right)^2\left(\frac{\sqrt{2}}{2}\right)^2 - \left(-\frac{\sqrt{3}}{2}\right)^2\left(\frac{\sqrt{2}}{2}\right)^2 = \left(\frac{1}{4}\right)\left(\frac{2}{4}\right) - \left(\frac{3}{4}\right)\left(\frac{2}{4}\right)$

$= \frac{1}{8} - \frac{3}{8} = -\frac{1}{4}$

3. $\sin(285°)\sin(75°) = \sin(240°+45°)\sin(30°+45°)$

$$\begin{aligned}
&\sin(240°+45°)\sin(30°+45°) = \\
&= \left(\sin 240°\cos 45° + \cos 240°\sin 45°\right)\left(\sin 30°\cos 45° + \cos 30°\sin 45°\right) \\
&= \left[\left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(-\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)\right]\left[\left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)\right] \\
&= \left(-\frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}\right)\left(\frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4}\right) = -\frac{\sqrt{12}}{16} - \frac{\sqrt{36}}{16} - \frac{\sqrt{4}}{16} - \frac{\sqrt{12}}{16} \\
&= -\frac{2\sqrt{3}}{16} - \frac{6}{16} - \frac{2}{16} - \frac{2\sqrt{3}}{16} = -\frac{\sqrt{3}}{8} - \frac{3}{8} - \frac{1}{8} - \frac{\sqrt{3}}{8} = -\frac{2\sqrt{3}}{8} - \frac{4}{8} = -\frac{\sqrt{3}}{4} - \frac{1}{2} = -\frac{1}{2}\left(\frac{\sqrt{3}}{2} + 1\right)
\end{aligned}$$

4.
$$\begin{aligned}
\sin(75°) + \sin(15°) &= \sin(45°+30°) + \sin(45°-30°) \\
&= \left[\sin(45°)\cos(30°) + \cos(45°)\sin(30°)\right] + \left[\sin(45°)\cos(30°) - \cos(45°)\sin(30°)\right] \\
&= 2\sin(45°)\cos(30°) \\
&= 2\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{6}}{2}
\end{aligned}$$

5.
$$\begin{aligned}
\cos(255°) - \cos(195°) &= \cos(225°+30°) - \cos(225°-30°) \\
&= \left[\cos(225°)\cos(30°) - \sin(225°)\sin(30°)\right] - \left[\cos(225°)\cos(30°) + \sin(225°)\sin(30°)\right] \\
&= -2\sin(225°)\sin(30°) \\
&= -2\left(-\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) = \frac{\sqrt{2}}{2}
\end{aligned}$$

6.
$$\begin{aligned}
\sin(255°) - \sin(15°) &= \sin(135°+120°) - \sin(135°-120°) \\
&= \left[\sin(135°)\cos(120°) + \cos(135°)\sin(120°)\right] - \left[\sin(135°)\cos(120°) - \cos(135°)\sin(120°)\right] \\
&= \sin(135°)\cos(120°) + \cos(135°)\sin(120°) - \sin(135°)\cos(120°) + \cos(135°)\sin(120°) \\
&= 2\cos(135°)\sin(120°) \\
&= 2\left(-\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = -\frac{\sqrt{6}}{2}
\end{aligned}$$

7.
$$\begin{aligned}
\sin(4\theta)\sin(2\theta) &= \frac{1}{2}\left[\cos(4\theta-2\theta) - \cos(4\theta+2\theta)\right] \\
&= \frac{1}{2}\left[\cos(2\theta) - \cos(6\theta)\right]
\end{aligned}$$

8.
$$\begin{aligned}
\cos(4\theta)\cos(2\theta) &= \frac{1}{2}\left[\cos(4\theta-2\theta) + \cos(4\theta+2\theta)\right] \\
&= \frac{1}{2}\left[\cos(2\theta) + \cos(6\theta)\right]
\end{aligned}$$

9.
$$\begin{aligned}
\sin(4\theta)\cos(2\theta) &= \frac{1}{2}\left[\sin(4\theta+2\theta) + \sin(4\theta-2\theta)\right] \\
&= \frac{1}{2}\left[\sin(6\theta) + \sin(2\theta)\right]
\end{aligned}$$

10.
$$\begin{aligned}
\sin(3\theta)\sin(5\theta) &= \frac{1}{2}\left[\cos(3\theta-5\theta) - \cos(3\theta+5\theta)\right] \\
&= \frac{1}{2}\left[\cos(-2\theta) - \cos(8\theta)\right] \\
&= \frac{1}{2}\left[\cos(2\theta) - \cos(8\theta)\right]
\end{aligned}$$

11. $\cos(3\theta)\cos(5\theta) = \frac{1}{2}[\cos(3\theta-5\theta)+\cos(3\theta+5\theta)]$
$= \frac{1}{2}[\cos(-2\theta)+\cos(8\theta)]$
$= \frac{1}{2}[\cos(2\theta)+\cos(8\theta)]$

12. $\sin(4\theta)\cos(6\theta) = \frac{1}{2}[\sin(4\theta+6\theta)+\sin(4\theta-6\theta)]$
$= \frac{1}{2}[\sin(10\theta)+\sin(-2\theta)]$
$= \frac{1}{2}[\sin(10\theta)-\sin(2\theta)]$

13. $\sin\theta\sin(2\theta) = \frac{1}{2}[\cos(\theta-2\theta)-\cos(\theta+2\theta)]$
$= \frac{1}{2}[\cos(-\theta)-\cos(3\theta)]$
$= \frac{1}{2}[\cos\theta-\cos(3\theta)]$

14. $\cos(3\theta)\cos(4\theta) = \frac{1}{2}[\cos(3\theta-4\theta)+\cos(3\theta+4\theta)]$
$= \frac{1}{2}[\cos(-\theta)+\cos(7\theta)]$
$= \frac{1}{2}[\cos\theta+\cos(7\theta)]$

15. $\sin\frac{3\theta}{2}\cos\frac{\theta}{2} = \frac{1}{2}\left[\sin\left(\frac{3\theta}{2}+\frac{\theta}{2}\right)+\sin\left(\frac{3\theta}{2}-\frac{\theta}{2}\right)\right]$
$= \frac{1}{2}[\sin(2\theta)+\sin\theta]$

16. $\sin\frac{\theta}{2}\cos\frac{5\theta}{2} = \frac{1}{2}\left[\sin\left(\frac{\theta}{2}+\frac{5\theta}{2}\right)+\sin\left(\frac{\theta}{2}-\frac{5\theta}{2}\right)\right]$
$= \frac{1}{2}[\sin(3\theta)+\sin(-2\theta)]$
$= \frac{1}{2}[\sin(3\theta)-\sin(2\theta)]$

17. $\sin(4\theta)-\sin(2\theta) = 2\sin\left(\frac{4\theta-2\theta}{2}\right)\cos\left(\frac{4\theta+2\theta}{2}\right)$
$= 2\sin\theta\cos(3\theta)$

18. $\sin(4\theta)+\sin(2\theta) = 2\sin\left(\frac{4\theta+2\theta}{2}\right)\cos\left(\frac{4\theta-2\theta}{2}\right)$
$= 2\sin(3\theta)\cos\theta$

19. $\cos(2\theta)+\cos(4\theta) = 2\cos\left(\frac{2\theta+4\theta}{2}\right)\cos\left(\frac{2\theta-4\theta}{2}\right)$
$= 2\cos(3\theta)\cos(-\theta)$
$= 2\cos(3\theta)\cos\theta$

20. $\cos(5\theta)-\cos(3\theta) = -2\sin\left(\frac{5\theta+3\theta}{2}\right)\sin\left(\frac{5\theta-3\theta}{2}\right)$
$= -2\sin(4\theta)\sin\theta$

21. $\sin\theta+\sin(3\theta) = 2\sin\left(\frac{\theta+3\theta}{2}\right)\cos\left(\frac{\theta-3\theta}{2}\right)$
$= 2\sin(2\theta)\cos(-\theta)$
$= 2\sin(2\theta)\cos\theta$

22. $\cos\theta+\cos(3\theta) = 2\cos\left(\frac{\theta+3\theta}{2}\right)\cos\left(\frac{\theta-3\theta}{2}\right)$
$= 2\cos(2\theta)\cos(-\theta)$
$= 2\cos(2\theta)\cos\theta$

23. $\cos\frac{\theta}{2}-\cos\frac{3\theta}{2} = -2\sin\left(\frac{\frac{\theta}{2}+\frac{3\theta}{2}}{2}\right)\sin\left(\frac{\frac{\theta}{2}-\frac{3\theta}{2}}{2}\right)$
$= -2\sin\theta\sin\left(-\frac{\theta}{2}\right)$
$= -2\sin\theta\left(-\sin\frac{\theta}{2}\right)$
$= 2\sin\theta\sin\frac{\theta}{2}$

24. $\sin\frac{\theta}{2}-\sin\frac{3\theta}{2} = 2\sin\left(\frac{\frac{\theta}{2}-\frac{3\theta}{2}}{2}\right)\cos\left(\frac{\frac{\theta}{2}+\frac{3\theta}{2}}{2}\right)$
$= 2\sin\left(-\frac{\theta}{2}\right)\cos\theta$
$= -2\sin\frac{\theta}{2}\cos\theta$

25. $\frac{\sin\theta+\sin(3\theta)}{2\sin(2\theta)} = \frac{2\sin\left(\frac{\theta+3\theta}{2}\right)\cos\left(\frac{\theta-3\theta}{2}\right)}{2\sin(2\theta)}$
$= \frac{2\sin(2\theta)\cos(-\theta)}{2\sin(2\theta)}$
$= \cos(-\theta)$
$= \cos\theta$

26. $\dfrac{\cos\theta+\cos(3\theta)}{2\cos(2\theta)} = \dfrac{2\cos\left(\dfrac{\theta+3\theta}{2}\right)\cos\left(\dfrac{\theta-3\theta}{2}\right)}{2\cos(2\theta)}$

$= \dfrac{2\cos(2\theta)\cos(-\theta)}{2\cos(2\theta)}$

$= \cos(-\theta)$

$= \cos\theta$

27. $\dfrac{\sin(4\theta)+\sin(2\theta)}{\cos(4\theta)+\cos(2\theta)} = \dfrac{2\sin\left(\dfrac{4\theta+2\theta}{2}\right)\cos\left(\dfrac{4\theta-2\theta}{2}\right)}{\cos(4\theta)+\cos(2\theta)}$

$= \dfrac{2\sin(3\theta)\cos\theta}{2\cos(3\theta)\cos\theta}$

$= \dfrac{\sin(3\theta)}{\cos(3\theta)}$

$= \tan(3\theta)$

28. $\dfrac{\cos\theta-\cos(3\theta)}{\sin(3\theta)-\sin\theta} = \dfrac{-2\sin\left(\dfrac{\theta+3\theta}{2}\right)\sin\left(\dfrac{\theta-3\theta}{2}\right)}{2\sin\left(\dfrac{3\theta-\theta}{2}\right)\cos\left(\dfrac{3\theta+\theta}{2}\right)}$

$= \dfrac{-2\sin(2\theta)\sin(-\theta)}{2\sin\theta\cos(2\theta)}$

$= \dfrac{-(-\sin\theta)\sin(2\theta)}{\sin\theta\cos(2\theta)}$

$= \tan(2\theta)$

29. $\dfrac{\cos\theta-\cos(3\theta)}{\sin\theta+\sin(3\theta)} = \dfrac{-2\sin\left(\dfrac{\theta+3\theta}{2}\right)\sin\left(\dfrac{\theta-3\theta}{2}\right)}{2\sin\left(\dfrac{\theta+3\theta}{2}\right)\cos\left(\dfrac{\theta-3\theta}{2}\right)}$

$= \dfrac{-2\sin(2\theta)\sin(-\theta)}{2\sin(2\theta)\cos(-\theta)}$

$= \dfrac{-(-\sin\theta)}{\cos\theta}$

$= \tan\theta$

30. $\dfrac{\cos\theta-\cos(5\theta)}{\sin\theta+\sin(5\theta)} = \dfrac{-2\sin\left(\dfrac{\theta+5\theta}{2}\right)\sin\left(\dfrac{\theta-5\theta}{2}\right)}{2\sin\left(\dfrac{\theta+5\theta}{2}\right)\cos\left(\dfrac{\theta-5\theta}{2}\right)}$

$= \dfrac{-2\sin(3\theta)\sin(-2\theta)}{2\sin(3\theta)\cos(-2\theta)}$

$= \dfrac{-(-\sin 2\theta)}{\cos(2\theta)}$

$= \tan(2\theta)$

31. $\sin\theta[\sin\theta+\sin(3\theta)]$

$= \sin\theta\left[2\sin\left(\dfrac{\theta+3\theta}{2}\right)\cos\left(\dfrac{\theta-3\theta}{2}\right)\right]$

$= \sin\theta[2\sin(2\theta)\cos(-\theta)]$

$= \cos\theta[2\sin(2\theta)\sin\theta]$

$= \cos\theta\left[2\cdot\dfrac{1}{2}[\cos\theta-\cos(3\theta)]\right]$

$= \cos\theta[\cos\theta-\cos(3\theta)]$

32. $\sin\theta[\sin(3\theta)+\sin(5\theta)]$

$= \sin\theta\left[2\sin\left(\dfrac{3\theta+5\theta}{2}\right)\cos\left(\dfrac{3\theta-5\theta}{2}\right)\right]$

$= \sin\theta[2\sin(4\theta)\cos(-\theta)]$

$= \cos\theta[2\sin(4\theta)\sin\theta]$

$= \cos\theta\left[2\cdot\dfrac{1}{2}[\cos(3\theta)-\cos(5\theta)]\right]$

$= \cos\theta[\cos(3\theta)-\cos(5\theta)]$

33. $\dfrac{\sin(4\theta)+\sin(8\theta)}{\cos(4\theta)+\cos(8\theta)}$

$= \dfrac{2\sin\left(\dfrac{4\theta+8\theta}{2}\right)\cos\left(\dfrac{4\theta-8\theta}{2}\right)}{2\cos\left(\dfrac{4\theta+8\theta}{2}\right)\cos\left(\dfrac{4\theta-8\theta}{2}\right)}$

$= \dfrac{2\sin(6\theta)\cos(-2\theta)}{2\cos(6\theta)\cos(-2\theta)}$

$= \dfrac{\sin(6\theta)}{\cos(6\theta)}$

$= \tan(6\theta)$

34. $$\frac{\sin(4\theta)-\sin(8\theta)}{\cos(4\theta)-\cos(8\theta)} = \frac{2\sin\left(\frac{4\theta-8\theta}{2}\right)\cos\left(\frac{4\theta+8\theta}{2}\right)}{-2\sin\left(\frac{4\theta+8\theta}{2}\right)\sin\left(\frac{4\theta-8\theta}{2}\right)} = \frac{2\sin(-2\theta)\cos(6\theta)}{-2\sin(6\theta)\sin(-2\theta)} = \frac{\cos(6\theta)}{-\sin(6\theta)} = -\cot(6\theta)$$

35. $$\frac{\sin(4\theta)+\sin(8\theta)}{\sin(4\theta)-\sin(8\theta)} = \frac{2\sin\left(\frac{4\theta+8\theta}{2}\right)\cos\left(\frac{4\theta-8\theta}{2}\right)}{-2\sin\left(\frac{4\theta-8\theta}{2}\right)\cos\left(\frac{4\theta+8\theta}{2}\right)} = \frac{2\sin(6\theta)\cos(-2\theta)}{2\sin(-2\theta)\cos(6\theta)} = \frac{\sin(6\theta)\cos(2\theta)}{-\sin(2\theta)\cos(6\theta)} = -\tan(6\theta)\cot(2\theta) = -\frac{\tan(6\theta)}{\tan(2\theta)}$$

36. $$\frac{\cos(4\theta)-\cos(8\theta)}{\cos(4\theta)+\cos(8\theta)} = \frac{-2\sin\left(\frac{4\theta+8\theta}{2}\right)\sin\left(\frac{4\theta-8\theta}{2}\right)}{2\cos\left(\frac{4\theta+8\theta}{2}\right)\cos\left(\frac{4\theta-8\theta}{2}\right)} = \frac{-2\sin(6\theta)\sin(-2\theta)}{2\cos(6\theta)\cos(-2\theta)} = -\frac{\sin(6\theta)}{\cos(6\theta)}\cdot\frac{\sin(-2\theta)}{\cos(-2\theta)} = -\tan(6\theta)\tan(-2\theta) = \tan(2\theta)\tan(6\theta)$$

37. $$\frac{\sin\alpha+\sin\beta}{\sin\alpha-\sin\beta} = \frac{2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)}{2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)} = \frac{\sin\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)}\cdot\frac{\cos\left(\frac{\alpha-\beta}{2}\right)}{\sin\left(\frac{\alpha-\beta}{2}\right)} = \tan\left(\frac{\alpha+\beta}{2}\right)\cot\left(\frac{\alpha-\beta}{2}\right)$$

38. $$\frac{\cos\alpha+\cos\beta}{\cos\alpha-\cos\beta} = \frac{2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)}{-2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)} = -\frac{\cos\left(\frac{\alpha+\beta}{2}\right)}{\sin\left(\frac{\alpha+\beta}{2}\right)}\cdot\frac{\cos\left(\frac{\alpha-\beta}{2}\right)}{\sin\left(\frac{\alpha-\beta}{2}\right)} = -\cot\left(\frac{\alpha+\beta}{2}\right)\cot\left(\frac{\alpha-\beta}{2}\right)$$

39. $$\frac{\sin\alpha+\sin\beta}{\cos\alpha+\cos\beta} = \frac{2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)}{2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)} = \frac{\sin\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha+\beta}{2}\right)} = \tan\left(\frac{\alpha+\beta}{2}\right)$$

40. $$\frac{\sin\alpha-\sin\beta}{\cos\alpha-\cos\beta} = \frac{2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)}{-2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)} = -\frac{\cos\left(\frac{\alpha+\beta}{2}\right)}{\sin\left(\frac{\alpha+\beta}{2}\right)} = -\cot\left(\frac{\alpha+\beta}{2}\right)$$

41. $$\begin{aligned}1+\cos(2\theta)+\cos(4\theta)+\cos(6\theta) &= \cos 0+\cos(6\theta)+\cos(2\theta)+\cos(4\theta)\\ &= 2\cos\left(\frac{0+6\theta}{2}\right)\cos\left(\frac{0-6\theta}{2}\right)+2\cos\left(\frac{2\theta+4\theta}{2}\right)\cos\left(\frac{2\theta-4\theta}{2}\right)\\ &= 2\cos(3\theta)\cos(-3\theta)+2\cos(3\theta)\cos(-\theta)\\ &= 2\cos^2(3\theta)+2\cos(3\theta)\cos\theta\\ &= 2\cos(3\theta)\left[\cos(3\theta)+\cos\theta\right]\\ &= 2\cos(3\theta)\left[2\cos\left(\frac{3\theta+\theta}{2}\right)\cos\left(\frac{3\theta-\theta}{2}\right)\right]\\ &= 2\cos(3\theta)\left[2\cos(2\theta)\cos\theta\right]\\ &= 4\cos\theta\cos(2\theta)\cos(3\theta)\end{aligned}$$

42. $$\begin{aligned}1-\cos(2\theta)+\cos(4\theta)-\cos(6\theta) &= \left[\cos 0-\cos(6\theta)\right]+\left[\cos(4\theta)-\cos(2\theta)\right]\\ &= -2\sin\left(\frac{0+6\theta}{2}\right)\sin\left(\frac{0-6\theta}{2}\right)-2\sin\left(\frac{2\theta+4\theta}{2}\right)\sin\left(\frac{2\theta-4\theta}{2}\right)\\ &= -2\sin(3\theta)\sin(-3\theta)-2\sin(3\theta)\sin(\theta)\\ &= 2\sin^2(3\theta)-2\sin(3\theta)\sin\theta\\ &= 2\sin(3\theta)\left[\sin(3\theta)-\sin\theta\right]\\ &= 2\sin(3\theta)\left[2\sin\left(\frac{3\theta-\theta}{2}\right)\cos\left(\frac{3\theta+\theta}{2}\right)\right]\\ &= 2\sin(3\theta)\left[2\sin\theta\cos(2\theta)\right]\\ &= 4\sin\theta\cos(2\theta)\sin(3\theta)\end{aligned}$$

43. $$\begin{aligned}\sin(2\theta)+\sin(4\theta) &= 0\\ \sin(2\theta)+2\sin(2\theta)\cos(2\theta) &= 0\\ \sin(2\theta)\left(1+2\cos(2\theta)\right) &= 0\end{aligned}$$

$\sin(2\theta)=0$ or $1+2\cos(2\theta)=0$

$\cos(2\theta)=-\frac{1}{2}$

$2\theta=0+2k\pi$ or $2\theta=\pi+2k\pi$ or

$\theta=k\pi$ $\qquad \theta=\frac{\pi}{2}+k\pi$

$2\theta=\frac{2\pi}{3}+2k\pi$ or $2\theta=\frac{4\pi}{3}+2k\pi$

$\theta=\frac{\pi}{3}+k\pi$ $\qquad \theta=\frac{2\pi}{3}+k\pi$

On the interval $0\le\theta<2\pi$, the solution set is $\left\{0,\ \frac{\pi}{3},\ \frac{\pi}{2},\ \frac{2\pi}{3},\ \pi,\ \frac{4\pi}{3},\ \frac{3\pi}{2},\ \frac{5\pi}{3}\right\}$.

44. $$\begin{aligned}\cos(2\theta)+\cos(4\theta) &= 0\\ 2\cos\left(\frac{2\theta+4\theta}{2}\right)\cos\left(\frac{2\theta-4\theta}{2}\right) &= 0\\ 2\cos(3\theta)\cos(-\theta) &= 0\\ 2\cos(3\theta)\cos\theta &= 0\end{aligned}$$

$\cos(3\theta)=0$ or $\cos\theta=0$

$3\theta=\frac{\pi}{2}+2k\pi$ or $3\theta=\frac{3\pi}{2}+2k\pi$ or

$\theta=\frac{\pi}{6}+\frac{2k\pi}{3}$ $\qquad \theta=\frac{\pi}{2}+\frac{2k\pi}{3}$

$\theta=\frac{\pi}{2}+2k\pi$ or $\theta=\frac{3\pi}{2}+2k\pi$

On the interval $0\le\theta<2\pi$, the solution set is $\left\{\frac{\pi}{6},\ \frac{\pi}{2},\ \frac{5\pi}{6},\ \frac{7\pi}{6},\ \frac{3\pi}{2},\ \frac{11\pi}{6}\right\}$.

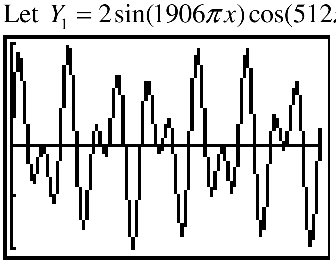
45.
$$\cos(4\theta)-\cos(6\theta)=0$$
$$-2\sin\left(\frac{4\theta+6\theta}{2}\right)\sin\left(\frac{4\theta-6\theta}{2}\right)=0$$
$$-2\sin(5\theta)\sin(-\theta)=0$$
$$2\sin(5\theta)\sin\theta=0$$
$$\sin(5\theta)=0 \quad\text{or}\quad \sin\theta=0$$
$$5\theta=0+2k\pi \quad\text{or}\quad 5\theta=\pi+2k\pi \quad\text{or}$$
$$\theta=\frac{2k\pi}{5} \qquad \theta=\frac{\pi}{5}+\frac{2k\pi}{5}$$
$$\theta=0+2k\pi \quad\text{or}\quad \theta=\pi+2k\pi$$
On the interval $0\le\theta<2\pi$, the solution set is $\left\{0,\frac{\pi}{5},\frac{2\pi}{5},\frac{3\pi}{5},\frac{4\pi}{5},\pi,\frac{6\pi}{5},\frac{7\pi}{5},\frac{8\pi}{5},\frac{9\pi}{5}\right\}$.

46.
$$\sin(4\theta)-\sin(6\theta)=0$$
$$2\sin\left(\frac{4\theta-6\theta}{2}\right)\cos\left(\frac{4\theta+6\theta}{2}\right)=0$$
$$2\sin(-\theta)\cos(5\theta)=0$$
$$-2\sin\theta\cos(5\theta)=0$$
$$\cos(5\theta)=0 \quad\text{or}\quad \sin\theta=0$$
$$\theta=0+2k\pi \quad\text{or}\quad \theta=\pi+2k\pi \quad\text{or}$$
$$5\theta=\frac{\pi}{2}+2k\pi \quad\text{or}\quad 5\theta=\frac{3\pi}{2}+2k\pi$$
$$\theta=\frac{\pi}{10}+\frac{2k\pi}{5} \qquad \theta=\frac{3\pi}{10}+\frac{2k\pi}{5}$$
On the interval $0\le\theta<2\pi$, the solution set is $\left\{0,\frac{\pi}{10},\frac{3\pi}{10},\frac{\pi}{2},\frac{7\pi}{10},\frac{9\pi}{10},\pi,\frac{11\pi}{10},\frac{13\pi}{10},\frac{3\pi}{2},\frac{17\pi}{10},\frac{19\pi}{10}\right\}$.

47. a.
$$y=\sin\left[2\pi(697)t\right]+\sin\left[2\pi(1209)t\right]$$
$$=2\sin\left(\frac{2\pi(697)t+2\pi(1209)t}{2}\right)\cos\left(\frac{2\pi(697)t-2\pi(1209)t}{2}\right)$$
$$=2\sin(1906\pi t)\cos(-512\pi t)$$
$$=2\sin(1906\pi t)\cos(512\pi t)$$

b. Because $|\sin\theta|\le 1$ and $|\cos\theta|\le 1$ for all θ, it follows that $|\sin(1906\pi t)|\le 1$ and $|\cos(512\pi t)|\le 1$ for all values of *t*. Thus, $y=2\sin(1906\pi t)\cos(512\pi t)\le 2\cdot1\cdot1=2$. That is, the maximum value of *y* is 2.

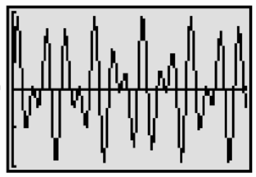
c. Let $Y_1=2\sin(1906\pi x)\cos(512\pi x)$. Window: x [0,0.01], y [-2, 2]



48. a.
$$y=\sin\left[2\pi(941)t\right]+\sin\left[2\pi(1477)t\right]$$
$$=2\sin\left(\frac{2\pi(941)t+2\pi(1477)t}{2}\right)\cos\left(\frac{2\pi(941)t-2\pi(1477)t}{2}\right)$$
$$=2\sin(2418\pi t)\cos(-536\pi t)$$
$$=2\sin(2418\pi t)\cos(536\pi t)$$

b. Because $|\sin\theta|\le 1$ and $|\cos\theta|\le 1$ for all θ, it follows that $|\sin(2418\pi t)|\le 1$ and $|\cos(2418\pi t)|\le 1$ for all values of *t*. Thus, $y=2\sin(2418\pi t)\cos(536\pi t)\le 2\cdot1\cdot1=2$. That is, the maximum value of *y* is 2.

c. Let $Y_1 = 2\sin(2418\pi x)\cos(536\pi x)$.



49. $I_u = I_x\cos^2\theta + I_y\sin^2\theta - 2I_{xy}\sin\theta\cos\theta$

$$= I_x\left(\frac{\cos 2\theta + 1}{2}\right) + I_y\left(\frac{1-\cos 2\theta}{2}\right) - I_{xy}2\sin\theta\cos\theta$$

$$= \frac{I_x\cos 2\theta}{2} + \frac{I_x}{2} + \frac{I_y}{2} - \frac{I_y\cos 2\theta}{2} - I_{xy}\sin 2\theta$$

$$= \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos 2\theta - I_{xy}\sin 2\theta$$

$I_v = I_x\sin^2\theta + I_y\cos^2\theta + 2I_{xy}\sin\theta\cos\theta$

$$= I_x\left(\frac{1-\cos 2\theta}{2}\right) + I_y\left(\frac{\cos 2\theta + 1}{2}\right) + I_{xy}2\sin\theta\cos\theta$$

$$= \frac{I_x}{2} - \frac{I_x\cos 2\theta}{2} + \frac{I_y\cos 2\theta}{2} + \frac{I_y}{2} + I_{xy}\sin 2\theta$$

$$= \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2}\cos 2\theta + I_{xy}\sin 2\theta$$

50. a. Since ϕ and v_0 are fixed, we need to maximize $\sin\theta\cos(\theta-\phi)$.

$$\sin\theta\cos(\theta-\phi) = \frac{1}{2}\left[\sin(\theta+(\theta-\phi)) + \sin(\theta-(\theta-\phi))\right]$$

$$= \frac{1}{2}\left[\sin(2\theta-\phi) + \sin\phi\right]$$

This quantity will be maximized when $\sin(2\theta-\phi) = 1$. So,

$$R_{\max} = \frac{2v_0^2\cdot\frac{1}{2}\cdot(1+\sin\phi)}{g\cos^2\phi} = \frac{v_0^2(1+\sin\phi)}{g(1-\sin^2\phi)} = \frac{v_0^2(1+\sin\phi)}{g(1-\sin\phi)(1+\sin\phi)} = \frac{v_0^2}{g(1-\sin\phi)}$$

b. $R_{\max} = \dfrac{(50)^2}{9.8(1-\sin 35°)} \approx 598.24$

The maximum range is about 598 meters.

51. $\sin(2\alpha)+\sin(2\beta)+\sin(2\gamma)$

$$\begin{aligned}
&= 2\sin\left(\frac{2\alpha+2\beta}{2}\right)\cos\left(\frac{2\alpha-2\beta}{2}\right)+\sin(2\gamma)\\
&= 2\sin(\alpha+\beta)\cos(\alpha-\beta)+2\sin\gamma\cos\gamma\\
&= 2\sin(\pi-\gamma)\cos(\alpha-\beta)+2\sin\gamma\cos\gamma\\
&= 2\sin\gamma\cos(\alpha-\beta)+2\sin\gamma\cos\gamma\\
&= 2\sin\gamma\left[\cos(\alpha-\beta)+\cos\gamma\right]\\
&= 2\sin\gamma\left[2\cos\left(\frac{\alpha-\beta+\gamma}{2}\right)\cos\left(\frac{\alpha-\beta-\gamma}{2}\right)\right]\\
&= 4\sin\gamma\cos\left(\frac{\pi-2\beta}{2}\right)\cos\left(\frac{2\alpha-\pi}{2}\right)\\
&= 4\sin\gamma\cos\left(\frac{\pi}{2}-\beta\right)\cos\left(\alpha-\frac{\pi}{2}\right)\\
&= 4\sin\gamma\sin\beta\sin\alpha\\
&= 4\sin\alpha\sin\beta\sin\gamma
\end{aligned}$$

52. $\tan\alpha+\tan\beta+\tan\gamma=\dfrac{\sin\alpha}{\cos\alpha}+\dfrac{\sin\beta}{\cos\beta}+\dfrac{\sin\gamma}{\cos\gamma}$

$$\begin{aligned}
&= \frac{\sin\alpha\cos\beta\cos\gamma+\sin\beta\cos\alpha\cos\gamma+\sin\gamma\cos\alpha\cos\beta}{\cos\alpha\cos\beta\cos\gamma}\\
&= \frac{\cos\gamma(\sin\alpha\cos\beta+\cos\alpha\sin\beta)+\sin\gamma\cos\alpha\cos\beta}{\cos\alpha\cos\beta\cos\gamma}\\
&= \frac{\cos\gamma\sin(\alpha+\beta)+\sin\gamma\cos\alpha\cos\beta}{\cos\alpha\cos\beta\cos\gamma}=\frac{\cos\gamma\sin(\pi-\gamma)+\sin\gamma\cos\alpha\cos\beta}{\cos\alpha\cos\beta\cos\gamma}\\
&= \frac{\cos\gamma\sin\gamma+\sin\gamma\cos\alpha\cos\beta}{\cos\alpha\cos\beta\cos\gamma}=\frac{\sin\gamma(\cos\gamma+\cos\alpha\cos\beta)}{\cos\alpha\cos\beta\cos\gamma}\\
&= \frac{\sin\gamma\left[\cos\left(\pi-(\alpha+\beta)\right)+\cos\alpha\cos\beta\right]}{\cos\alpha\cos\beta\cos\gamma}=\frac{\sin\gamma\left[-\cos(\alpha+\beta)+\cos\alpha\cos\beta\right]}{\cos\alpha\cos\beta\cos\gamma}\\
&= \frac{\sin\gamma\left(-\cos\alpha\cos\beta+\sin\alpha\sin\beta+\cos\alpha\cos\beta\right)}{\cos\alpha\cos\beta\cos\gamma}\\
&= \frac{\sin\gamma(\sin\alpha\sin\beta)}{\cos\alpha\cos\beta\cos\gamma}=\tan\alpha\tan\beta\tan\gamma
\end{aligned}$$

53. Add the sum formulas for $\sin(\alpha+\beta)$ and $\sin(\alpha-\beta)$ and solve for $\sin\alpha\cos\beta$:

$$\begin{aligned}
\sin(\alpha+\beta)&=\sin\alpha\cos\beta+\cos\alpha\sin\beta\\
\sin(\alpha-\beta)&=\sin\alpha\cos\beta-\cos\alpha\sin\beta\\
\sin(\alpha+\beta)+\sin(\alpha-\beta)&=2\sin\alpha\cos\beta\\
\sin\alpha\cos\beta&=\frac{1}{2}\left[\sin(\alpha+\beta)+\sin(\alpha-\beta)\right]
\end{aligned}$$

54. $2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)$

$$= 2\cdot\frac{1}{2}\left[\sin\left(\frac{\alpha-\beta}{2}+\frac{\alpha+\beta}{2}\right)+\sin\left(\frac{\alpha-\beta}{2}-\frac{\alpha+\beta}{2}\right)\right]$$
$$= \sin\left(\frac{2\alpha}{2}\right)+\sin\left(\frac{-2\beta}{2}\right)$$
$$= \sin\alpha+\sin(-\beta)$$
$$= \sin\alpha-\sin\beta$$

Thus, $\sin\alpha-\sin\beta = 2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)$.

55. $2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

$$= 2\cdot\frac{1}{2}\left[\cos\left(\frac{\alpha+\beta}{2}-\frac{\alpha-\beta}{2}\right)+\cos\left(\frac{\alpha+\beta}{2}+\frac{\alpha-\beta}{2}\right)\right]$$
$$= \cos\left(\frac{2\beta}{2}\right)+\cos\left(\frac{2\alpha}{2}\right)$$
$$= \cos\beta+\cos\alpha$$

Thus, $\cos\alpha+\cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$.

56. $-2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$

$$= -2\cdot\frac{1}{2}\left[\cos\left(\frac{\alpha+\beta}{2}-\frac{\alpha-\beta}{2}\right)-\cos\left(\frac{\alpha+\beta}{2}+\frac{\alpha-\beta}{2}\right)\right]$$
$$= -\left[\cos\left(\frac{2\beta}{2}\right)-\cos\left(\frac{2\alpha}{2}\right)\right]$$
$$= \cos\alpha-\cos\beta$$

Thus, $\cos\alpha-\cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$.

Chapter 7 Review Exercises

1. $\sin^{-1} 1$

Find the angle θ, $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$, whose sine equals 1.

$$\sin\theta = 1, \quad -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$$
$$\theta = \frac{\pi}{2}$$

Thus, $\sin^{-1}(1) = \frac{\pi}{2}$.

2. $\cos^{-1} 0$

Find the angle θ, $0\le\theta\le\pi$, whose cosine equals 0.

$$\cos\theta = 0, \quad 0\le\theta\le\pi$$
$$\theta = \frac{\pi}{2}$$

Thus, $\cos^{-1}(0) = \frac{\pi}{2}$.

3. $\tan^{-1} 1$

Find the angle θ, $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$, whose tangent equals 1.

$$\tan\theta = 1, \quad -\frac{\pi}{2}<\theta<\frac{\pi}{2}$$
$$\theta = \frac{\pi}{4}$$

Thus, $\tan^{-1}(1) = \frac{\pi}{4}$.

4. $\sin^{-1}\left(-\frac{1}{2}\right)$

Find the angle θ, $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$, whose sine equals $-\frac{1}{2}$.

$$\sin\theta = -\frac{1}{2}, \quad -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$$
$$\theta = -\frac{\pi}{6}$$

Thus, $\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$.

5. $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

Find the angle θ, $0\le\theta\le\pi$, whose cosine equals $-\frac{\sqrt{3}}{2}$.

$$\cos\theta = -\frac{\sqrt{3}}{2}, \quad 0\le\theta\le\pi$$
$$\theta = \frac{5\pi}{6}$$

Thus, $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$.

6. $\tan^{-1}\left(-\sqrt{3}\right)$

Find the angle θ, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, whose tangent equals $-\sqrt{3}$.

$\tan\theta = -\sqrt{3}, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$\theta = -\frac{\pi}{3}$

Thus, $\tan^{-1}\left(-\sqrt{3}\right) = -\frac{\pi}{3}$.

7. $\sec^{-1}\sqrt{2}$

Find the angle θ, $0 \le \theta \le \pi$, whose secant equals $\sqrt{2}$.

$\sec\theta = \sqrt{2}, \quad 0 \le \theta \le \pi$

$\theta = \frac{\pi}{4}$

Thus, $\sec^{-1}\sqrt{2} = \frac{\pi}{4}$.

8. $\cot^{-1}(-1)$

Find the angle θ, $0 < \theta < \pi$, whose cotangent equals -1.

$\cot\theta = -1, \quad 0 < \theta < \pi$

$\theta = \frac{3\pi}{4}$

Thus, $\cot^{-1}(-1) = \frac{3\pi}{4}$.

9. $\sin^{-1}\left(\sin\left(\frac{3\pi}{8}\right)\right)$ follows the form of the equation $f^{-1}\left(f\left(x\right)\right) = \sin^{-1}\left(\sin\left(x\right)\right) = x$. Since $\frac{3\pi}{8}$ is in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, we can apply the equation directly and get $\sin^{-1}\left(\sin\left(\frac{3\pi}{8}\right)\right) = \frac{3\pi}{8}$.

10. $\cos^{-1}\left(\cos\frac{3\pi}{4}\right)$ follows the form of the equation $f^{-1}\left(f\left(x\right)\right) = \cos^{-1}\left(\cos\left(x\right)\right) = x$. Since $\frac{3\pi}{4}$ is in the interval $\left[0, \pi\right]$, we can apply the equation directly and get $\cos^{-1}\left(\cos\frac{3\pi}{4}\right) = \frac{3\pi}{4}$.

11. $\tan^{-1}\left(\tan\left(\frac{2\pi}{3}\right)\right)$ follows the form of the equation $f^{-1}\left(f\left(x\right)\right) = \tan^{-1}\left(\tan\left(x\right)\right) = x$ but we cannot use the formula directly since $\frac{2\pi}{3}$ is not in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. We need to find an angle θ in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ for which $\tan\left(\frac{2\pi}{3}\right) = \tan\theta$. The angle $\frac{2\pi}{3}$ is in quadrant II so tangent is negative. The reference angle of $\frac{2\pi}{3}$ is $\frac{\pi}{3}$ and we want θ to be in quadrant IV so tangent will still be negative. Thus, we have $\tan\left(\frac{2\pi}{3}\right) = \tan\left(-\frac{\pi}{3}\right)$. Since $-\frac{\pi}{3}$ is in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, we can apply the equation above and get

$\tan^{-1}\left(\tan\left(\frac{2\pi}{3}\right)\right) = \tan^{-1}\left(\tan\left(-\frac{\pi}{3}\right)\right) = -\frac{\pi}{3}$.

12. $\cos^{-1}\left(\cos\left(\frac{15\pi}{7}\right)\right)$ follows the form of the equation $f^{-1}\left(f\left(x\right)\right) = \cos^{-1}\left(\cos\left(x\right)\right) = x$, but we cannot use the formula directly since $\frac{15\pi}{7}$ is not in the interval $\left[0, \pi\right]$. We need to find an angle θ in the interval $\left[0, \pi\right]$ for which $\cos\left(\frac{15\pi}{7}\right) = \cos\theta$. The angle $\frac{15\pi}{7}$ is in quadrant I so the reference angle of $\frac{15\pi}{7}$ is $\frac{\pi}{7}$. Thus, we have $\cos\left(\frac{15\pi}{7}\right) = \cos\frac{\pi}{7}$. Since $\frac{\pi}{7}$ is in the interval $[0,\pi]$, we can apply the equation above and get $\cos^{-1}\left(\cos\left(\frac{15\pi}{7}\right)\right)=\cos^{-1}\left(\cos\frac{\pi}{7}\right)=\frac{\pi}{7}$.

13. $\sin^{-1}\left(\sin\left(-\frac{8\pi}{9}\right)\right)$ follows the form of the equation $f^{-1}(f(x))=\sin^{-1}(\sin(x))=x$, but we cannot use the formula directly since $-\frac{8\pi}{9}$ is not in the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$. We need to find an angle θ in the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ for which $\sin\left(-\frac{8\pi}{9}\right)=\sin\theta$. The angle $-\frac{8\pi}{9}$ is in quadrant III so sine is negative. The reference angle of $-\frac{8\pi}{9}$ is $\frac{\pi}{9}$ and we want θ to be in quadrant IV so sine will still be negative. Thus, we have $\sin\left(-\frac{8\pi}{9}\right)=\sin\left(-\frac{\pi}{9}\right)$. Since $-\frac{\pi}{9}$ is in the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, we can apply the equation above and get $\sin^{-1}\left(\sin\left(-\frac{8\pi}{9}\right)\right)=\sin^{-1}\left(\sin\left(-\frac{\pi}{9}\right)\right)=-\frac{\pi}{9}$.

14. $\sin\left(\sin^{-1}0.9\right)$ follows the form of the equation $f\left(f^{-1}(x)\right)=\sin\left(\sin^{-1}(x)\right)=x$. Since 0.9 is in the interval $[-1,1]$, we can apply the equation directly and get $\sin\left(\sin^{-1}0.9\right)=0.9$.

15. $\cos\left(\cos^{-1}0.6\right)$ follows the form of the equation $f\left(f^{-1}(x)\right)=\cos\left(\cos^{-1}(x)\right)=x$. Since 0.6 is in the interval $[-1,1]$, we can apply the equation directly and get $\cos\left(\cos^{-1}0.6\right)=0.6$.

16. $\tan\left(\tan^{-1}5\right)$ follows the form of the equation $f\left(f^{-1}(x)\right)=\tan\left(\tan^{-1}(x)\right)=x$. Since 5 is a real number, we can apply the equation directly and get $\tan\left(\tan^{-1}5\right)=5$.

17. Since there is no angle θ such that $\cos\theta=-1.6$, the quantity $\cos^{-1}(-1.6)$ is not defined. Thus, $\cos\left(\cos^{-1}(-1.6)\right)$ is not defined.

18. $\sin^{-1}\left(\cos\frac{2\pi}{3}\right)=\sin^{-1}\left(-\frac{1}{2}\right)=-\frac{\pi}{6}$

19. $\cos^{-1}\left(\tan\frac{3\pi}{4}\right)=\cos^{-1}(-1)=\pi$

20. $\tan\left[\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right]$

Find the angle θ, $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$, whose sine equals $-\frac{\sqrt{3}}{2}$.

$$\sin\theta=-\frac{\sqrt{3}}{2},\qquad -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$$
$$\theta=-\frac{\pi}{3}$$

So, $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)=-\frac{\pi}{3}$.

Thus, $\tan\left[\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right]=\tan\left(-\frac{\pi}{3}\right)=-\sqrt{3}$.

21. $\sec\left(\tan^{-1}\frac{\sqrt{3}}{3}\right)$

Find the angle θ, $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$, whose tangent is $\frac{\sqrt{3}}{3}$

$$\tan\theta=\frac{\sqrt{3}}{3},\qquad -\frac{\pi}{2}<\theta<\frac{\pi}{2}$$
$$\theta=\frac{\pi}{6}$$

So, $\tan^{-1}\frac{\sqrt{3}}{3}=\frac{\pi}{6}$.

Thus, $\sec\left(\tan^{-1}\frac{\sqrt{3}}{3}\right)=\sec\left(\frac{\pi}{6}\right)=\frac{2\sqrt{3}}{3}$.

22. $\sin\left(\cot^{-1}\frac{3}{4}\right)$

Since $\cot\theta=\frac{3}{4}$, $0<\theta<\pi$, θ is in quadrant I.

Let $x=3$ and $y=4$. Solve for r: $9+16=r^2$

$$r^2=25$$
$$r=5$$

Thus, $\sin\left(\tan^{-1}\frac{3}{4}\right)=\sin\theta=\frac{y}{r}=\frac{4}{5}$.

23. $\tan\left[\sin^{-1}\left(-\frac{4}{5}\right)\right]$

Since $\sin\theta=-\frac{4}{5}$, $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$, let $y=-4$ and $r=5$. Solve for x: $x^2+16=25$

$$x^2=9$$
$$x=\pm3$$

Since θ is in quadrant IV, $x=3$.

Thus, $\tan\left[\sin^{-1}\left(-\frac{4}{5}\right)\right]=\tan\theta=\frac{y}{x}=\frac{-4}{3}=-\frac{4}{3}$

24. $f(x)=2\sin(3x)$

$$y=2\sin(3x)$$
$$x=2\sin(3y)$$
$$\frac{x}{2}=\sin(3y)$$
$$3y=\sin^{-1}\left(\frac{x}{2}\right)$$
$$y=\frac{1}{3}\sin^{-1}\left(\frac{x}{2}\right)=f^{-1}(x)$$

The domain of $f(x)$ equals the range of $f^{-1}(x)$ and is $-\frac{\pi}{6}\le x\le\frac{\pi}{6}$, or $\left[-\frac{\pi}{6},\frac{\pi}{6}\right]$ in interval notation. To find the domain of $f^{-1}(x)$ we note that the argument of the inverse sine function is $\frac{x}{2}$ and that it must lie in the interval $[-1,1]$. That is,

$$-1\le\frac{x}{2}\le1$$
$$-2\le x\le2$$

The domain of $f^{-1}(x)$ is $\{x\,|-2\le x\le2\}$, or $[-2,2]$ in interval notation. Recall that the domain of a function is the range of its inverse and the domain of the inverse is the range of the function. Therefore, the range of $f(x)$ is $[-2,2]$.

25. $f(x)=-\cos x+3$

$$y=-\cos x+3$$
$$x=-\cos y+3$$
$$x-3=-\cos y$$
$$3-x=\cos y$$
$$y=\cos^{-1}(3-x)=f^{-1}(x)$$

The domain of $f(x)$ equals the range of $f^{-1}(x)$ and is $0\le x\le\pi$, or $[0,\pi]$ in interval notation. To find the domain of $f^{-1}(x)$ we note that the argument of the inverse cosine function is $3-x$ and that it must lie in the interval $[-1,1]$. That is,

$$-1\le3-x\le1$$
$$-4\le-x\le-2$$
$$4\ge x\ge2$$
$$2\le x\le4$$

The domain of $f^{-1}(x)$ is $\{x\,|\,2\le x\le4\}$, or $[2,4]$ in interval notation. Recall that the domain of a function is the range of its inverse and the domain of the inverse is the range of the function. Therefore, the range of $f(x)$ is $[2,4]$.

26. Let $\theta=\sin^{-1}u$ so that $\sin\theta=u$, $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$, $-1\le u\le1$. Then,

$$\cos\left(\sin^{-1}u\right)=\cos\theta=\sqrt{\cos^2\theta}$$
$$=\sqrt{1-\sin^2\theta}=\sqrt{1-u^2}$$

27. Let $\theta = \csc^{-1} u$ so that $\csc\theta = u$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ and $\theta \ne 0$, $|u| \ge 1$. Then,

$$\tan\left(\csc^{-1} u\right) = \tan\theta = \sqrt{\tan^2\theta}$$
$$= \sqrt{\sec^2\theta - 1} = \sqrt{\frac{1}{\csc^2\theta - 1}}$$
$$= \frac{1}{\sqrt{u^2 - 1}}$$

28.
$$\tan\theta\cot\theta - \sin^2\theta = \tan\theta \cdot \frac{1}{\tan\theta} - \sin^2\theta$$
$$= 1 - \sin^2\theta$$
$$= \cos^2\theta$$

29.
$$\sin^2\theta(1+\cot^2\theta) = \sin^2\theta \cdot \csc^2\theta$$
$$= \sin^2\theta \cdot \frac{1}{\sin^2\theta}$$
$$= 1$$

30.
$$5\cos^2\theta + 3\sin^2\theta = 2\cos^2\theta + 3\cos^2\theta + 3\sin^2\theta$$
$$= 2\cos^2\theta + 3\left(\cos^2\theta + \sin^2\theta\right)$$
$$= 2\cos^2\theta + 3 \cdot 1$$
$$= 3 + 2\cos^2\theta$$

31.
$$\frac{1-\cos\theta}{\sin\theta} + \frac{\sin\theta}{1-\cos\theta} = \frac{(1-\cos\theta)^2 + \sin^2\theta}{\sin\theta(1-\cos\theta)}$$
$$= \frac{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta}{\sin\theta(1-\cos\theta)}$$
$$= \frac{1 - 2\cos\theta + 1}{\sin\theta(1-\cos\theta)}$$
$$= \frac{2 - 2\cos\theta}{\sin\theta(1-\cos\theta)}$$
$$= \frac{2(1-\cos\theta)}{\sin\theta(1-\cos\theta)}$$
$$= \frac{2}{\sin\theta}$$
$$= 2\csc\theta$$

32.
$$\frac{\cos\theta}{\cos\theta - \sin\theta} = \frac{\cos\theta}{\cos\theta - \sin\theta} \cdot \frac{\frac{1}{\cos\theta}}{\frac{1}{\cos\theta}}$$
$$= \frac{1}{1 - \frac{\sin\theta}{\cos\theta}}$$
$$= \frac{1}{1 - \tan\theta}$$

33.
$$\frac{\csc\theta}{1+\csc\theta} = \frac{\frac{1}{\sin\theta}}{1 + \frac{1}{\sin\theta}} \cdot \frac{\sin\theta}{\sin\theta}$$
$$= \frac{1}{\sin\theta + 1}$$
$$= \frac{1}{1+\sin\theta} \cdot \frac{1-\sin\theta}{1-\sin\theta}$$
$$= \frac{1-\sin\theta}{1-\sin^2\theta}$$
$$= \frac{1-\sin\theta}{\cos^2\theta}$$

34.
$$\csc\theta - \sin\theta = \frac{1}{\sin\theta} - \sin\theta$$
$$= \frac{1-\sin^2\theta}{\sin\theta}$$
$$= \frac{\cos^2\theta}{\sin\theta}$$
$$= \cos\theta \cdot \frac{\cos\theta}{\sin\theta}$$
$$= \cos\theta\cot\theta$$

35.
$$\frac{1-\sin\theta}{\sec\theta} = \cos\theta(1-\sin\theta)$$
$$= \cos\theta(1-\sin\theta) \cdot \frac{1+\sin\theta}{1+\sin\theta}$$
$$= \frac{\cos\theta\left(1-\sin^2\theta\right)}{1+\sin\theta}$$
$$= \frac{\cos\theta\left(\cos^2\theta\right)}{1+\sin\theta}$$
$$= \frac{\cos^3\theta}{1+\sin\theta}$$

36.
$$\cot\theta - \tan\theta = \frac{\cos\theta}{\sin\theta} - \frac{\sin\theta}{\cos\theta}$$
$$= \frac{\cos^2\theta - \sin^2\theta}{\sin\theta\cos\theta}$$
$$= \frac{1 - \sin^2\theta - \sin^2\theta}{\sin\theta\cos\theta}$$
$$= \frac{1 - 2\sin^2\theta}{\sin\theta\cos\theta}$$

37. $\dfrac{\cos(\alpha+\beta)}{\cos\alpha\sin\beta}=\dfrac{\cos\alpha\cos\beta-\sin\alpha\sin\beta}{\cos\alpha\sin\beta}$
$=\dfrac{\cos\alpha\cos\beta}{\cos\alpha\sin\beta}-\dfrac{\sin\alpha\sin\beta}{\cos\alpha\sin\beta}$
$=\dfrac{\cos\beta}{\sin\beta}-\dfrac{\sin\alpha}{\cos\alpha}$
$=\cot\beta-\tan\alpha$

38. $\dfrac{\cos(\alpha-\beta)}{\cos\alpha\cos\beta}=\dfrac{\cos\alpha\cos\beta+\sin\alpha\sin\beta}{\cos\alpha\cos\beta}$
$=\dfrac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta}+\dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}$
$=1+\tan\alpha\tan\beta$

39. $(1+\cos\theta)\tan\dfrac{\theta}{2}=(1+\cos\theta)\cdot\dfrac{\sin\theta}{1+\cos\theta}=\sin\theta$

40. $2\cot\theta\cot(2\theta)=2\cdot\dfrac{\cos\theta}{\sin\theta}\cdot\dfrac{\cos(2\theta)}{\sin(2\theta)}$
$=\dfrac{2\cos\theta\left(\cos^2\theta-\sin^2\theta\right)}{\sin\theta\left(2\sin\theta\cos\theta\right)}$
$=\dfrac{\cos^2\theta-\sin^2\theta}{\sin^2\theta}$
$=\dfrac{\cos^2\theta}{\sin^2\theta}-\dfrac{\sin^2\theta}{\sin^2\theta}$
$=\cot^2\theta-1$

41. $1-8\sin^2\theta\cos^2\theta=1-2\left(2\sin\theta\cos\theta\right)^2$
$=1-2\sin^2(2\theta)$
$=\cos(2\cdot2\theta)$
$=\cos(4\theta)$

42. $\dfrac{\sin(3\theta)\cos\theta-\sin\theta\cos(3\theta)}{\sin(2\theta)}=\dfrac{\sin(3\theta-\theta)}{\sin(2\theta)}$
$=\dfrac{\sin(2\theta)}{\sin(2\theta)}$
$=1$

43. $\dfrac{\sin(2\theta)+\sin(4\theta)}{\cos(2\theta)+\cos(4\theta)}=\dfrac{2\sin\left(\frac{2\theta+4\theta}{2}\right)\cos\left(\frac{2\theta-4\theta}{2}\right)}{2\cos\left(\frac{2\theta+4\theta}{2}\right)\cos\left(\frac{2\theta+4\theta}{2}\right)}$
$=\dfrac{2\sin(3\theta)\cos(-\theta)}{2\cos(3\theta)\cos(-\theta)}$
$=\dfrac{\sin(3\theta)}{\cos(3\theta)}$
$=\tan(3\theta)$

44. $\dfrac{\cos(2\theta)-\cos(4\theta)}{\cos(2\theta)+\cos(4\theta)}-\tan\theta\tan(3\theta)$
$=\dfrac{-2\sin(3\theta)\sin(-\theta)}{2\cos(3\theta)\cos(-\theta)}-\tan\theta\tan(3\theta)$
$=\dfrac{2\sin(3\theta)\sin\theta}{2\cos(3\theta)\cos\theta}-\tan\theta\tan(3\theta)$
$=\tan(3\theta)\tan\theta-\tan\theta\tan(3\theta)$
$=0$

45. $\sin165^\circ=\sin(120^\circ+45^\circ)$
$=\sin120^\circ\cdot\cos45^\circ+\cos120^\circ\cdot\sin45^\circ$
$=\left(\dfrac{\sqrt{3}}{2}\right)\cdot\left(\dfrac{\sqrt{2}}{2}\right)+\left(-\dfrac{1}{2}\right)\cdot\left(\dfrac{\sqrt{2}}{2}\right)$
$=\dfrac{\sqrt{6}}{4}-\dfrac{\sqrt{2}}{4}$
$=\dfrac{1}{4}\left(\sqrt{6}-\sqrt{2}\right)$

46. $\tan105^\circ=\tan(60^\circ+45^\circ)$
$=\dfrac{\tan60^\circ+\tan45^\circ}{1-\tan60^\circ\tan45^\circ}$
$=\dfrac{\sqrt{3}+1}{1-\sqrt{3}\cdot1}$
$=\dfrac{\sqrt{3}+1}{1-\sqrt{3}}\cdot\dfrac{1+\sqrt{3}}{1+\sqrt{3}}$
$=\dfrac{1+2\sqrt{3}+3}{1-3}$
$=\dfrac{4+2\sqrt{3}}{-2}$
$=-2-\sqrt{3}$

47. $\cos\frac{5\pi}{12}=\cos\left(\frac{3\pi}{12}+\frac{2\pi}{12}\right)$
$=\cos\frac{\pi}{4}\cdot\cos\frac{\pi}{6}-\sin\frac{\pi}{4}\cdot\sin\frac{\pi}{6}$
$=\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}-\frac{\sqrt{2}}{2}\cdot\frac{1}{2}$
$=\frac{\sqrt{6}}{4}-\frac{\sqrt{2}}{4}$
$=\frac{1}{4}\left(\sqrt{6}-\sqrt{2}\right)$

48. $\sin\left(-\frac{\pi}{12}\right)=\sin\left(\frac{2\pi}{12}-\frac{3\pi}{12}\right)$
$=\sin\frac{\pi}{6}\cdot\cos\frac{\pi}{4}-\cos\frac{\pi}{6}\cdot\sin\frac{\pi}{4}$
$=\frac{1}{2}\cdot\frac{\sqrt{2}}{2}-\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}$
$=\frac{\sqrt{2}}{4}-\frac{\sqrt{6}}{4}$
$=\frac{1}{4}\left(\sqrt{2}-\sqrt{6}\right)$

49. $\cos 80^\circ\cdot\cos 20^\circ+\sin 80^\circ\cdot\sin 20^\circ=\cos\left(80^\circ-20^\circ\right)$
$=\cos 60^\circ$
$=\frac{1}{2}$

50. $\sin 70^\circ\cdot\cos 40^\circ-\cos 70^\circ\cdot\sin 40^\circ=\sin\left(70^\circ-40^\circ\right)$
$=\sin 30^\circ$
$=\frac{1}{2}$

51. $\tan\frac{\pi}{8}=\tan\left(\frac{\frac{\pi}{4}}{2}\right)=\sqrt{\frac{1-\cos\frac{\pi}{4}}{1+\cos\frac{\pi}{4}}}=\sqrt{\frac{1-\frac{\sqrt{2}}{2}}{1+\frac{\sqrt{2}}{2}}}$
$=\sqrt{\frac{2-\sqrt{2}}{2+\sqrt{2}}}$
$=\sqrt{\frac{2-\sqrt{2}}{2+\sqrt{2}}\cdot\frac{2-\sqrt{2}}{2-\sqrt{2}}}$
$=\sqrt{\frac{\left(2-\sqrt{2}\right)^2}{4}}$
$=\frac{2-\sqrt{2}}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}}$
$=\frac{2\sqrt{2}-2}{2}$
$=\sqrt{2}-1$

52. $\sin\frac{5\pi}{8}=\sin\left(\frac{\frac{5\pi}{4}}{2}\right)=\sqrt{\frac{1-\cos\frac{5\pi}{4}}{2}}=\sqrt{\frac{1-\left(-\frac{\sqrt{2}}{2}\right)}{2}}$
$=\sqrt{\frac{2+\sqrt{2}}{4}}$
$=\frac{\sqrt{2+\sqrt{2}}}{2}$

53. $\sin\alpha=\frac{4}{5},\ 0<\alpha<\frac{\pi}{2};\quad \sin\beta=\frac{5}{13},\ \frac{\pi}{2}<\beta<\pi$

$\cos\alpha=\frac{3}{5}$, $\tan\alpha=\frac{4}{3}$, $\cos\beta=-\frac{12}{13}$, $\tan\beta=-\frac{5}{12}$,

$0<\frac{\alpha}{2}<\frac{\pi}{4},\quad \frac{\pi}{4}<\frac{\beta}{2}<\frac{\pi}{2}$

a. $\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$
$=\left(\frac{4}{5}\right)\cdot\left(-\frac{12}{13}\right)+\left(\frac{3}{5}\right)\cdot\left(\frac{5}{13}\right)$
$=\frac{-48+15}{65}=-\frac{33}{65}$

b. $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$
$=\left(\frac{3}{5}\right)\cdot\left(-\frac{12}{13}\right)-\left(\frac{4}{5}\right)\cdot\left(\frac{5}{13}\right)$
$=\frac{-36-20}{65}=-\frac{56}{65}$

c. $\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$
$$=\left(\frac{4}{5}\right)\cdot\left(-\frac{12}{13}\right)-\left(\frac{3}{5}\right)\cdot\left(\frac{5}{13}\right)$$
$$=\frac{-48-15}{65}=-\frac{63}{65}$$

d. $\tan(\alpha+\beta)=\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$
$$=\frac{\frac{4}{3}+\left(-\frac{5}{12}\right)}{1-\left(\frac{4}{3}\right)\cdot\left(-\frac{5}{12}\right)}$$
$$=\frac{\frac{11}{12}}{\frac{14}{9}}=\frac{11}{12}\cdot\frac{9}{14}=\frac{33}{56}$$

e. $\sin(2\alpha)=2\sin\alpha\cos\alpha=2\cdot\frac{4}{5}\cdot\frac{3}{5}=\frac{24}{25}$

f. $\cos(2\beta)=\cos^2\beta-\sin^2\beta$
$$=\left(-\frac{12}{13}\right)^2-\left(\frac{5}{13}\right)^2=\frac{144}{169}-\frac{25}{169}=\frac{119}{169}$$

g. $\sin\dfrac{\beta}{2}=\sqrt{\dfrac{1-\cos\beta}{2}}$
$$=\sqrt{\frac{1-\left(-\frac{12}{13}\right)}{2}}$$
$$=\sqrt{\frac{\frac{25}{13}}{2}}=\sqrt{\frac{25}{26}}=\frac{5}{\sqrt{26}}=\frac{5\sqrt{26}}{26}$$

h. $\cos\dfrac{\alpha}{2}=\sqrt{\dfrac{1+\cos\alpha}{2}}$
$$=\sqrt{\frac{1+\frac{3}{5}}{2}}=\sqrt{\frac{\frac{8}{5}}{2}}=\sqrt{\frac{4}{5}}=\frac{2}{\sqrt{5}}=\frac{2\sqrt{5}}{5}$$

54. $\sin\alpha=-\frac{3}{5},\ \pi<\alpha<\frac{3\pi}{2};\ \cos\beta=\frac{12}{13},\ \frac{3\pi}{2}<\beta<2\pi$

$\cos\alpha=-\frac{4}{5},\ \tan\alpha=\frac{3}{4},\ \sin\beta=-\frac{5}{13},\ \tan\beta=-\frac{5}{12},$

$\frac{\pi}{2}<\frac{\alpha}{2}<\frac{3\pi}{4},\quad \frac{3\pi}{4}<\frac{\beta}{2}<\pi$

a. $\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$
$$=\left(-\frac{3}{5}\right)\cdot\left(\frac{12}{13}\right)+\left(-\frac{4}{5}\right)\cdot\left(-\frac{5}{13}\right)$$
$$=\frac{-36+20}{65}$$
$$=-\frac{16}{65}$$

b. $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$
$$=\left(-\frac{4}{5}\right)\cdot\left(\frac{12}{13}\right)-\left(-\frac{3}{5}\right)\cdot\left(-\frac{5}{13}\right)$$
$$=\frac{-48-15}{65}$$
$$=-\frac{63}{65}$$

c. $\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$
$$=\left(-\frac{3}{5}\right)\cdot\left(\frac{12}{13}\right)-\left(-\frac{4}{5}\right)\cdot\left(-\frac{5}{13}\right)$$
$$=\frac{-36-20}{65}$$
$$=-\frac{56}{65}$$

d. $\tan(\alpha+\beta)=\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$
$$=\frac{\frac{3}{4}+\left(-\frac{5}{12}\right)}{1-\frac{3}{4}\left(-\frac{5}{12}\right)}=\frac{\frac{1}{3}}{\frac{21}{16}}=\frac{1}{3}\cdot\frac{16}{21}=\frac{16}{63}$$

e. $\sin(2\alpha)=2\sin\alpha\cos\alpha$
$$=2\cdot\left(-\frac{3}{5}\right)\cdot\left(-\frac{4}{5}\right)=\frac{24}{25}$$

f. $\cos(2\beta)=\cos^2\beta-\sin^2\beta$
$$=\left(\frac{12}{13}\right)^2-\left(-\frac{5}{13}\right)^2$$
$$=\frac{144}{169}-\frac{25}{169}=\frac{119}{169}$$

g. $\sin\dfrac{\beta}{2}=\sqrt{\dfrac{1-\cos\beta}{2}}$
$$=\sqrt{\frac{1-\frac{12}{13}}{2}}=\sqrt{\frac{\frac{1}{13}}{2}}=\sqrt{\frac{1}{26}}=\frac{1}{\sqrt{26}}=\frac{\sqrt{26}}{26}$$

h. $\cos\frac{\alpha}{2} = -\sqrt{\frac{1+\cos\alpha}{2}}$
$= -\sqrt{\frac{1+\left(-\frac{4}{5}\right)}{2}}$
$= -\sqrt{\frac{\frac{1}{5}}{2}} = -\sqrt{\frac{1}{10}} = -\frac{1}{\sqrt{10}} = -\frac{\sqrt{10}}{10}$

55. $\tan\alpha = \frac{3}{4}, \pi < \alpha < \frac{3\pi}{2};\ \tan\beta = \frac{12}{5}, 0 < \beta < \frac{\pi}{2}$

$\sin\alpha = -\frac{3}{5}, \cos\alpha = -\frac{4}{5}, \sin\beta = \frac{12}{13}, \cos\beta = \frac{5}{13},$
$\frac{\pi}{2} < \frac{\alpha}{2} < \frac{3\pi}{4},\ 0 < \frac{\beta}{2} < \frac{\pi}{4}$

a. $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
$= \left(-\frac{3}{5}\right)\cdot\left(\frac{5}{13}\right) + \left(-\frac{4}{5}\right)\cdot\left(\frac{12}{13}\right)$
$= -\frac{15}{65} - \frac{48}{65}$
$= -\frac{63}{65}$

b. $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
$= \left(-\frac{4}{5}\right)\cdot\left(\frac{5}{13}\right) - \left(-\frac{3}{5}\right)\cdot\left(\frac{12}{13}\right)$
$= -\frac{20}{65} + \frac{36}{65}$
$= \frac{16}{65}$

c. $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$
$= \left(-\frac{3}{5}\right)\cdot\left(\frac{5}{13}\right) - \left(-\frac{4}{5}\right)\cdot\left(\frac{12}{13}\right)$
$= -\frac{15}{65} + \frac{48}{65}$
$= \frac{33}{65}$

d. $\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$
$= \frac{\frac{3}{4} + \frac{12}{5}}{1 - \left(\frac{3}{4}\right)\left(\frac{12}{5}\right)}$
$= \frac{\frac{63}{20}}{-\frac{4}{5}} = \frac{63}{20}\left(-\frac{5}{4}\right) = -\frac{63}{16}$

e. $\sin(2\alpha) = 2\sin\alpha\cos\alpha = 2\left(-\frac{3}{5}\right)\left(-\frac{4}{5}\right) = \frac{24}{25}$

f. $\cos(2\beta) = \cos^2\beta - \sin^2\beta$
$= \left(\frac{5}{13}\right)^2 - \left(\frac{12}{13}\right)^2 = \frac{25}{169} - \frac{144}{169} = -\frac{119}{169}$

g. $\sin\frac{\beta}{2} = \sqrt{\frac{1-\cos\beta}{2}}$
$= \sqrt{\frac{1-\frac{5}{13}}{2}} = \sqrt{\frac{\frac{8}{13}}{2}} = \sqrt{\frac{4}{13}} = \frac{2}{\sqrt{13}} = \frac{2\sqrt{13}}{13}$

h. $\cos\frac{\alpha}{2} = -\sqrt{\frac{1+\cos\alpha}{2}}$
$= -\sqrt{\frac{1+\left(-\frac{4}{5}\right)}{2}}$
$= -\sqrt{\frac{\frac{1}{5}}{2}} = -\sqrt{\frac{1}{10}} = -\frac{1}{\sqrt{10}} = -\frac{\sqrt{10}}{10}$

56. $\sec\alpha = 2, -\frac{\pi}{2} < \alpha < 0;\ \sec\beta = 3, \frac{3\pi}{2} < \beta < 2\pi$

$\sin\alpha = -\frac{\sqrt{3}}{2}, \cos\alpha = \frac{1}{2}, \tan\alpha = -\sqrt{3},$
$\sin\beta = -\frac{2\sqrt{2}}{3}, \cos\beta = \frac{1}{3}, \tan\beta = -2\sqrt{2},$
$-\frac{\pi}{4} < \frac{\alpha}{2} < 0,\quad \frac{3\pi}{4} < \frac{\beta}{2} < \pi$

a. $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
$= -\frac{\sqrt{3}}{2}\left(\frac{1}{3}\right) + \frac{1}{2}\left(-\frac{2\sqrt{2}}{3}\right)$
$= \frac{-\sqrt{3} - 2\sqrt{2}}{6}$

b. $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
$= \frac{1}{2}\cdot\frac{1}{3} - \left(-\frac{\sqrt{3}}{2}\right)\left(-\frac{2\sqrt{2}}{3}\right)$
$= \frac{1-2\sqrt{6}}{6}$

c. $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$
$= -\frac{\sqrt{3}}{2}\cdot\frac{1}{3} - \frac{1}{2}\cdot\left(-\frac{2\sqrt{2}}{3}\right)$
$= \frac{-\sqrt{3} + 2\sqrt{2}}{6}$

d. $\tan(\alpha+\beta)=\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$

$$=\frac{-\sqrt{3}+\left(-2\sqrt{2}\right)}{1-\left(-\sqrt{3}\right)\left(-2\sqrt{2}\right)}$$
$$=\left(\frac{-\sqrt{3}-2\sqrt{2}}{1-2\sqrt{6}}\right)\cdot\left(\frac{1+2\sqrt{6}}{1+2\sqrt{6}}\right)$$
$$=\frac{-9\sqrt{3}-8\sqrt{2}}{-23}$$
$$=\frac{8\sqrt{2}+9\sqrt{3}}{23}$$

e. $\sin(2\alpha)=2\sin\alpha\cos\alpha=2\left(-\dfrac{\sqrt{3}}{2}\right)\left(\dfrac{1}{2}\right)=-\dfrac{\sqrt{3}}{2}$

f. $\cos(2\beta)=\cos^2\beta-\sin^2\beta$

$$=\left(\frac{1}{3}\right)^2-\left(-\frac{2\sqrt{2}}{3}\right)^2=\frac{1}{9}-\frac{8}{9}=-\frac{7}{9}$$

g. $\sin\dfrac{\beta}{2}=\sqrt{\dfrac{1-\cos\beta}{2}}$

$$=\sqrt{\frac{1-\frac{1}{3}}{2}}=\sqrt{\frac{\frac{2}{3}}{2}}=\sqrt{\frac{1}{3}}=\frac{1}{\sqrt{3}}=\frac{\sqrt{3}}{3}$$

h. $\cos\dfrac{\alpha}{2}=\sqrt{\dfrac{1+\cos\alpha}{2}}$

$$=\sqrt{\frac{1+\frac{1}{2}}{2}}=\sqrt{\frac{\frac{3}{2}}{2}}=\sqrt{\frac{3}{4}}=\frac{\sqrt{3}}{2}$$

57. $\sin\alpha=-\dfrac{2}{3},\ \pi<\alpha<\dfrac{3\pi}{2};\ \cos\beta=-\dfrac{2}{3},\ \pi<\beta<\dfrac{3\pi}{2}$

$\cos\alpha=-\dfrac{\sqrt{5}}{3},\ \tan\alpha=\dfrac{2\sqrt{5}}{5},\ \sin\beta=-\dfrac{\sqrt{5}}{3},$

$\tan\beta=\dfrac{\sqrt{5}}{2},\ \dfrac{\pi}{2}<\dfrac{\alpha}{2}<\dfrac{3\pi}{4},\ \dfrac{\pi}{2}<\dfrac{\beta}{2}<\dfrac{3\pi}{4}$

a. $\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$

$$=\left(-\frac{2}{3}\right)\left(-\frac{2}{3}\right)+\left(-\frac{\sqrt{5}}{3}\right)\left(-\frac{\sqrt{5}}{3}\right)$$
$$=\frac{4}{9}+\frac{5}{9}$$
$$=1$$

b. $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$

$$=\left(-\frac{\sqrt{5}}{3}\right)\left(-\frac{2}{3}\right)-\left(-\frac{2}{3}\right)\left(-\frac{\sqrt{5}}{3}\right)$$
$$=\frac{2\sqrt{5}}{9}-\frac{2\sqrt{5}}{9}$$
$$=0$$

c. $\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$

$$=\left(-\frac{2}{3}\right)\left(-\frac{2}{3}\right)-\left(-\frac{\sqrt{5}}{3}\right)\left(-\frac{\sqrt{5}}{3}\right)$$
$$=\frac{4}{9}-\frac{5}{9}$$
$$=-\frac{1}{9}$$

d. $\tan(\alpha+\beta)=\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$

$$=\frac{\frac{2\sqrt{5}}{5}+\frac{\sqrt{5}}{2}}{1-\frac{2\sqrt{5}}{5}\cdot\frac{\sqrt{5}}{2}}$$
$$=\frac{\frac{4\sqrt{5}+5\sqrt{5}}{10}}{1-1}$$
$$=\frac{\frac{9\sqrt{5}}{10}}{0};\quad \text{Undefined}$$

e. $\sin(2\alpha)=2\sin\alpha\cos\alpha$

$$=2\left(-\frac{2}{3}\right)\left(-\frac{\sqrt{5}}{3}\right)=\frac{4\sqrt{5}}{9}$$

f. $\cos(2\beta)=\cos^2\beta-\sin^2\beta$

$$=\left(-\frac{2}{3}\right)^2-\left(-\frac{\sqrt{5}}{3}\right)^2=\frac{4}{9}-\frac{5}{9}=-\frac{1}{9}$$

g. $\sin\dfrac{\beta}{2}=\sqrt{\dfrac{1-\cos\beta}{2}}$

$$=\sqrt{\frac{1-\left(-\frac{2}{3}\right)}{2}}=\sqrt{\frac{\frac{5}{3}}{2}}=\sqrt{\frac{5}{6}}=\frac{\sqrt{30}}{6}$$

h. $\cos\frac{\alpha}{2} = -\sqrt{\frac{1+\cos\alpha}{2}} = -\sqrt{\frac{1+\left(-\frac{\sqrt{5}}{3}\right)}{2}}$

$= -\sqrt{\frac{\frac{3-\sqrt{5}}{3}}{2}}$

$= -\sqrt{\frac{3-\sqrt{5}}{6}}$

$= -\frac{\sqrt{6\left(3-\sqrt{5}\right)}}{6}$

$= -\frac{\sqrt{6}\sqrt{3-\sqrt{5}}}{6}$

58. $\cos\left(\sin^{-1}\frac{3}{5} - \cos^{-1}\frac{1}{2}\right)$

Let $\alpha = \sin^{-1}\frac{3}{5}$ and $\beta = \cos^{-1}\frac{1}{2}$. α is in quadrant I; β is in quadrant I. Then $\sin\alpha = \frac{3}{5}$, $0 \le \alpha \le \frac{\pi}{2}$, and $\cos\beta = \frac{1}{2}$, $0 \le \beta \le \frac{\pi}{2}$.

$\cos\alpha = \sqrt{1-\sin^2\alpha}$

$= \sqrt{1-\left(\frac{3}{5}\right)^2} = \sqrt{1-\frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$

$\sin\beta = \sqrt{1-\cos^2\beta}$

$= \sqrt{1-\left(\frac{1}{2}\right)^2} = \sqrt{1-\frac{1}{4}} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$

$\cos\left(\sin^{-1}\frac{3}{5} - \cos^{-1}\frac{1}{2}\right) = \cos(\alpha-\beta)$

$= \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$= \frac{4}{5}\cdot\frac{1}{2} + \frac{3}{5}\cdot\frac{\sqrt{3}}{2}$

$= \frac{4}{10} + \frac{3\sqrt{3}}{10} = \frac{4+3\sqrt{3}}{10}$

59. $\sin\left(\cos^{-1}\frac{5}{13} - \cos^{-1}\frac{4}{5}\right)$

Let $\alpha = \cos^{-1}\frac{5}{13}$ and $\beta = \cos^{-1}\frac{4}{5}$. α is in quadrant I; β is in quadrant I. Then $\cos\alpha = \frac{5}{13}$, $0 \le \alpha \le \frac{\pi}{2}$, and $\cos\beta = \frac{4}{5}$, $0 \le \beta \le \frac{\pi}{2}$.

$\sin\alpha = \sqrt{1-\cos^2\alpha}$

$= \sqrt{1-\left(\frac{5}{13}\right)^2} = \sqrt{1-\frac{25}{169}} = \sqrt{\frac{144}{169}} = \frac{12}{13}$

$\sin\beta = \sqrt{1-\cos^2\beta}$

$= \sqrt{1-\left(\frac{4}{5}\right)^2} = \sqrt{1-\frac{16}{25}} = \sqrt{\frac{9}{25}} = \frac{3}{5}$

$\sin\left(\cos^{-1}\frac{5}{13} - \cos^{-1}\frac{4}{5}\right) = \sin(\alpha-\beta)$

$= \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$= \frac{12}{13}\cdot\frac{4}{5} - \frac{5}{13}\cdot\frac{3}{5}$

$= \frac{48}{65} - \frac{15}{65} = \frac{33}{65}$

60. $\tan\left[\sin^{-1}\left(-\frac{1}{2}\right) - \tan^{-1}\frac{3}{4}\right]$

Let $\alpha = \sin^{-1}\left(-\frac{1}{2}\right)$ and $\beta = \tan^{-1}\frac{3}{4}$. α is in quadrant IV; β is in quadrant I. Then, $\sin\alpha = -\frac{1}{2}$, $0 \le \alpha \le \frac{\pi}{2}$, and $\tan\beta = \frac{3}{4}$, $0 < \beta < \frac{\pi}{2}$.

$\cos\alpha = \sqrt{1-\sin^2\alpha}$

$= \sqrt{1-\left(-\frac{1}{2}\right)^2} = \sqrt{1-\frac{1}{4}} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$

$\tan\alpha = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$

$$\tan\left[\sin^{-1}\left(-\frac{1}{2}\right)-\tan^{-1}\left(\frac{3}{4}\right)\right]=\tan(\alpha-\beta)$$
$$=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$
$$=\frac{-\frac{\sqrt{3}}{3}-\frac{3}{4}}{1+\left(-\frac{\sqrt{3}}{3}\right)\left(\frac{3}{4}\right)}$$
$$=\frac{\frac{-4\sqrt{3}-9}{12}}{1-\frac{3\sqrt{3}}{12}}$$
$$=\frac{-9-4\sqrt{3}}{12-3\sqrt{3}}\cdot\frac{12+3\sqrt{3}}{12+3\sqrt{3}}$$
$$=\frac{-144-75\sqrt{3}}{117}$$
$$=\frac{-48-25\sqrt{3}}{39}$$
$$=-\frac{48+25\sqrt{3}}{39}$$

61. $\cos\left[\tan^{-1}(-1)+\cos^{-1}\left(-\frac{4}{5}\right)\right]$

Let $\alpha=\tan^{-1}(-1)$ and $\beta=\cos^{-1}\left(-\frac{4}{5}\right)$. α is in quadrant IV; β is in quadrant II. Then

$\tan\alpha=-1,\ -\frac{\pi}{2}<\alpha<0$, and $\cos\beta=-\frac{4}{5}$, $\frac{\pi}{2}\le\beta\le\pi$.

$$\sec\alpha=\sqrt{1+\tan^2\alpha}=\sqrt{1+(-1)^2}=\sqrt{2}$$
$$\cos\alpha=\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}$$
$$\sin\alpha=-\sqrt{1-\cos^2\alpha}$$
$$=-\sqrt{1-\left(\frac{\sqrt{2}}{2}\right)^2}=-\sqrt{1-\frac{1}{2}}=-\sqrt{\frac{1}{2}}=-\frac{\sqrt{2}}{2}$$
$$\sin\beta=\sqrt{1-\cos^2\beta}$$
$$=\sqrt{1-\left(-\frac{4}{5}\right)^2}=\sqrt{1-\frac{16}{25}}=\sqrt{\frac{9}{25}}=\frac{3}{5}$$

$$\cos\left[\tan^{-1}(-1)+\cos^{-1}\left(-\frac{4}{5}\right)\right]=\cos(\alpha+\beta)$$
$$=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$
$$=\left(\frac{\sqrt{2}}{2}\right)\left(-\frac{4}{5}\right)-\left(-\frac{\sqrt{2}}{2}\right)\left(\frac{3}{5}\right)$$
$$=\frac{-4\sqrt{2}}{10}+\frac{3\sqrt{2}}{10}$$
$$=-\frac{\sqrt{2}}{10}$$

62. $\sin\left[2\cos^{-1}\left(-\frac{3}{5}\right)\right]$

Let $\alpha=\cos^{-1}\left(-\frac{3}{5}\right)$. α is in quadrant II. Then

$\cos\alpha=-\frac{3}{5},\ \frac{\pi}{2}\le\alpha\le\pi$.

$$\sin\alpha=\sqrt{1-\cos^2\alpha}$$
$$=\sqrt{1-\left(-\frac{3}{5}\right)^2}=\sqrt{1-\frac{9}{25}}=\sqrt{\frac{16}{25}}=\frac{4}{5}$$
$$\sin\left[2\cos^{-1}\left(-\frac{3}{5}\right)\right]=\sin 2\alpha$$
$$=2\sin\alpha\cos\alpha$$
$$=2\left(\frac{4}{5}\right)\left(-\frac{3}{5}\right)=-\frac{24}{25}$$

63. $\cos\left(2\tan^{-1}\frac{4}{3}\right)$

Let $\alpha=\tan^{-1}\frac{4}{3}$. α is in quadrant I. Then

$\tan\alpha=\frac{4}{3},\ 0<\alpha<\frac{\pi}{2}$.

$$\sec\alpha=\sqrt{\tan^2\alpha+1}$$
$$=\sqrt{\left(\frac{4}{3}\right)^2+1}=\sqrt{\frac{16}{9}+1}=\sqrt{\frac{25}{9}}=\frac{5}{3}$$
$$\cos\alpha=\frac{3}{5}$$
$$\cos\left(2\tan^{-1}\frac{4}{3}\right)=\cos(2\alpha)$$
$$=2\cos^2\alpha-1$$
$$=2\left(\frac{3}{5}\right)^2-1=2\left(\frac{9}{25}\right)-1=-\frac{7}{25}$$

64. $\cos\theta = \frac{1}{2}$

$\theta = \frac{\pi}{3} + 2k\pi$ or $\theta = \frac{5\pi}{3} + 2k\pi$, k is any integer

On $0 \le \theta < 2\pi$, the solution set is $\left\{\frac{\pi}{3}, \frac{5\pi}{3}\right\}$.

65. $\tan\theta + \sqrt{3} = 0$

$\tan\theta = -\sqrt{3}$

$\theta = \frac{2\pi}{3} + k\pi$, k is any integer

On the interval $0 \le \theta < 2\pi$, the solution set is $\left\{\frac{2\pi}{3}, \frac{5\pi}{3}\right\}$.

66. $\sin(2\theta) + 1 = 0$

$\sin(2\theta) = -1$

$2\theta = \frac{3\pi}{2} + 2k\pi$

$\theta = \frac{3\pi}{4} + k\pi$, k is any integer

On the interval $0 \le \theta < 2\pi$, the solution set is $\left\{\frac{3\pi}{4}, \frac{7\pi}{4}\right\}$

67. $\tan(2\theta) = 0$

$2\theta = 0 + k\pi$

$\theta = \frac{k\pi}{2}$, where k is any integer

On the interval $0 \le \theta < 2\pi$, the solution set is $\left\{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\right\}$.

68. $\sec^2\theta = 4$

$\sec\theta = \pm 2$

$\cos\theta = \pm\frac{1}{2}$

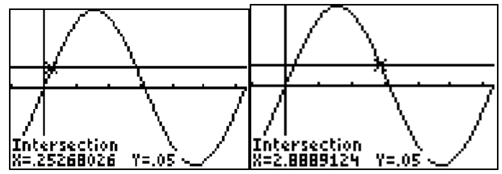
$\theta = \frac{\pi}{3} + k\pi$ or $\theta = \frac{2\pi}{3} + k\pi$,

where k is any integer

On the interval $0 \le \theta < 2\pi$, the solution set is $\left\{\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}\right\}$.

69. $0.2\sin\theta = 0.05$

Find the intersection of $Y_1 = 0.2\sin\theta$ and $Y_2 = 0.05$:



On the interval $0 \le \theta < 2\pi$, $x \approx 0.25$ or $x \approx 2.89$

The solution set is $\{0.25, 2.89\}$.

70. $\sin\theta + \sin(2\theta) = 0$

$\sin\theta + 2\sin\theta\cos\theta = 0$

$\sin\theta(1 + 2\cos\theta) = 0$

$1 + 2\cos\theta = 0$ or $\sin\theta = 0$

$\cos\theta = -\frac{1}{2}$ $\quad \theta = 0, \pi$

$\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$

On $0 \le \theta < 2\pi$, the solution set is $\left\{0, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}\right\}$.

71. $\sin(2\theta) - \cos\theta - 2\sin\theta + 1 = 0$

$2\sin\theta\cos\theta - \cos\theta - 2\sin\theta + 1 = 0$

$\cos\theta(2\sin\theta - 1) - 1(2\sin\theta - 1) = 0$

$(2\sin\theta - 1)(\cos\theta - 1) = 0$

$\sin\theta = \frac{1}{2}$ or $\cos\theta = 1$

$\theta = 0$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

On $0 \le \theta < 2\pi$, the solution set is $\left\{0, \frac{\pi}{6}, \frac{5\pi}{6}\right\}$.

72. $2\sin^2\theta - 3\sin\theta + 1 = 0$

$(2\sin\theta - 1)(\sin\theta - 1) = 0$

$2\sin\theta - 1 = 0$ or $\sin\theta - 1 = 0$

$\sin\theta = 1$

$\sin\theta = \frac{1}{2}$ $\quad \theta = \frac{\pi}{2}$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$

On $0 \le \theta < 2\pi$, the solution set is $\left\{\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}\right\}$.

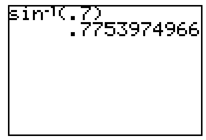
73. $$4\sin^2\theta = 1 + 4\cos\theta$$
$$4\left(1-\cos^2\theta\right) = 1 + 4\cos\theta$$
$$4 - 4\cos^2\theta = 1 + 4\cos\theta$$
$$4\cos^2\theta + 4\cos\theta - 3 = 0$$
$$(2\cos\theta - 1)(2\cos\theta + 3) = 0$$
$$2\cos\theta - 1 = 0 \quad \text{or} \quad 2\cos\theta + 3 = 0$$
$$\cos\theta = \frac{1}{2} \qquad \cos\theta = -\frac{3}{2}$$
$$\theta = \frac{\pi}{3}, \frac{5\pi}{3} \qquad \text{(not possible)}$$
On $0 \le \theta < 2\pi$, the solution set is $\left\{\frac{\pi}{3}, \frac{5\pi}{3}\right\}$.

74. $$\sin(2\theta) = \sqrt{2}\cos\theta$$
$$2\sin\theta\cos\theta = \sqrt{2}\cos\theta$$
$$2\sin\theta\cos\theta - \sqrt{2}\cos\theta = 0$$
$$\cos\theta\left(2\sin\theta - \sqrt{2}\right) = 0$$
$$\cos\theta = 0 \quad \text{or} \quad 2\sin\theta - \sqrt{2} = 0$$
$$\theta = \frac{\pi}{2}, \frac{3\pi}{2} \qquad \sin\theta = \frac{\sqrt{2}}{2}$$
$$\theta = \frac{\pi}{4}, \frac{3\pi}{4}$$
On $0 \le \theta < 2\pi$, the solution set is $\left\{\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{3\pi}{2}\right\}$.

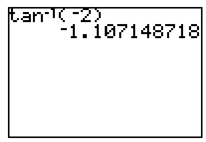
75. $\sin\theta - \cos\theta = 1$

Divide each side by $\sqrt{2}$:
$$\frac{1}{\sqrt{2}}\sin\theta - \frac{1}{\sqrt{2}}\cos\theta = \frac{1}{\sqrt{2}}$$
Rewrite in the difference of two angles form where $\cos\phi = \frac{1}{\sqrt{2}}$, $\sin\phi = \frac{1}{\sqrt{2}}$, and $\phi = \frac{\pi}{4}$:
$$\sin\theta\cos\phi - \cos\theta\sin\phi = \frac{1}{\sqrt{2}}$$
$$\sin(\theta - \phi) = \frac{\sqrt{2}}{2}$$
$$\theta - \phi = \frac{\pi}{4} \quad \text{or} \quad \theta - \phi = \frac{3\pi}{4}$$
$$\theta - \frac{\pi}{4} = \frac{\pi}{4} \quad \text{or} \quad \theta - \frac{\pi}{4} = \frac{3\pi}{4}$$
$$\theta = \frac{\pi}{2} \quad \text{or} \quad \theta = \pi$$
On $0 \le \theta < 2\pi$, the solution set is $\left\{\frac{\pi}{2}, \pi\right\}$.

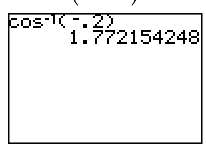
76. $\sin^{-1}(0.7) \approx 0.78$



77. $\tan^{-1}(-2) \approx -1.11$

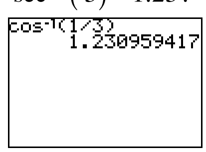


78. $\cos^{-1}(-0.2) \approx 1.77$



79. $\sec^{-1}(3) = \cos^{-1}\left(\frac{1}{3}\right)$

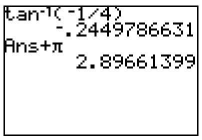
We seek the angle θ, $0 \le \theta \le \pi$, whose cosine equals $\frac{1}{3}$. Now $\cos\theta = \frac{1}{3}$, so θ lies in quadrant I. The calculator yields $\cos^{-1}\frac{1}{3} \approx 1.23$, which is an angle in quadrant I, so $\sec^{-1}(3) \approx 1.23$.



80. $\cot^{-1}(-4) = \tan^{-1}\left(-\frac{1}{4}\right)$

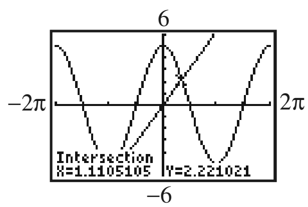
We seek the angle θ, $0 \le \theta \le \pi$, whose tangent equals $-\frac{1}{4}$. Now $\tan\theta = -\frac{1}{4}$, so θ lies in quadrant II. The calculator yields

$\tan^{-1}\left(-\frac{1}{4}\right) \approx -0.24$, which is an angle in quadrant IV. Since θ lies in quadrant II, $\theta \approx -0.24 + \pi \approx 2.90$. Therefore, $\cot^{-1}(-4) \approx 2.90$.



81. $2x = 5\cos x$

Find the intersection of $Y_1 = 2x$ and $Y_2 = 5\cos x$:

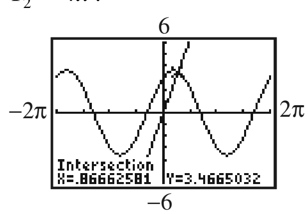


$x \approx 1.11$

The solution set is $\{1.11\}$.

82. $2\sin x + 3\cos x = 4x$

Find the intersection of $Y_1 = 2\sin x + 3\cos x$ and $Y_2 = 4x$:

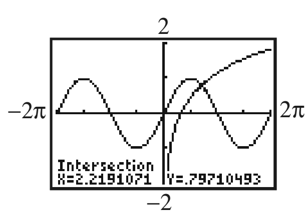


$x \approx 0.87$.

The solution set is $\{0.87\}$.

83. $\sin x = \ln x$

Find the intersection of $Y_1 = \sin x$ and $Y_2 = \ln x$:



$x \approx 2.22$

The solution set is $\{2.22\}$.

84. $-3\sin^{-1} x = \pi$

$$\sin^{-1} x = -\frac{\pi}{3}$$

$$x = \sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

The solution set is $\left\{-\frac{\sqrt{3}}{2}\right\}$.

85. $2\cos^{-1} x + \pi = 4\cos^{-1} x$

$$-2\cos^{-1} x + \pi = 0$$

$$-2\cos^{-1} x = -\pi$$

$$\cos^{-1} x = \frac{\pi}{2}$$

$$x = \cos\frac{\pi}{2} = 0$$

The solution set is $\{0\}$.

86. Using a half-angle formula:

$$\sin 15° = \sin\left(\frac{30°}{2}\right) = \sqrt{\frac{1-\cos 30°}{2}} = \sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}} = \sqrt{\frac{2-\sqrt{3}}{4}} = \frac{\sqrt{2-\sqrt{3}}}{2}$$

Note: since 15° lies in quadrant I, we have $\sin 15° > 0$.

Using a difference formula:

$$\sin 15° = \sin(45° - 30°)$$
$$= \sin(45°)\cos(30°) - \cos(45°)\sin(30°)$$
$$= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2}$$
$$= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \frac{\sqrt{6}-\sqrt{2}}{4} = \frac{1}{4}\left(\sqrt{6}-\sqrt{2}\right)$$

Verifying equality:

$$\begin{aligned}\frac{1}{4}\left(\sqrt{6}-\sqrt{2}\right)&=\frac{\sqrt{6}-\sqrt{2}}{4}\\&=\frac{\sqrt{2}\cdot\sqrt{3}-\sqrt{2}}{4}\\&=\frac{\sqrt{2}\left(\sqrt{3}-1\right)}{4}\\&=\sqrt{\left(\frac{\sqrt{2}\left(\sqrt{3}-1\right)}{4}\right)^2}\\&=\sqrt{\frac{2\left(3-2\sqrt{3}+1\right)}{16}}\\&=\sqrt{\frac{2\left(4-2\sqrt{3}\right)}{16}}\\&=\sqrt{\frac{2\cdot2\left(2-\sqrt{3}\right)}{16}}\\&=\sqrt{\frac{2-\sqrt{3}}{4}}\\&=\frac{\sqrt{2-\sqrt{3}}}{2}\end{aligned}$$

87. Given the value of $\cos\theta$, the most efficient Double-angle Formula to use is $\cos(2\theta)=2\cos^2\theta-1$.

Chapter 7 Test

1. Let $\theta=\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$. We seek the angle θ, such that $0\le\theta\le\pi$ and $\theta\ne\frac{\pi}{2}$, whose secant equals $\frac{2}{\sqrt{3}}$. The only value in the restricted range with a secant of $\frac{2}{\sqrt{3}}$ is $\frac{\pi}{6}$. Thus, $\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)=\frac{\pi}{6}$.

2. Let $\theta=\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$. We seek the angle θ, such that $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$, whose sine equals $-\frac{\sqrt{2}}{2}$. The only value in the restricted range with a sine of $-\frac{2}{\sqrt{2}}$ is $-\frac{\pi}{4}$. Thus, $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)=-\frac{\pi}{4}$.

3. $\sin^{-1}\left(\sin\frac{11\pi}{5}\right)$ follows the form of the equation $f^{-1}\left(f\left(x\right)\right)=\sin^{-1}\left(\sin\left(x\right)\right)=x$, but because $\frac{11\pi}{5}$ is not in the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, we cannot directly use the equation.
We need to find an angle θ in the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ for which $\sin\frac{11\pi}{5}=\sin\theta$. The angle $\frac{11\pi}{5}$ is in quadrant I. The reference angle of $\frac{11\pi}{5}$ is $\frac{\pi}{5}$ and $\sin\frac{11\pi}{5}=\sin\frac{\pi}{5}$. Since $\frac{\pi}{5}$ is in the interval $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, we can apply the equation above and get $\sin^{-1}\left(\sin\frac{11\pi}{5}\right)=\frac{\pi}{5}$.

4. $\tan\left(\tan^{-1}\frac{7}{3}\right)$ follows the form $f\left(f^{-1}\left(x\right)\right)=\tan\left(\tan^{-1}x\right)=x$. Since the domain of the inverse tangent is all real numbers, we can directly apply this equation to get $\tan\left(\tan^{-1}\frac{7}{3}\right)=\frac{7}{3}$.

5. $\cot\left(\csc^{-1}\sqrt{10}\right)$

Since $\csc^{-1}\theta=\frac{r}{y}=\sqrt{10}$, $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$, let $r=\sqrt{10}$ and $y=1$. Solve for *x*:

$$\begin{aligned}x^2+1^2&=\left(\sqrt{10}\right)^2\\x^2+1&=10\\x^2&=9\\x&=3\end{aligned}$$

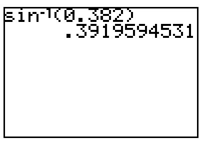
θ is in quadrant I.

Thus, $\cot\left(\csc^{-1}\sqrt{10}\right)=\cot\theta=\frac{x}{y}=\frac{3}{1}=3$.

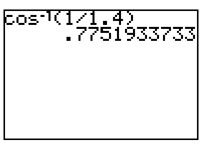
6. Let $\theta = \cos^{-1}\left(-\frac{3}{4}\right)$.

$$\sec\left[\cos^{-1}\left(-\frac{3}{4}\right)\right] = \sec\theta = \frac{1}{\cos\theta} = \frac{1}{\cos\left[\cos^{-1}\left(-\frac{3}{4}\right)\right]} = \frac{1}{-\frac{3}{4}} = -\frac{4}{3}$$

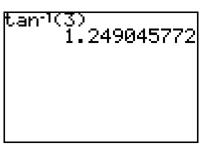
7. $\sin^{-1}(0.382) \approx 0.39$ radian



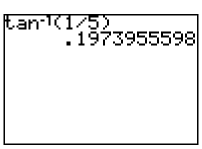
8. $\sec^{-1} 1.4 = \cos^{-1}\left(\frac{1}{1.4}\right) \approx 0.78$ radian



9. $\tan^{-1} 3 \approx 1.25$ radians



10. $\cot^{-1} 5 = \tan^{-1}\left(\frac{1}{5}\right) \approx 0.20$ radian



11.
$$\begin{aligned}
&\frac{\csc\theta + \cot\theta}{\sec\theta + \tan\theta} \\
&= \frac{\csc\theta + \cot\theta}{\sec\theta + \tan\theta} \cdot \frac{\csc\theta - \cot\theta}{\csc\theta - \cot\theta} \\
&= \frac{\csc^2\theta - \cot^2\theta}{(\sec\theta + \tan\theta)(\csc\theta - \cot\theta)} \\
&= \frac{1}{(\sec\theta + \tan\theta)(\csc\theta - \cot\theta)} \\
&= \frac{1}{(\sec\theta + \tan\theta)(\csc\theta - \cot\theta)} \cdot \frac{\sec\theta - \tan\theta}{\sec\theta - \tan\theta} \\
&= \frac{\sec\theta - \tan\theta}{(\sec^2\theta - \tan^2\theta)(\csc\theta - \cot\theta)} \\
&= \frac{\sec\theta - \tan\theta}{\csc\theta - \cot\theta}
\end{aligned}$$

12.
$$\begin{aligned}
\sin\theta\tan\theta + \cos\theta &= \sin\theta \cdot \frac{\sin\theta}{\cos\theta} + \cos\theta \\
&= \frac{\sin^2\theta}{\cos\theta} + \frac{\cos^2\theta}{\cos\theta} \\
&= \frac{\sin^2\theta + \cos^2\theta}{\cos\theta} \\
&= \frac{1}{\cos\theta} \\
&= \sec\theta
\end{aligned}$$

13.
$$\begin{aligned}
\tan\theta + \cot\theta &= \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} \\
&= \frac{\sin^2\theta}{\sin\theta\cos\theta} + \frac{\cos^2\theta}{\sin\theta\cos\theta} \\
&= \frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta} \\
&= \frac{1}{\sin\theta\cos\theta} \\
&= \frac{2}{2\sin\theta\cos\theta} \\
&= \frac{2}{\sin(2\theta)} \\
&= 2\csc(2\theta)
\end{aligned}$$

14. $\dfrac{\sin(\alpha+\beta)}{\tan\alpha+\tan\beta}$

$= \dfrac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\dfrac{\sin\alpha}{\cos\alpha}+\dfrac{\sin\beta}{\cos\beta}}$

$= \dfrac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\dfrac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}+\dfrac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}$

$= \dfrac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\dfrac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}$

$= \dfrac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{1}\cdot\dfrac{\cos\alpha\cos\beta}{\sin\alpha\cos\beta+\cos\alpha\sin\beta}$

$= \cos\alpha\cos\beta$

15. $\sin(3\theta)$

$= \sin(\theta+2\theta)$

$= \sin\theta\cos(2\theta)+\cos\theta\sin(2\theta)$

$= \sin\theta\left(\cos^2\theta-\sin^2\theta\right)+\cos\theta\cdot 2\sin\theta\cos\theta$

$= \sin\theta\cos^2\theta-\sin^3\theta+2\sin\theta\cos^2\theta$

$= 3\sin\theta\cos^2\theta-\sin^3\theta$

$= 3\sin\theta\left(1-\sin^2\theta\right)-\sin^3\theta$

$= 3\sin\theta-3\sin^3\theta-\sin^3\theta$

$= 3\sin\theta-4\sin^3\theta$

16. $\dfrac{\tan\theta-\cot\theta}{\tan\theta+\cot\theta} = \dfrac{\dfrac{\sin\theta}{\cos\theta}-\dfrac{\cos\theta}{\sin\theta}}{\dfrac{\sin\theta}{\cos\theta}+\dfrac{\cos\theta}{\sin\theta}}$

$= \dfrac{\dfrac{\sin^2\theta-\cos^2\theta}{\sin\theta\cos\theta}}{\dfrac{\sin^2\theta+\cos^2\theta}{\sin\theta\cos\theta}}$

$= \dfrac{\sin^2\theta-\cos^2\theta}{\sin^2\theta+\cos^2\theta}$

$= \dfrac{-\cos(2\theta)}{1}$

$= -\left(2\cos^2\theta-1\right)$

$= 1-2\cos^2\theta$

17. $\cos 15^\circ = \cos(45^\circ-30^\circ)$

$= \cos 45^\circ\cos 30^\circ+\sin 45^\circ\sin 30^\circ$

$= \dfrac{\sqrt{2}}{2}\cdot\dfrac{\sqrt{3}}{2}+\dfrac{\sqrt{2}}{2}\cdot\dfrac{1}{2}$

$= \dfrac{\sqrt{2}}{4}\left(\sqrt{3}+1\right)$

$= \dfrac{\sqrt{6}+\sqrt{2}}{4}$ or $\dfrac{1}{4}\left(\sqrt{6}+\sqrt{2}\right)$

18. $\tan 75^\circ = \tan(45^\circ+30^\circ)$

$= \dfrac{\tan 45^\circ+\tan 30^\circ}{1-\tan 45^\circ\tan 30^\circ}$

$= \dfrac{1+\dfrac{\sqrt{3}}{3}}{1-1\cdot\dfrac{\sqrt{3}}{3}}$

$= \dfrac{3+\sqrt{3}}{3-\sqrt{3}}$

$= \dfrac{3+\sqrt{3}}{3-\sqrt{3}}\cdot\dfrac{3+\sqrt{3}}{3+\sqrt{3}}$

$= \dfrac{9+6\sqrt{3}+3}{3^2-3}$

$= \dfrac{12+6\sqrt{3}}{6}$

$= 2+\sqrt{3}$

19. $\sin\left(\dfrac{1}{2}\cos^{-1}\dfrac{3}{5}\right)$

Let $\theta=\cos^{-1}\dfrac{3}{5}$. Since $0<\theta<\dfrac{\pi}{2}$ (from the range of $\cos^{-1}x$),

$\sin\left(\dfrac{1}{2}\theta\right)=\sqrt{\dfrac{1-\cos\theta}{2}}$

$= \sqrt{\dfrac{1-\cos\left(\cos^{-1}\dfrac{3}{5}\right)}{2}} = \sqrt{\dfrac{1-\frac{3}{5}}{2}}$

$= \sqrt{\dfrac{1}{5}} = \dfrac{\sqrt{5}}{5}$

20. $\tan\left(2\sin^{-1}\dfrac{6}{11}\right)$

Let $\theta=\sin^{-1}\dfrac{6}{11}$. Then $\sin\theta=\dfrac{6}{11}$ and θ lies in quadrant I. Since $\sin\theta=\dfrac{y}{r}=\dfrac{6}{11}$, let $y=6$ and $r=11$, and solve for *x*: $x^2+6^2=11^2$

$x^2=85$

$x=\sqrt{85}$

$$\tan\theta = \frac{y}{x} = \frac{6}{\sqrt{85}} = \frac{6\sqrt{85}}{85}$$

$$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2\left(\frac{6\sqrt{85}}{85}\right)}{1-\left(\frac{6\sqrt{85}}{85}\right)^2}$$

$$= \frac{\frac{12\sqrt{85}}{85}}{1-\frac{36}{85}} = \frac{12\sqrt{85}}{85}\cdot\frac{85}{49}$$

$$= \frac{12\sqrt{85}}{49}$$

21. $\cos\left(\sin^{-1}\frac{2}{3}+\tan^{-1}\frac{3}{2}\right)$

Let $\alpha = \sin^{-1}\frac{2}{3}$ and $\beta = \tan^{-1}\frac{3}{2}$. Then $\sin\alpha = \frac{2}{3}$ and $\tan\beta = \frac{3}{2}$, and both α and β lie in quadrant I. Since $\sin\alpha = \frac{y_1}{r_1} = \frac{2}{3}$, let $y_1 = 2$ and $r_1 = 3$. Solve for x_1: $x_1^2 + 2^2 = 3^2$

$$x_1^2 + 4 = 9$$
$$x_1^2 = 5$$
$$x_1 = \sqrt{5}$$

Thus, $\cos\alpha = \frac{x_1}{r_1} = \frac{\sqrt{5}}{3}$.

Since $\tan\beta = \frac{y_2}{x_2} = \frac{3}{2}$, let $x_2 = 2$ and $y_2 = 3$.

Solve for x_2: $2^2 + 3^2 = r_2^2$

$$4 + 9 = r_2^2$$
$$r_2^2 = 13$$
$$r_2 = \sqrt{13}$$

Thus, $\sin\beta = \frac{y_2}{r_2} = \frac{3}{\sqrt{13}}$.

Therefore, $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$$= \frac{\sqrt{5}}{3}\cdot\frac{2}{\sqrt{13}} - \frac{2}{3}\cdot\frac{3}{\sqrt{13}}$$
$$= \frac{2\sqrt{5}-6}{3\sqrt{13}}$$
$$= \frac{2\sqrt{13}\left(\sqrt{5}-3\right)}{39}$$

22. Let $\alpha = 75°$, $\beta = 15°$.

Since $\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta)+\sin(\alpha-\beta)\right]$,

$$\sin 75°\cos 15° = \frac{1}{2}\left[\sin(90°)+\sin(60°)\right]$$
$$= \frac{1}{2}\left[1+\frac{\sqrt{3}}{2}\right] = \frac{1}{4}\left(2+\sqrt{3}\right) = \frac{2+\sqrt{3}}{4}$$

23. $\sin 75° + \sin 15°$

$$= 2\sin\left(\frac{75°+15°}{2}\right)\cos\left(\frac{75°-15°}{2}\right)$$
$$= 2\sin(45°)\cos(30°) = 2\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{6}}{2}$$

24. $\cos 65°\cos 20° + \sin 65°\sin 20°$

$$= \cos(65°-20°) = \cos(45°) = \frac{\sqrt{2}}{2}$$

25. $4\sin^2\theta - 3 = 0$

$$4\sin^2\theta = 3$$
$$\sin^2\theta = \frac{3}{4}$$
$$\sin\theta = \pm\frac{\sqrt{3}}{2}$$

On the interval $[0, 2\pi)$, the sine function takes on a value of $\frac{\sqrt{3}}{2}$ when $\theta = \frac{\pi}{3}$ or $\theta = \frac{2\pi}{3}$. The sine takes on a value of $-\frac{\sqrt{3}}{2}$ when $\theta = \frac{4\pi}{3}$ and $\theta = \frac{5\pi}{3}$. The solution set is $\left\{\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}\right\}$.

26. $-3\cos\left(\frac{\pi}{2}-\theta\right) = \tan\theta$

$$-3\sin\theta = \tan\theta$$
$$0 = \frac{\sin\theta}{\cos\theta} + 3\sin\theta$$
$$0 = \sin\theta\left(\frac{1}{\cos\theta}+3\right)$$

$\sin\theta = 0$ or $\frac{1}{\cos\theta} + 3 = 0$

$$\cos\theta = -\frac{1}{3}$$

On the interval $[0, 2\pi)$, the sine function takes on a value of 0 when $\theta = 0$ or $\theta = \pi$. The cosine function takes on a value of $-\frac{1}{3}$ in the second and third quadrants when $\theta = \pi - \cos^{-1}\frac{1}{3}$ and $\theta = \pi + \cos^{-1}\frac{1}{3}$. That is $\theta \approx 1.911$ and $\theta \approx 4.373$. The solution set is $\{0, 1.911, \pi, 4.373\}$.

27. $\cos^2\theta + 2\sin\theta\cos\theta - \sin^2\theta = 0$

$$\left(\cos^2\theta - \sin^2\theta\right) + 2\sin\theta\cos\theta = 0$$
$$\cos(2\theta) + \sin(2\theta) = 0$$
$$\sin(2\theta) = -\cos(2\theta)$$
$$\tan(2\theta) = -1$$

The tangent function takes on the value -1 when its argument is $\frac{3\pi}{4} + k\pi$. Thus, we need

$$2\theta = \frac{3\pi}{4} + k\pi$$
$$\theta = \frac{3\pi}{8} + k\frac{\pi}{2}$$
$$\theta = \frac{\pi}{8}(3 + 4k)$$

On the interval $[0, 2\pi)$, the solution set is $\left\{\frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}, \frac{15\pi}{8}\right\}$.

28. $\sin(\theta + 1) = \cos\theta$

$$\sin\theta\cos 1 + \cos\theta\sin 1 = \cos\theta$$
$$\frac{\sin\theta\cos 1 + \cos\theta\sin 1}{\cos\theta} = \frac{\cos\theta}{\cos\theta}$$
$$\tan\theta\cos 1 + \sin 1 = 1$$
$$\tan\theta\cos 1 = 1 - \sin 1$$
$$\tan\theta = \frac{1 - \sin 1}{\cos 1}$$

Therefore, $\theta = \tan^{-1}\left(\frac{1 - \sin 1}{\cos 1}\right) \approx 0.285$ or

$$\theta = \pi + \tan^{-1}\left(\frac{1 - \sin 1}{\cos 1}\right) \approx 3.427$$

The solution set is $\{0.285, 3.427\}$.

29. $4\sin^2\theta + 7\sin\theta = 2$

$$4\sin^2\theta + 7\sin\theta - 2 = 0$$

Let $u = \sin\theta$. Then,

$$4u^2 + 7u - 2 = 0$$
$$(4u - 1)(u + 2) = 0$$

$$4u - 1 = 0 \quad \text{or} \quad u + 2 = 0$$
$$4u = 1 \qquad\qquad u = -2$$
$$u = \frac{1}{4}$$

Substituting back in terms of θ, we have

$$\sin\theta = \frac{1}{4} \quad \text{or} \quad \sin\theta = -2$$

The second equation has no solution since $-1 \le \sin\theta \le 1$ for all values of θ. Therefore, we only need to find values of θ between 0 and 2π such that $\sin\theta = \frac{1}{4}$. These will occur in the first and second quadrants. Thus, $\theta = \sin^{-1}\frac{1}{4} \approx 0.253$ and $\theta = \pi - \sin^{-1}\frac{1}{4} \approx 2.889$.

The solution set is $\{0.253, 2.889\}$.

Chapter 7 Cumulative Review

1. $3x^2 + x - 1 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1^2 - 4(3)(-1)}}{2(3)} = \frac{-1 \pm \sqrt{1 + 12}}{6} = \frac{-1 \pm \sqrt{13}}{6}$$

The solution set is $\left\{\frac{-1 - \sqrt{13}}{6}, \frac{-1 + \sqrt{13}}{6}\right\}$.

2. Line containing points $(-2, 5)$ and $(4, -1)$:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1 - 5}{4 - (-2)} = \frac{-6}{6} = -1$$

Using $y - y_1 = m(x - x_1)$ with point $(4, -1)$,

$$y - (-1) = -1(x - 4)$$
$$y + 1 = -1(x - 4)$$
$$y + 1 = -x + 4$$
$$y = -x + 3 \quad \text{or} \quad x + y = 3$$

Distance between points $(-2,5)$ and $(4,-1)$:

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$
$$=\sqrt{(4-(-2))^2+(-1-5)^2}$$
$$=\sqrt{6^2+(-6)^2}=\sqrt{72}=\sqrt{36\cdot 2}=6\sqrt{2}$$

Midpoint of segment with endpoints $(-2,5)$ and $(4,-1)$:

$$\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)=\left(\frac{-2+4}{2},\frac{5+(-1)}{2}\right)=(1,2)$$

3. $3x+y^2=9$

x-intercept: $3x+0^2=9$; $(3,0)$

$$3x=9$$
$$x=3$$

y-intercepts: $3(0)+y^2=9$; $(0,-3)$, $(0,3)$

$$y^2=9$$
$$y=\pm 3$$

Tests for symmetry:

x-axis: Replace y with $-y$: $3x+(-y)^2=9$

$$3x+y^2=9$$

Since we obtain the original equation, the graph is symmetric with respect to the x-axis.

y-axis: Replace x with $-x$: $3(-x)+y^2=9$

$$-3x+y^2=9$$

Since we do not obtain the original equation, the graph is not symmetric with respect to the y-axis.

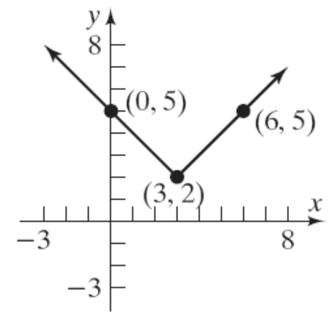
Origin: Replace x with $-x$ and y with $-y$:

$$3(-x)+(-y)^2=9$$
$$-3x+y^2=9$$

Since we do not obtain the original equation, the graph is not symmetric with respect to the origin.

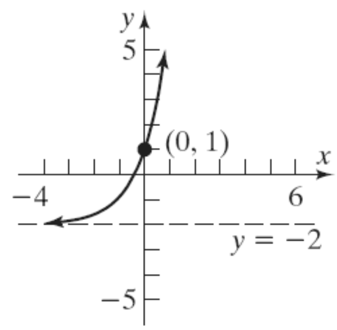
4. $y=|x-3|+2$

Using the graph of $y=|x|$, shift horizontally to the right 3 units and vertically up 2 units.



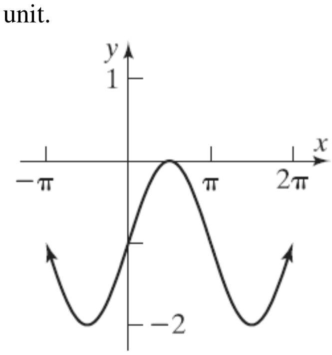
5. $y=3e^x-2$

Using the graph of $y=e^x$, stretch vertically by a factor of 3, and shift down 2 units.

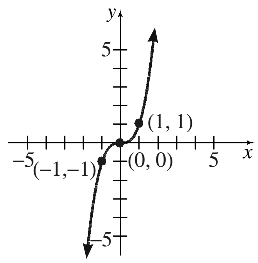


6. $y=\cos\left(x-\frac{\pi}{2}\right)-1$

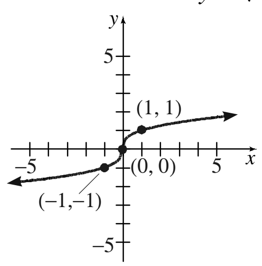
Using the graph of $y=\cos x$, horizontally shift to the right $\frac{\pi}{2}$ units, and vertically shift down 1 unit.

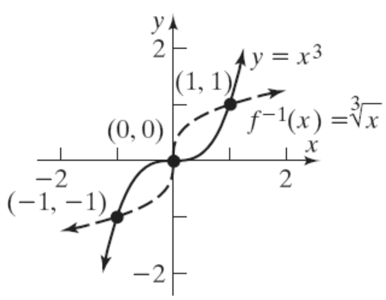


7. a. $y = x^3$

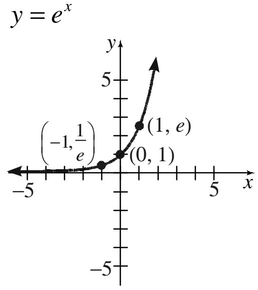


Inverse function: $y = \sqrt[3]{x}$

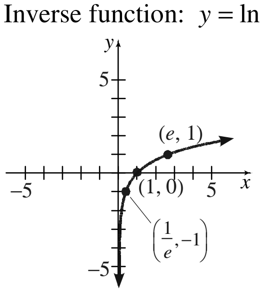


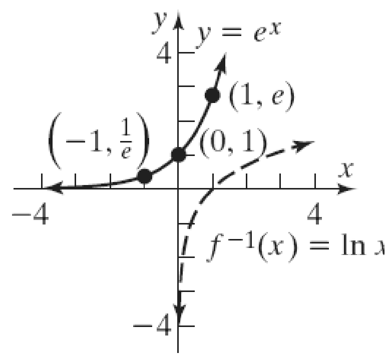


b. $y = e^x$

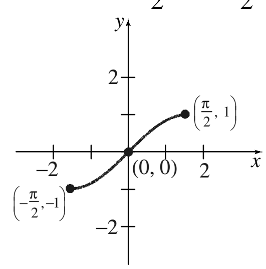


Inverse function: $y = \ln x$

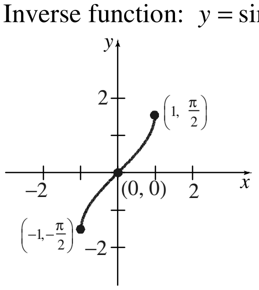


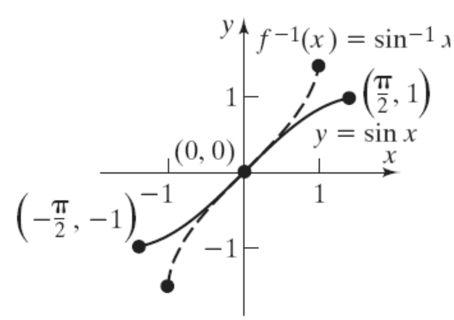


c. $y = \sin x$, $-\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2}$

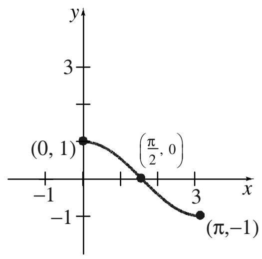


Inverse function: $y = \sin^{-1} x$

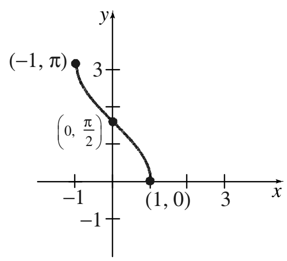


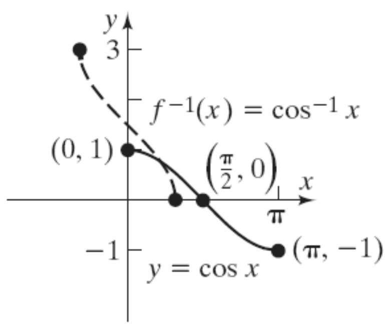


d. $y = \cos x$, $0 \le x \le \pi$



Inverse function: $y = \cos^{-1} x$





8. $\sin\theta = -\frac{1}{3}, \pi < \theta < \frac{3\pi}{2}$, so θ lies in Quadrant III.

a. In Quadrant III, $\cos\theta < 0$

$$\cos\theta = -\sqrt{1-\sin^2\theta} = -\sqrt{1-\left(-\frac{1}{3}\right)^2}$$
$$= -\sqrt{1-\frac{1}{9}} = -\sqrt{\frac{8}{9}}$$
$$= -\frac{2\sqrt{2}}{3}$$

b. $$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-\frac{1}{3}}{-\frac{2\sqrt{2}}{3}}$$
$$= -\frac{1}{3}\left(-\frac{3}{2\sqrt{2}}\right) = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$$

c. $$\sin(2\theta) = 2\sin\theta\cos\theta = 2\left(-\frac{1}{3}\right)\left(-\frac{2\sqrt{2}}{3}\right)$$
$$= \frac{4\sqrt{2}}{9}$$

d. $$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$
$$= \left(-\frac{2\sqrt{2}}{3}\right)^2 - \left(\frac{1}{3}\right)^2 = \frac{8}{9} - \frac{1}{9} = \frac{7}{9}$$

e. Since $\pi < \theta < \frac{3\pi}{2}$, we have that $\frac{\pi}{2} < \frac{\theta}{2} < \frac{3\pi}{4}$.

Thus, $\frac{1}{2}\theta$ lies in Quadrant II and $\sin\left(\frac{1}{2}\theta\right) > 0$.

$$\sin\left(\frac{1}{2}\theta\right) = \sqrt{\frac{1-\cos\theta}{2}} = \sqrt{\frac{1-\left(-\frac{2\sqrt{2}}{3}\right)}{2}}$$
$$= \sqrt{\frac{\frac{3+2\sqrt{2}}{3}}{2}} = \sqrt{\frac{3+2\sqrt{2}}{6}}$$

f. Since $\frac{1}{2}\theta$ lies in Quadrant II, $\cos\left(\frac{1}{2}\theta\right) < 0$.

$$\cos\left(\frac{1}{2}\theta\right) = -\sqrt{\frac{1+\cos\theta}{2}} = -\sqrt{\frac{1+\left(-\frac{2\sqrt{2}}{3}\right)}{2}}$$
$$= -\sqrt{\frac{\frac{3-2\sqrt{2}}{3}}{2}} = -\sqrt{\frac{3-2\sqrt{2}}{6}}$$

9. $\cos\left(\tan^{-1} 2\right)$

Let $\theta = \tan^{-1} 2$. Then $\tan\theta = \frac{y}{x} = \frac{2}{1}$,

$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Let $x = 1$ and $y = 2$.

Solve for *r*: $r^2 = x^2 + y^2$
$$r^2 = 1^2 + 2^2$$
$$r^2 = 5$$
$$r = \sqrt{5}$$

θ is in quadrant I.

$$\cos\left(\tan^{-1} 2\right) = \cos\theta = \frac{x}{r} = \frac{1}{\sqrt{5}} = \frac{1}{\sqrt{5}}\cdot\frac{\sqrt{5}}{\sqrt{5}} = \frac{\sqrt{5}}{5}$$

10. $\sin\alpha = \frac{1}{3}, \frac{\pi}{2} < \alpha < \pi;\ \cos\beta = -\frac{1}{3}, \pi < \beta < \frac{3\pi}{2}$

a. Since $\frac{\pi}{2} < \alpha < \pi$, we know that α lies in Quadrant II and $\cos\alpha < 0$.

$$\cos\alpha = -\sqrt{1-\sin^2\alpha} = -\sqrt{1-\left(\frac{1}{3}\right)^2} = -\sqrt{1-\frac{1}{9}} = -\sqrt{\frac{8}{9}} = -\frac{2\sqrt{2}}{3}$$

b. $\pi < \beta < \frac{3\pi}{2}$, we know that β lies in Quadrant III and $\sin\beta < 0$.

$$\sin\beta = -\sqrt{1-\cos^2\beta} = -\sqrt{1-\left(-\frac{1}{3}\right)^2} = -\sqrt{1-\frac{1}{9}} = -\sqrt{\frac{8}{9}} = -\frac{2\sqrt{2}}{3}$$

c. $$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha = \left(-\frac{2\sqrt{2}}{3}\right)^2 - \left(\frac{1}{3}\right)^2 = \frac{8}{9} - \frac{1}{9} = \frac{7}{9}$$

d. $$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta = -\frac{2\sqrt{2}}{3}\left(-\frac{1}{3}\right) - \frac{1}{3}\left(-\frac{2\sqrt{2}}{3}\right) = \frac{2\sqrt{2}}{9} + \frac{2\sqrt{2}}{9} = \frac{4\sqrt{2}}{9}$$

e. Since $\pi < \beta < \frac{3\pi}{2}$, we have that $\frac{\pi}{2} < \frac{\beta}{2} < \frac{3\pi}{4}$. Thus, $\frac{\beta}{2}$ lies in Quadrant II and $\sin\frac{\beta}{2} > 0$.

$$\sin\frac{\beta}{2} = \sqrt{\frac{1-\cos\beta}{2}} = \sqrt{\frac{1-\left(-\frac{1}{3}\right)}{2}} = \sqrt{\frac{\frac{4}{3}}{2}} = \sqrt{\frac{4}{6}} = \frac{2}{\sqrt{6}} = \frac{2\sqrt{6}}{6} = \frac{\sqrt{6}}{3}$$

11. $f(x) = 2x^5 - x^4 - 4x^3 + 2x^2 + 2x - 1$

a. $f(x)$ has at most 5 real zeros.

Possible rational zeros:

$p = \pm 1;\ q = \pm 1, \pm 2;\ \frac{p}{q} = \pm 1, \pm\frac{1}{2}$

Using the Bounds on Zeros Theorem:

$f(x) = 2\left(x^5 - 0.5x^4 - 2x^3 + x^2 + x - 0.5\right)$

$a_4 = -0.5, a_3 = -2, a_2 = 1, a_1 = 1, a_0 = -0.5$

$\text{Max}\ \{1, |-0.5| + |1| + |1| + |-2| + |-0.5|\}$
$= \text{Max}\ \{1, 5\} = 5$

$1 + \text{Max}\ \{|-0.5|, |1|, |1|, |-2|, |-0.5|\}$
$= 1 + 2 = 3$

The smaller of the two numbers is 3. Thus, every zero of f must lie between -3 and 3.

Use synthetic division with –1:

−1)	2	−1	−4	2	2	−1
		−2	3	1	−3	1
	2	−3	−1	3	−1	0

Since the remainder is 0, $x-(-1) = x+1$ is a factor. The other factor is the quotient: $2x^4 - 3x^3 - x^2 + 3x - 1$.

Use synthetic division with 1 on the quotient:

1)	2	−3	−1	3	−1
		2	−1	−2	1
	2	−1	−2	1	0

Since the remainder is 0, $x-1$ is a factor. The other factor is the quotient: $2x^3 - x^2 - 2x + 1$.

Factoring:

$$2x^3 - x^2 - 2x + 1 = x^2(2x-1) - 1(2x-1) = (2x-1)(x^2-1) = (2x-1)(x-1)(x+1)$$

Therefore,

$$f(x) = (2x-1)(x-1)^2(x+1)^2 = 2\left(x-\frac{1}{2}\right)(x-1)^2(x+1)^2$$

The real zeros are -1 and 1 (both with multiplicity 2) and $\frac{1}{2}$ (multiplicity 1).

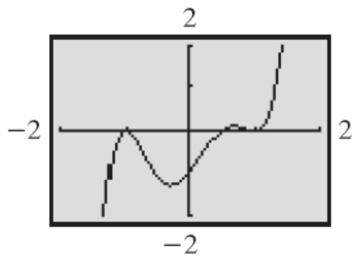
b. x-intercepts: $1, \frac{1}{2}, -1$

y-intercept: -1

The intercepts are $(0,-1)$, $(1,0)$, $\left(\frac{1}{2},0\right)$, and $(-1,0)$

c. f resembles the graph of $y = 2x^5$ for large $|x|$.

d. Let $Y_1 = 2x^5 - x^4 - 4x^3 + 2x^2 + 2x - 1$

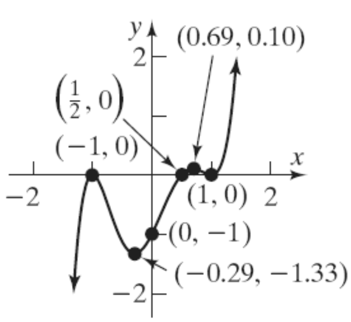


e. Four turning points exist. Use the MAXIMUM and MINIMUM features to locate local maxima at $(-1,0)$, $(0.69,0.10)$ and local minima at $(1,0)$, $(-0.29,-1.33)$.

f. To graph by hand, we determine some additional information about the intervals between the x-intercepts:

Interval	$(-\infty,-1)$	$(-1,0.5)$	$(0.5,1)$	$(1,\infty)$
Test number	−2	0	0.7	2
Value of f	−45	−1	≈ 0.1	27
Location	Below x-axis	Below x-axis	Above x-axis	Above x-axis
Point	$(-2,-45)$	$(0,-1)$	$(0.7,0.1)$	$(2,27)$

f is above the x-axis for $(0.5,1)$ and $(1,\infty)$, and below the x-axis for $(-\infty,-1)$ and $(-1,0.5)$.



g. f is increasing on $(-\infty,-1)$, $(-0.29,0.69)$, and $(1,\infty)$. f is decreasing on $(-1,-0.29)$ and $(0.69,1)$.

12. $f(x) = 2x^2 + 3x + 1$; $g(x) = x^2 + 3x + 2$

a.
$$f(x) = 0$$
$$2x^2 + 3x + 1 = 0$$
$$(2x+1)(x+1) = 0$$
$$x = -\frac{1}{2} \text{ or } x = -1$$

The solution set is $\left\{-1, -\frac{1}{2}\right\}$.

b.
$$f(x) = g(x)$$
$$2x^2 + 3x + 1 = x^2 + 3x + 2$$
$$x^2 - 1 = 0$$
$$(x+1)(x-1) = 0$$
$$x = -1 \text{ or } x = 1$$

The solution set is $\{-1, 1\}$.

c.
$$f(x) > 0$$
$$2x^2 + 3x + 1 > 0$$
$$(2x+1)(x+1) > 0$$
$$f(x) = (2x+1)(x+1)$$

The zeros of f are $x = -\frac{1}{2}$ and $x = -1$

Interval	$(-\infty,-1)$	$\left(-1,-\frac{1}{2}\right)$	$\left(-\frac{1}{2},\infty\right)$
Test number	−2	−0.75	0
Value of f	3	−0.125	1
Conclusion	Positive	Negative	Positive

The solution set is $(-\infty,-1) \cup \left(-\frac{1}{2},\infty\right)$.

d. $f(x) \ge g(x)$

$2x^2 + 3x + 1 \ge x^2 + 3x + 2$

$x^2 - 1 \ge 0$

$(x+1)(x-1) \ge 0$

$p(x) = (x-1)(x+1)$

The zeros of p are $x = -1$ and $x = 1$.

Interval	$(-\infty, -1)$	$(-1, 1)$	$(1, \infty)$
Test number	–2	0	2
Value of p	3	–1	3
Conclusion	Positive	Negative	Positive

The solution set is $(-\infty, -1] \cup [1, \infty)$.

Chapter 7 Projects

Project I – Internet Based Project

Project II

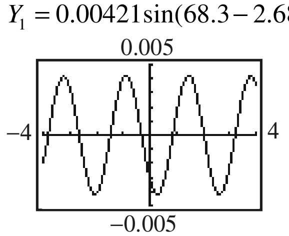
a. Amplitude = 0.00421 m

b. $\omega = 2.68$ radians/sec

c. $f = \dfrac{\omega}{2\pi} = \dfrac{2.68}{2\pi} \approx 0.4265$ vibrations/sec

d. $\lambda = \dfrac{2\pi}{k} = \dfrac{2\pi}{68.3} \approx 0.09199 \text{ m}$

e. If $x = 1$, the resulting equation is $y = 0.00421\sin(68.3 - 2.68t)$. To graph, let $Y_1 = 0.00421\sin(68.3 - 2.68x)$.



f. Note: $(kx - \omega t) + (kx - \omega t + \phi) = 2kx - 2\omega t + \phi$ and $(kx - \omega t) - (kx - \omega t + \phi) = -\phi$.

$$y_1 + y_2 = y_m \sin(kx - \omega t) + y_m \sin(kx - \omega t + \phi)$$
$$= y_m[\sin(kx - \omega t) + \sin(kx - \omega t + \phi)]$$
$$= y_m\left[2\sin\left(\frac{2kx - 2wt + \phi}{2}\right)\cos\left(\frac{-\phi}{2}\right)\right]$$
$$= 2y_m \sin\left(\frac{2kx - 2wt + \phi}{2}\right)\cos\left(\frac{\phi}{2}\right)$$

g. $y_m = 0.0045$, $\phi = 2.5$, $\lambda = 0.09$, $f = 2.3$

Let $x = 1$:

$\lambda = 0.09 = \dfrac{2\pi}{k}$ $\qquad$ $f = 2.3 = \dfrac{\omega}{2\pi}$

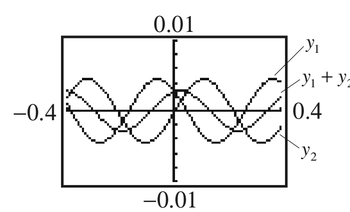
$k = \dfrac{200\pi}{9} \approx 69.8$ $\qquad$ $\omega = 4.6\pi \approx 14.45$

$$y_1 = y_m \sin(kx - \omega t)$$
$$= 0.0045\sin(69.8 \cdot 1 - 14.45t)$$
$$= 0.0045\sin(69.8 - 14.45t)$$

$$y_2 = y_m \sin(kx - \omega t + \phi)$$
$$= 0.0045\sin(69.8 \cdot 1 - 14.45t + 2.5)$$
$$= 0.0045\sin(72.3 - 14.45t)$$

$$y_1 + y_2 = 2y_m \sin\left(\frac{2kx - 2\omega t + \phi}{2}\right)\cos\left(\frac{\phi}{2}\right)$$
$$= 2 \cdot 0.0045\sin\left(\frac{2 \cdot 69.8 \cdot 1 - 2 \cdot 14.45t + 2.5}{2}\right)\cos\left(\frac{2.5}{2}\right)$$
$$= 0.009\sin\left(\frac{142.1 - 28.9t}{2}\right)\cos(1.25)$$
$$= 0.009\sin(71.05 - 14.45t)\cos(1.25)$$

h. Let $Y_1 = 0.0045\sin(69.8 - 14.45x)$, $Y_2 = 0.0045\sin(72.3 - 14.45x)$, and $Y_3 = 0.009\sin(71.05 - 14.45x)\cos(1.25)$.



i. $y_m = 0.0045$, $\phi = 0.4$, $\lambda = 0.09$, $f = 2.3$

Let $x = 1$:

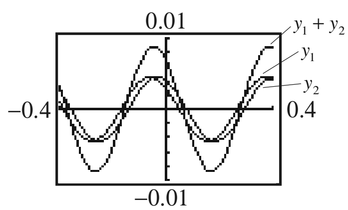
$\lambda = 0.09 = \dfrac{2\pi}{k}$ $\qquad$ $f = 2.3 = \dfrac{\omega}{2\pi}$

$k = \dfrac{200\pi}{9} = 69.8$ $\qquad$ $\omega = 4.6\pi = 14.45$

$$y_1 = 0.0045\sin(69.8 - 14.45t)$$
$$\begin{aligned} y_2 &= y_m \sin(kx - \omega t + \phi) \\ &= 0.0045\sin(69.8 \cdot 1 - 14.45t + 0.4) \\ &= 0.0045\sin(70.2 - 14.45t) \end{aligned}$$

$$\begin{aligned} y_1 + y_2 &= 2y_m \sin\left(\frac{2kx - 2\omega t + \phi}{2}\right)\cos\left(\frac{\phi}{2}\right) \\ &= 2 \cdot 0.0045\sin\left(\frac{2 \cdot 69.8 \cdot 1 - 2 \cdot 14.45t + 0.4}{2}\right)\cos\left(\frac{0.4}{2}\right) \\ &= 0.009\sin\left(\frac{140 - 28.9t}{2}\right)\cos(0.2) \\ &= 0.009\sin\left(70 - 14.45t\right)\cos(0.2) \end{aligned}$$

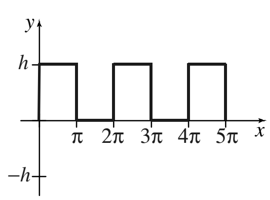
Let $Y_1 = 0.0045\sin(69.8 - 14.45x)$,
$Y_2 = 0.0045\sin(70.2 - 14.45x)$, and
$Y_3 = 0.009\sin\left(70 - 14.45x\right)\cos(0.2)$.



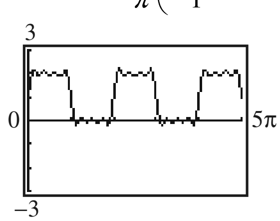
j. The phase shift causes the amplitude of $y_1 + y_2$ to increase from $0.009\cos(1.25) \approx 0.003$ to $0.009\cos(0.2) \approx 0.009$.

Project III

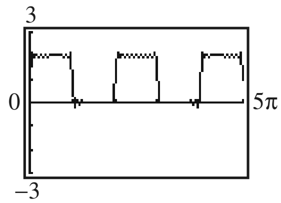
a.



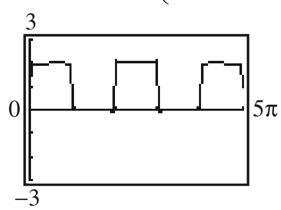
b. Let $Y_1 = 1 + \frac{4}{\pi}\left(\frac{\sin x}{1} + \frac{\sin(3x)}{3} + \frac{\sin(5x)}{5} + \frac{\sin(7x)}{7}\right)$



c. Let $Y_1 = 1 + \frac{4}{\pi}\left(\frac{\sin x}{1} + \frac{\sin(3x)}{3} + ... + \frac{\sin(17x)}{17}\right)$



d. Let $Y_1 = 1 + \frac{4}{\pi}\left(\frac{\sin x}{1} + \frac{\sin(3x)}{3} + ... + \frac{\sin(37x)}{37}\right)$



e. The best one is the one with the most terms.

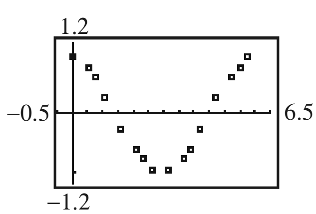
Project IV

a. $f(x) = \sin x$ (see table column 2)

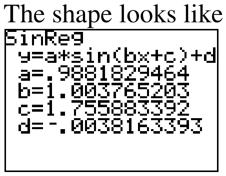
x	$f(x)$	$g(x)$	$h(x)$	$k(x)$	$m(x)$
0	0	0.954	–0.311	–0.749	6.085
$\frac{\pi}{6}$	$\frac{1}{2}$	0.791	–0.703	2.437	4.011
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	0.607	–1.341	1.387	–3.052
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	0.256	–0.978	0.588	–1.243
$\frac{\pi}{2}$	1	–0.256	–0.670	–0.063	0.413
$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	–0.607	–0.703	0.153	8.507
$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	–0.791	–0.623	2.380	–6.822
$\frac{5\pi}{6}$	$\frac{1}{2}$	–0.954	0	0.594	–2.695
π	0	–0.954	0.311	–0.817	1.536
$\frac{7\pi}{6}$	$-\frac{1}{2}$	–0.791	–0.117	–0.013	–5.248
$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	–0.607	1.341	–1.387	3.052
$\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2}$	–0.256	0.978	–0.588	1.243
$\frac{3\pi}{2}$	–1	0.256	0.670	0.063	–0.705
$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2}$	0.607	0.703	–0.306	
$\frac{7\pi}{4}$	$-\frac{\sqrt{2}}{2}$	0.791	0.623		
$\frac{11\pi}{6}$	$-\frac{1}{2}$	0.954			
2π	1				

b. $g(x) = \dfrac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}$ (see table column 3)

c.



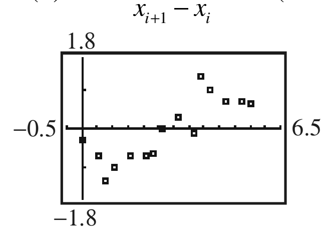
The shape looks like a sinusoidal graph.



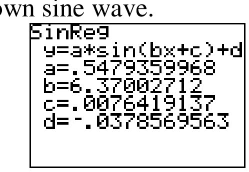
Rounding a, b, c, and d to the nearest tenth, we have that $y = \sin(x + 1.8)$.

Barring error due to rounding and approximation, this looks like $y = \cos x$

d. $h(x) = \dfrac{g(x_{i+1}) - g(x_i)}{x_{i+1} - x_i}$ (see table column 4)

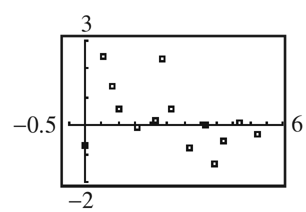


The shape is sinusoidal. It looks like an upside-down sine wave.

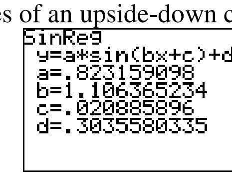


Rounding a, b, c, and d to the nearest tenth, we have that $y = 0.5\sin(6.4x)$.

e. $k(x) = \dfrac{h(x_{i+1}) - h(x_i)}{x_{i+1} - x_i}$ (see table column 5)



This curve is losing its sinusoidal features, although it still looks like one. It takes on the features of an upside-down cosine curve



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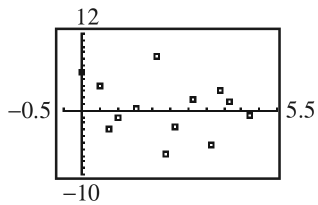
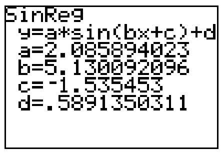
Rounding a, b, c, and d to the nearest tenth, we have that $y = 0.8\sin(1.1x) + 0.3$.

Note: The rounding error is getting greater and greater.

f. $m(x) = \dfrac{k(x_{i+1}) - k(x_i)}{x_{i+1} - x_i}$ (see table column 6)



The sinusoidal features are gone.



Rounding a, b, c, and d to the nearest tenth, we have that $y = 2.1\sin(5.1x - 1.5) + 0.6$.

g. It would seem that the curves would be less “involved”, but the rounding error has become incredibly great that the points are nowhere near accurate at this point in calculating the differences.