Chapter 4
Polynomial and Rational Functions

Section 4.1

1. $(-2,0)$, $(2,0)$, and $(0,9)$
x-intercepts: let $y=0$ and solve for x

$$\begin{aligned} 9x^2+4(0)&=36\\ 9x^2&=36\\ x^2&=4\\ x&=\pm 2 \end{aligned}$$

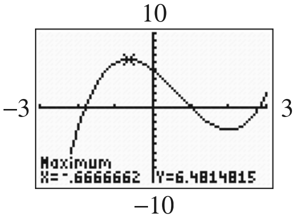
y-intercepts: let $x=0$ and solve for y

$$\begin{aligned} 9(0)^2+4y&=36\\ 4y&=36\\ y&=9 \end{aligned}$$

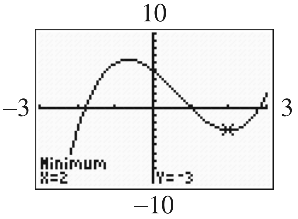
2. Yes; it has the form
$a_nx^n+a_{n-1}x^{n-1}+...+a_1x+a_0$ where each a_i is a real number and *n* is a positive integer.; degree 3

3. down; 4

4. Local maximum 6.48 at $x=-0.67$;
Local minimum -3 at $x=2$.



Maximum
X=-.6666662 Y=6.4814815



Minimum
X=2 Y=-3

5. True: $f(x)=0$ indicates that $y=0$ which indicates that the point is an x-intercept.

6. The point $(5,0)$ is on the graph and is the x-intercept of g.

7. smooth; continuous

8. touches

9. $(-1,1)$, $(0,0)$, and $(1,1)$

10. a. r is a real zero of a polynomial function f.
b. r is an x-intercept of the graph of f.
c. $x-r$ is a factor of f.

11. turning points

12. $y=3x^4$

13. ∞; $-\infty$

14. As x increases in the positive direction, $f(x)$ decreases without bound.

15. $f(x)=4x+x^3$ is a polynomial function of degree 3.

16. $f(x)=5x^2+4x^4$ is a polynomial function of degree 4.

17. $g(x)=\dfrac{1-x^2}{2}=\dfrac{1}{2}-\dfrac{1}{2}x^2$ is a polynomial function of degree 2.

18. $h(x)=3-\dfrac{1}{2}x$ is a polynomial function of degree 1.

19. $f(x)=1-\dfrac{1}{x}=1-x^{-1}$ is not a polynomial function because it contains a negative exponent.

20. $f(x)=x(x-1)=x^2-x$ is a polynomial function of degree 2.

21. $g(x)=x^{3/2}-x^2+2$ is not a polynomial function because it contains a fractional exponent.

22. $h(x)=\sqrt{x}\left(\sqrt{x}-1\right)=x-x^{1/2}$ is not a polynomial function because it contains fractional exponents.

23. $F(x)=5x^4-\pi x^3+\dfrac{1}{2}$ is a polynomial function of degree 4.

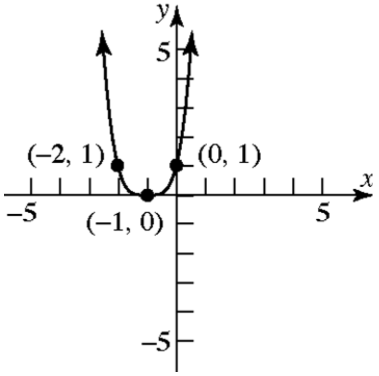
24. $F(x)=\dfrac{x^2-5}{x^3}=x^{-1}-5x^{-3}$ is not a polynomial function because it contains a negative exponent.

25. $G(x)=2(x-1)^2(x^2+1)=2(x^2-2x+1)(x^2+1)$
$=2(x^4+x^2-2x^3-2x+x^2+1)$
$=2(x^4-2x^3+2x^2-2x+1)$
$=2x^4-4x^3+4x^2-4x+2$
is a polynomial function of degree 4.

26. $G(x)=-3x^2(x+2)^3=-3x^2(x^3+6x^2+12x+8)$
$=-3x^5-18x^4-36x^3-24x^2$
is a polynomial function of degree 5.

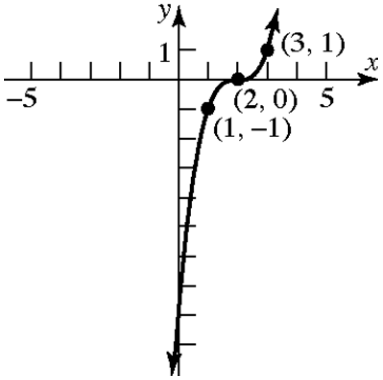
27. $f(x)=(x+1)^4$

Using the graph of $y=x^4$, shift the graph horizontally, 1 unit to the left.



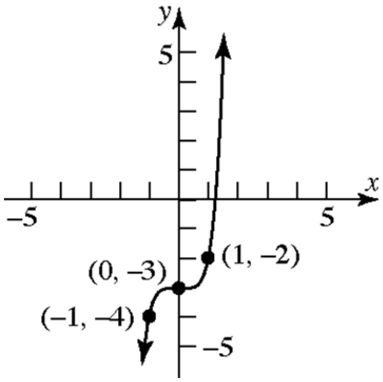
28. $f(x)=(x-2)^5$

Using the graph of $y=x^5$, shift the graph horizontally to the right 2 units.



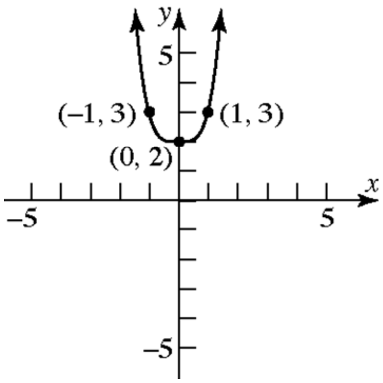
29. $f(x)=x^5-3$

Using the graph of $y=x^5$, shift the graph vertically, 3 units down.



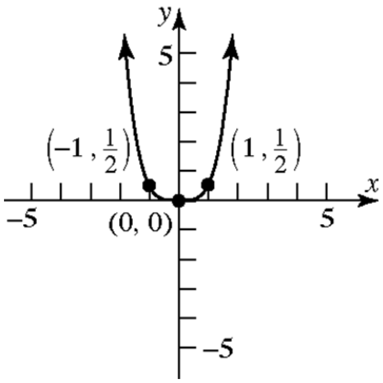
30. $f(x)=x^4+2$

Using the graph of $y=x^4$, shift the graph vertically up 2 units.



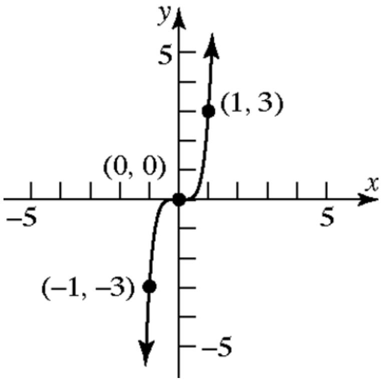
31. $f(x)=\frac{1}{2}x^4$

Using the graph of $y=x^4$, compress the graph vertically by a factor of $\frac{1}{2}$.



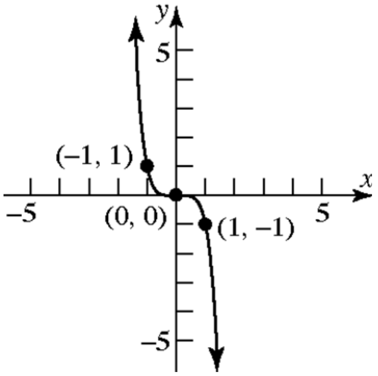
32. $f(x)=3x^5$

Using the graph of $y=x^5$, stretch the graph vertically by a factor of 3.



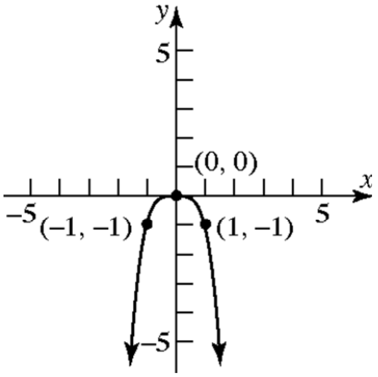
33. $f(x) = -x^5$

Using the graph of $y = x^5$, reflect the graph about the x-axis.



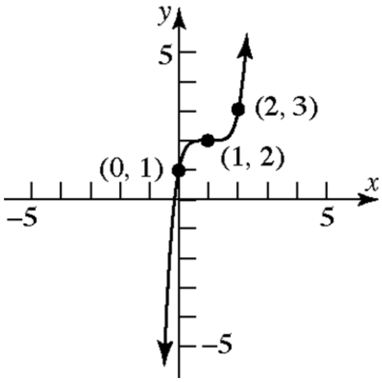
34. $f(x) = -x^4$

Using the graph of $y = x^4$, reflect the graph about the x-axis.



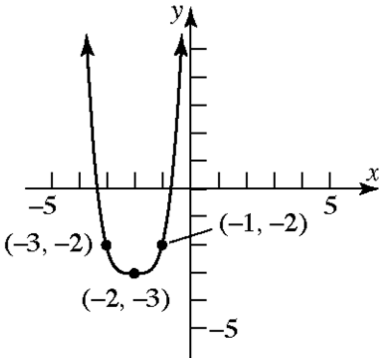
35. $f(x) = (x-1)^5 + 2$

Using the graph of $y = x^5$, shift the graph horizontally, 1 unit to the right, and shift vertically 2 units up.



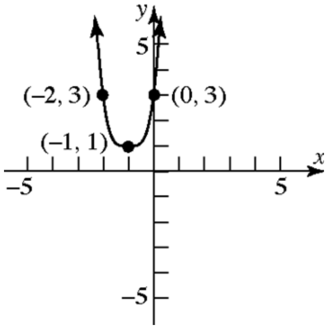
36. $f(x) = (x+2)^4 - 3$

Using the graph of $y = x^4$, shift the graph horizontally left 2 units, and shift vertically down 3 units.



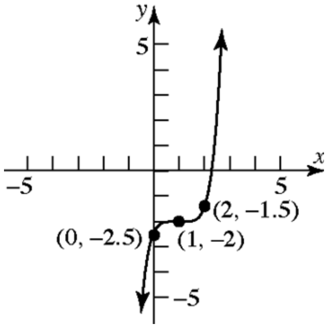
37. $f(x) = 2(x+1)^4 + 1$

Using the graph of $y = x^4$, shift the graph horizontally, 1 unit to the left, stretch vertically by a factor of 2, and shift vertically 1 unit up.



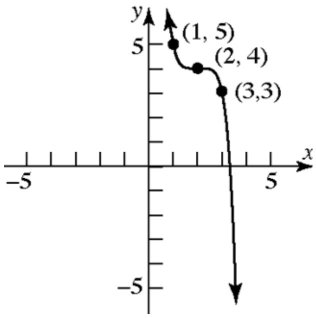
38. $f(x) = \frac{1}{2}(x-1)^5 - 2$

Using the graph of $y = x^5$, shift the graph horizontally 1 unit to the right, compress vertically by a factor of $\frac{1}{2}$, and shift vertically down 2 units.



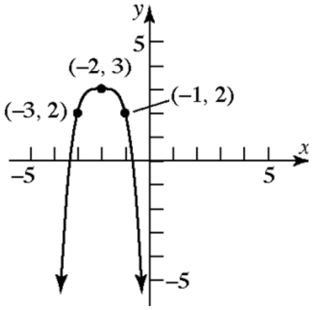
39. $f(x) = 4-(x-2)^5 = -(x-2)^5 + 4$

Using the graph of $y = x^5$, shift the graph horizontally, 2 units to the right, reflect about the x-axis, and shift vertically 4 units up.



40. $f(x) = 3-(x+2)^4 = -(x+2)^4 + 3$

Using the graph of $y = x^4$, shift the graph horizontally, 2 units to the left, reflect about the x-axis, and shift vertically 3 units up.



41. $f(x) = a\left(x-(-1)\right)(x-1)(x-3)$

For $a = 1$:

$f(x) = (x+1)(x-1)(x-3) = \left(x^2-1\right)(x-3)$

$= x^3 - 3x^2 - x + 3$

42. $f(x) = a\left(x-(-2)\right)(x-2)(x-3)$

For $a = 1$:

$f(x) = (x+2)(x-2)(x-3) = \left(x^2-4\right)(x-3)$

$= x^3 - 3x^2 - 4x + 12$

43. $f(x) = a\left(x-(-3)\right)(x-0)(x-4)$

For $a = 1$:

$f(x) = (x+3)(x)(x-4) = \left(x^2+3x\right)(x-4)$

$= x^3 - 4x^2 + 3x^2 - 12x$

$= x^3 - x^2 - 12x$

44. $f(x) = a\left(x-(-4)\right)(x-0)(x-2)$

For $a = 1$:

$f(x) = (x+4)(x)(x-2) = \left(x^2+4x\right)(x-2)$

$= x^3 - 2x^2 + 4x^2 - 8x$

$= x^3 + 2x^2 - 8x$

45. $f(x) = a\left(x-(-4)\right)\left(x-(-1)\right)(x-2)(x-3)$

For $a = 1$:

$f(x) = (x+4)(x+1)(x-2)(x-3)$

$= \left(x^2+5x+4\right)\left(x^2-5x+6\right)$

$= x^4 - 5x^3 + 6x^2 + 5x^3 - 25x^2 + 30x + 4x^2 - 20x + 24$

$= x^4 - 15x^2 + 10x + 24$

46. $f(x) = a\left(x-(-3)\right)\left(x-(-1)\right)(x-2)(x-5)$

For $a = 1$:

$f(x) = (x+3)(x+1)(x-2)(x-5)$

$= \left(x^2+4x+3\right)\left(x^2-7x+10\right)$

$= x^4 - 7x^3 + 10x^2 + 4x^3 - 28x^2$

$+ 40x + 3x^2 - 21x + 30$

$= x^4 - 3x^3 - 15x^2 + 19x + 30$

47. $f(x) = a\left(x-(-1)\right)\left(x-3\right)^2$

For $a = 1$:

$f(x) = (x+1)(x-3)^2$

$= \left(x+1\right)\left(x^2-6x+9\right)$

$= x^3 - 6x^2 + 9x + x^2 - 6x + 9$

$= x^3 - 5x^2 + 3x + 9$

48. $f(x)=a\left(x-(-2)\right)^2\left(x-4\right)$

For $a=1$:

$$f(x)=(x+2)^2(x-4)$$
$$=\left(x^2+4x+4\right)\left(x-4\right)$$
$$=x^3-4x^2+4x^2-16x+4x-16$$
$$=x^3-12x-16$$

49. a. The real zeros of $f(x)=3(x-7)(x+3)^2$ are: 7, with multiplicity one; and –3, with multiplicity two.

b. The graph crosses the x-axis at 7 (odd multiplicity) and touches it at –3 (even multiplicity).

c. $n-1=3-1=2$

d. The function resembles $y=3x^3$ for large values of $|x|$.

50. a. The real zeros of $f(x)=4(x+4)(x+3)^3$ are: –4, with multiplicity one; and –3, with multiplicity three.

b. The graph crosses the *x*-axis at –4 and at –3 (odd multiplicities).

c. $n-1=4-1=3$

d. The function resembles $y=4x^4$ for large values of $|x|$.

51. a. The real zeros of $f(x)=4\left(x^2+1\right)(x-2)^3$ is: 2, with multiplicity three. $x^2+1=0$ has no real solution.

b. The graph crosses the *x*-axis at 2 (odd multiplicity).

c. $n-1=5-1=4$

d. The function resembles $y=4x^5$ for large values of $|x|$.

52. a. The real zeros of $f(x)=2\left(x-3\right)(x^2+4)^3$ is: 3, with multiplicity one. $x^2+4=0$ has no real solution.

b. The graph crosses the x-axis at 3

c. $n-1=7-1=6$

d. The function resembles $y=2x^7$ for large values of $|x|$.

53. a. The real zero of $f(x)=-2\left(x+\frac{1}{2}\right)^2\left(x+4\right)^3$ are: $-\frac{1}{2}$, with multiplicity two; -4 with multiplicity 3.

b. The graph touches the x-axis at $-\frac{1}{2}$ (even multiplicity) and crosses the x-axis at -4 (odd multiplicity).

c. $n-1=5-1=4$

d. The function resembles $y=-2x^5$ for large values of $|x|$.

54. a. The real zeros of $f(x)=\left(x-\frac{1}{3}\right)^2\left(x-1\right)^3$ are: $\frac{1}{3}$, with multiplicity two; and 1, with multiplicity 3.

b. The graph touches the *x*-axis at $\frac{1}{3}$ (even multiplicity), and crosses the x-axis at 1 (odd multiplicity).

c. $n-1=5-1=4$

d. The function resembles $y=x^5$ for large values of $|x|$.

55. a. The real zeros of $f(x)=(x-5)^3(x+4)^2$ are: 5, with multiplicity three; and –4, with multiplicity two.

b. The graph crosses the x-axis at 5 (odd multiplicity) and touches it at –4 (even multiplicity).

c. $n-1=5-1=4$

d. The function resembles $y=x^5$ for large values of $|x|$.

56. a. The real zeros of $f(x)=\left(x+\sqrt{3}\right)^2(x-2)^4$ are: $-\sqrt{3}$, with multiplicity two; and 2, with multiplicity four.

b. The graph touches the x-axis at $-\sqrt{3}$ and at 2 (even multiplicities).

c. $n-1=6-1=5$

d. The function resembles $y=x^6$ for large values of $|x|$.

57. a. $f(x)=3\left(x^2+8\right)\left(x^2+9\right)^2$ has no real zeros. $x^2+8=0$ and $x^2+9=0$ have no real solutions.

b. The graph neither touches nor crosses the x-axis.

c. $n-1=6-1=5$

d. The function resembles $y=3x^6$ for large values of $|x|$.

58. a. $f(x)=-2\left(x^2+3\right)^3$ has no real zeros. $x^2+3=0$ has no real solutions.

b. The graph neither touches nor crosses the x-axis.

c. $n-1=6-1=5$

d. The function resembles $y=-2x^6$ for large values of $|x|$.

59. a. The real zeros of $f(x)=-2x^2(x^2-2)$ are: $-\sqrt{2}$ and $\sqrt{2}$ with multiplicity one; and 0, with multiplicity two.

b. The graph touches the x-axis at 0 (even multiplicity) and crosses the x-axis at $-\sqrt{2}$ and $\sqrt{2}$ (odd multiplicities).

c. $n-1=4-1=3$

d. The function resembles $y=-2x^4$ for large values of $|x|$.

60. a. The real zeros of $f(x)=4x(x^2-3)$ are: $-\sqrt{3}$, $\sqrt{3}$ and 0, with multiplicity one.

b. The graph crosses the x-axis at $-\sqrt{3}$, $\sqrt{3}$ and 0 (odd multiplicities).

c. $n-1=3-1=2$

d. The function resembles $y=4x^3$ for large values of $|x|$.

61. The graph could be the graph of a polynomial function.; zeros: $-1, 1, 2$; min degree = 3

62. The graph could be the graph of a polynomial.; zeros: $-1, 2$; min degree = 4

63. The graph cannot be the graph of a polynomial.; not continuous at $x=-1$

64. The graph cannot be the graph of a polynomial.; not smooth at $x=0$

65. The graph crosses the x-axis at $x=0$, $x=1$, and $x=2$. Thus, each of these zeros has an odd multiplicity. Using one for each of these multiplicities, a possible function is $f(x)=ax(x-1)(x-2)$**.** Since the y-intercept is 0, we know $f(0)=0$**.** Thus, *a* can be any positive constant. Using $a=1$**,** the function is $f(x)=x(x-1)(x-2)$**.**

66. The graph crosses the x-axis at $x=0$ and $x=2$ and touches at $x=1$. Thus, 0 and 2 each have odd multiplicities while 1 has an even multiplicity. Using one for each odd multiplicity and two for the even multiplicity, a possible function is $f(x)=ax(x-1)^2(x-2)$**.** Since the y-intercept is 0, we know $f(0)=0$**.** Thus, *a* can be any positive constant. Using $a=1$**,** the function is $f(x)=x(x-1)^2(x-2)$**.**

67. The graph crosses the *x*-axis at $x=-1$ and $x=2$ and touches it at $x=1$. Thus, -1 and 2 each have odd multiplicities while 1 has an even multiplicity. Using one for each odd multiplicity and two for the even multiplicity, a possible funtion is $f(x)=ax(x-1)^2(x-2)$. Since the *y*-intercept is 1, we know $f(0)=1$. Thus,

$$\begin{aligned} a(0+1)(0-1)(0-2)&=1\\ a(1)(-1)(-2)&=1\\ 2a&=1\\ a&=\frac{1}{2}\end{aligned}$$

The function is $f(x)=-\frac{1}{2}(x+1)(x-1)^2(x-2)$.

68. The graph crosses the *x*-axis at $x=-1$, $x=1$, and $x=2$. Thus, each of these zeros has an odd multiplicity. Using one for each of these multiplicities, a possible funtion is $f(x)=a(x+1)(x-1)(x-2)$. Since the *y*-intercept is -1, we know $f(0)=-1$. Thus,

$$a(0+1)(0-1)(0-2)=-1$$
$$a(1)(-1)(-2)=-1$$
$$2a=-1$$
$$a=-\frac{1}{2}$$

The function is $f(x)=-\frac{1}{2}(x+1)(x-1)(x-2)$.

69. $f(x)=x^2(x-3)$

Step 1: Degree is 3. The function resembles $y=x^3$ for large values of $|x|$.

Step 2: y-intercept: $f(0)=0^2(0-3)=0$

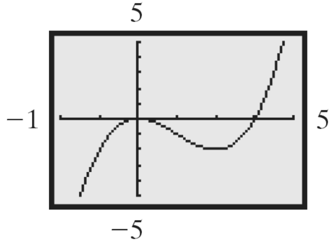
x-intercepts: solve $f(x)=0$

$0=x^2(x-3)$

$x=0,\ x=3$

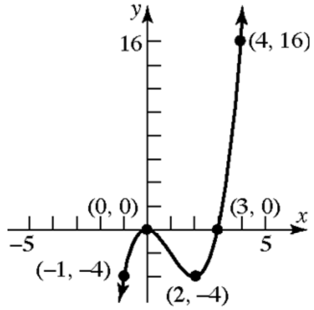
Step 3: Real zeros: 0 with multiplicity two, 3 with multiplicity one. The graph touches the x-axis at $x=0$ and crosses the x-axis at $x=3$.

Step 4: Graphing utility:



Step 5: 2 turning points; local maximum: (0, 0); local minimum: (2, –4)

Step 6: Graphing by hand:



Step 7: Domain: $(-\infty,\infty)$; Range: $(-\infty,\infty)$

Step 8: Increasing on $(-\infty, 0)$ and $(2, \infty)$; decreasing on $(0, 2)$

70. $f(x)=x(x+2)^2$

Step 1: Degree is 3. The function resembles $y=x^3$ for large values of $|x|$.

Step 2: y-intercept: $f(0)=0(0+2)^2=0$

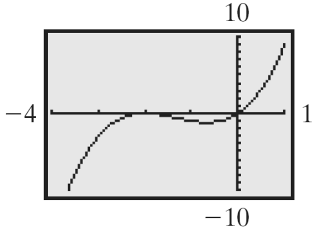
x-intercepts: solve $f(x)=0$

$0=x(x+2)^2$

$x=0,-2$

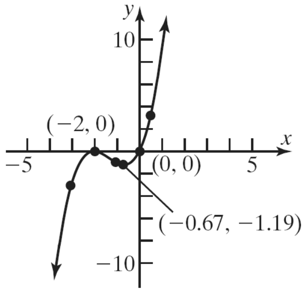
Step 3: Real zeros: -2 with multiplicity two, 0 with multiplicity one. The graph touches the x-axis at $x=-2$ and crosses the x-axis at $x=0$.

Step 4: Graphing utility:



Step 5: 2 turning points;
local maximum: (–2, 0);
local minimum: (–0.67, –1.19)

Step 6: Graphing by hand:



Step 7: Domain: $(-\infty,\infty)$; Range: $(-\infty,\infty)$

Step 8: Increasing on $(-\infty,-2)$ and $(-0.67,\infty)$; decreasing on $(-2,-0.67)$.

71. $f(x)=(x+4)(x-2)^2$

Step 1: Degree is 3. The function resembles $y=x^3$ for large values of $|x|$.

Step 2: y-intercept: $f(0)=(0+4)(0-2)^2=16$

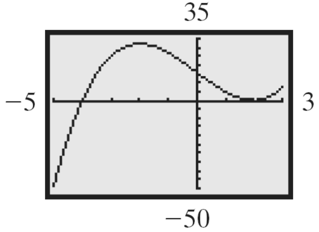
x-intercepts: solve $f(x)=0$

$0=(x+4)(x-2)^2$

$x=-4, 2$

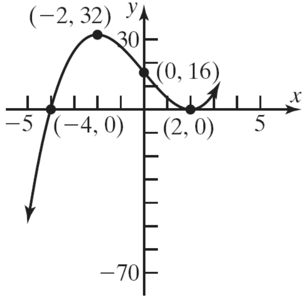
Step 3: Real zeros: –4 with multiplicity one, 2 with multiplicity two. The graph crosses the x-axis at $x = -4$ and touches the x-axis at $x = 2$

Step 4: Graphing utility:



Step 5: 2 turning points;
local maximum: $(-2, 32)$
local minimum: $(2, 0)$

Step 6: Graphing by hand;



Step 7: Domain: $(-\infty, \infty)$; Range: $(-\infty, \infty)$

Step 8: Increasing on $(-\infty, -2)$ and $(2, \infty)$; decreasing on $(-2, 2)$

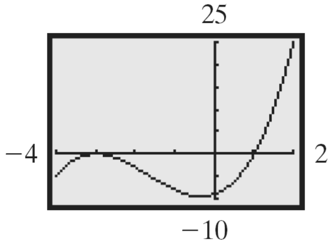
72. $f(x) = (x-1)(x+3)^2$

Step 1: Degree is 3. The function resembles $y = x^3$ for large values of $|x|$.

Step 2: y-intercept: $f(0) = (0-1)(0+3)^2 = -9$
x-intercepts: solve $f(x) = 0$
$0 = (x-1)(x+3)^2$
$x = 1, -3$

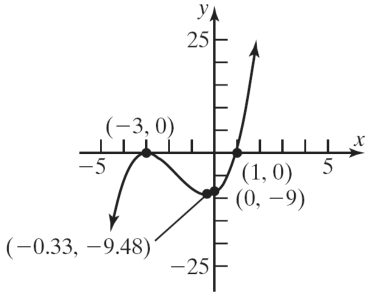
Step 3: Real zeros: -3 with multiplicity two, 1 with multiplicity one. The graph touches the x-axis at $x = -3$ and crosses it at $x = 1$.

Step 4: Graphing utility;



Step 5: 2 turning points;
local maximum: $(-3, 0)$
local minimum: $(-0.33, -9.48)$

Step 6: Graphing by hand;



Step 7: Domain: $(-\infty, \infty)$; Range: $(-\infty, \infty)$

Step 8: Increasing on $(-\infty, -3)$ and $(-0.33, \infty)$; decreasing on $(-3, -0.33)$

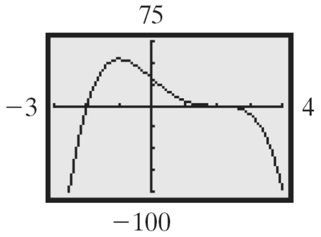
73. $f(x) = -2(x+2)(x-2)^3$

Step 1: Degree is 4. The function resembles $y = -2x^4$ for large values of $|x|$.

Step 2: y-intercept: $f(0) = -2(0+2)(0-2)^3 = 32$
x-intercepts: solve $f(x) = 0$
$0 = -2(x+2)(x-2)^3$
$x = -2, 2$

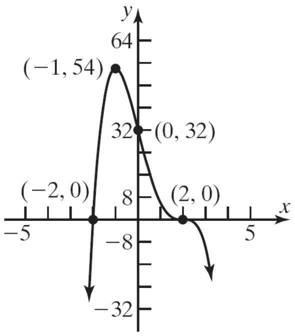
Step 3: Real zeros: -2 with multiplicity one, 2 with multiplicity three. The graph crosses the x-axis at $x = -2$ and $x = 2$.

Step 4: Graphing utility;



Step 5: 1 turning point; local maximum: $(-1, 54)$

Step 6: Graphing by hand;



Step 7: Domain: $(-\infty, \infty)$; Range: $(-\infty, 54]$

Step 8: Increasing on $(-\infty, -1)$;
decreasing on $(-1, \infty)$

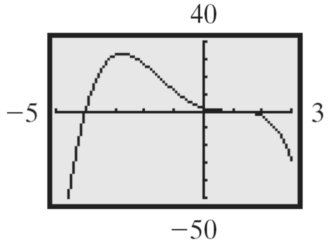
74. $f(x) = -\frac{1}{2}(x+4)(x-1)^3$

Step 1: Degree is 4. The function resembles $y = -\frac{1}{2}x^4$ for large value of $|x|$.

Step 2: y-intercept: $f(0) = -\frac{1}{2}(0+4)(0-1)^3 = 2$
x-intercepts: solve $f(x) = 0$
$0 = -\frac{1}{2}(x+4)(x-1)^3$
$x = -4, 1$

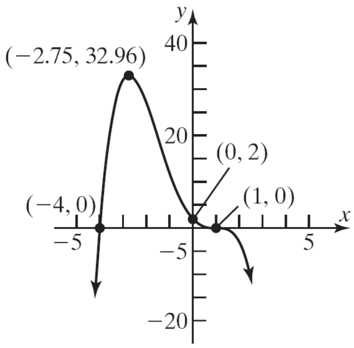
Step 3: Real zeros: -4 with multiplicity one, 1 with multiplicity three. The graph crosses the x-axis at $x = -4$ and $x = 1$.

Step 4: Graphing utility;



Step 5: 2 turning points:
local maximum at $(-2.75, 32.96)$

Step 6: Graphing by hand;



Step 7: Domain: $(-\infty, \infty)$; Range: $(-\infty, 32.96]$

Step 8: Increasing on $(-\infty, -2.75)$;
decreasing on $(-2.75, \infty)$

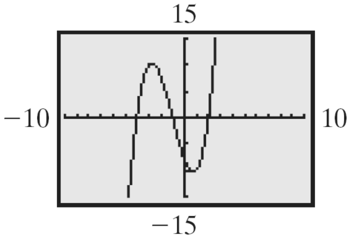
75. $f(x) = (x+1)(x-2)(x+4)$

Step 1: Degree is 3. The function resembles $y = x^3$ for large values of $|x|$.

Step 2: y-intercept: $f(0) = (0+1)(0-2)(0+4)$
$= -8$
x-intercepts: solve $f(x) = 0$
$0 = (x+1)(x-2)(x+4)$
$x = -1, 2, -4$

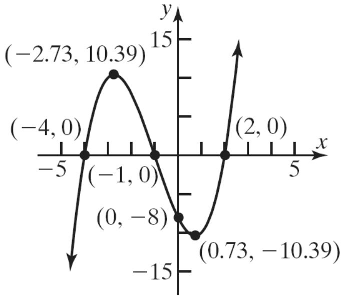
Step 3: Real zeros: -4 with multiplicity one, -1 with multiplicity one, 2 with multiplicity one. The graph crosses the x-axis at $x = -4, -1, 2$.

Step 4: Graphing utility:



Step 5: 2 turning points;
local maximum: (–2.73, 10.39);
local minimum: (0.73, –10.39)

Step 6: Graphing by hand;



Step 7: Domain: $(-\infty, \infty)$; Range: $(-\infty, \infty)$

Step 8: Increasing on $(-\infty, -2.73)$ and $(0.73, \infty)$; decreasing on $(-2.73, 0.73)$

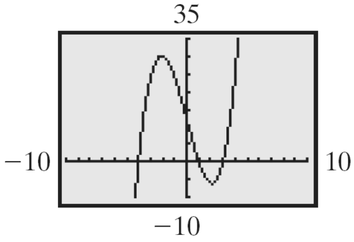
76. $f(x) = (x-1)(x+4)(x-3)$

Step 1: Degree is 3. The function resembles $y = x^3$ for large values of $|x|$

Step 2: y-intercept: $f(0) = (0-1)(0+4)(0-3)$ $= -12$
x-intercepts: solve $f(x) = 0$
$0 = (x-1)(x+4)(x-3)$
$x = 1, -4, 3$

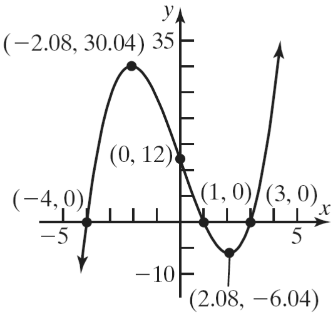
Step 3: Real zeros: –4 with multiplicity one, 1 with multiplicity one, 3 with multiplicity one. The graph crosses the x-axis at $x = -4, 1, 3$.

Step 4: Graphing utility:



Step 5: 2 turning points;
local maximum: (–2.08, 30.04);
local minimum: (2.08, –6.04)

Step 6: Graphing by hand;



Step 7: Domain: $(-\infty, \infty)$; Range: $(-\infty, \infty)$

Step 8: Increasing on $(-\infty, -2.08)$ and $(2.08, \infty)$; decreasing on $(-2.08, 2.08)$.

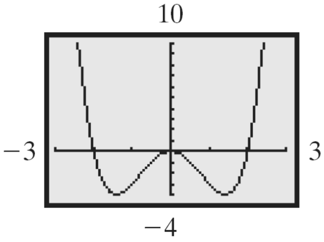
77. $f(x) = x^2(x-2)(x+2)$

Step 1: Degree is 4. The function resembles $y = x^4$ for large values of $|x|$.

Step 2: y-intercept: $f(x) = 0^2(0-2)(0+2) = 0$
x-intercepts: solve $f(x) = 0$
$0 = x^2(x-2)(x+2)$
$x = 0, 2, -2$

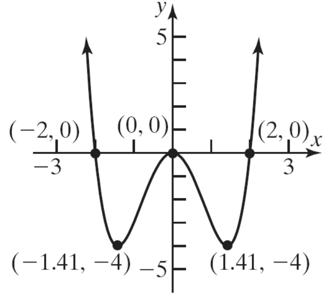
Step 3: Real zeros: –2 with multiplicity one, 0 with multiplicity two, 2 with multiplicity one. The graph crosses the x-axis at $x = -2$ and $x = -2$, and touches it at $x = 0$.

Step 4: Graphing utility:



Step 5: 3 turning points;
local maximum: (0, 0);
local minima: (–1.41, –4), (1.41, –4)

Step 6: graphing by hand



Step 7: Domain: $(-\infty, \infty)$; Range: $[-4, \infty)$

Step 8: Increasing on $(-1.41, 0)$ and $(1.41, \infty)$; decreasing on $(-\infty, -1.41)$ and $(0, 1.41)$.

78. $f(x) = x^2(x-3)(x+4)$

Step 1: Degree is 4. The graph of the function resembles $y = x^4$ for large values of $|x|$.

Step 2: y-intercept: $f(0) = 0^2(0-3)(0+4) = 0$

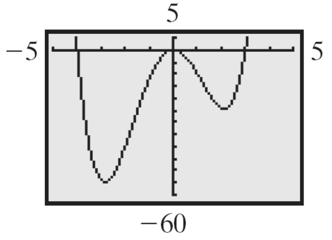
x-intercept: solve $f(x) = 0$

$x^2(x-3)(x+4) = 0$

$x = 0$ or $x = 3$ or $x = -4$

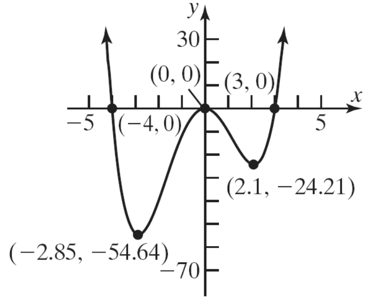
Step 3: Real zeros: –4 with multiplicity one, 0 with multiplicity two, 3 with multiplicity one. The graph crosses the x-axis at x = 3 and x = –4, and touches it at x = 0.

Step 4: Graphing utility:



Step 5: 3 turning points;
local maximum: (0, 0);
local minima: (–2.85, –54.64), (2.10, –24.21)

Step 6: Graphing by hand:



Step 7: Domain: $(-\infty, \infty)$; Range: $[-54.64, \infty)$

Step 8: Increasing on $(-2.85, 0)$ and $(2.10, \infty)$; decreasing on $(-\infty, -2.85)$ and $(0, 2.10)$.

79. $f(x) = (x+1)^2(x-2)^2$

Step 1: Degree is 4. The graph of the function resembles $y = x^4$ for large values of $|x|$.

Step 2: y-intercept: $f(0) = (0+1)^2(0-2)^2 = 4$

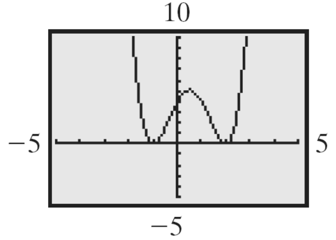
x-intercepts: solve $f(x) = 0$

$(x+1)^2(x-2)^2 = 0$

$x = -1$ or $x = 2$

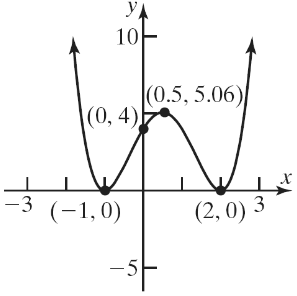
Step 3: Real zeros: −1 with multiplicity two, 2 with multiplicity two. The graph touches the x-axis at $x = -1$ and $x = 2$.

Step 4: Graphing utility:



Step 5: 3 turning points;
local maximum: (0.5, 5.06);
local minima: $(-1, 0)$, $(2, 0)$

Step 6: Graphing by hand:



Step 7: Domain: $(-\infty, \infty)$; Range: $[0, \infty)$

Step 8: Increasing on $(-1, 0.5)$ and $(2, \infty)$; decreasing on $(-\infty, -1)$ and $(0.5, 2)$.

80. $f(x) = (x+1)^3(x-3)$

Step 1: Degree is 4. The graph of the function resembles $y = x^4$ for large values of $|x|$.

Step 2: y-intercept: $f(0) = (0+1)^3(0-3) = -3$

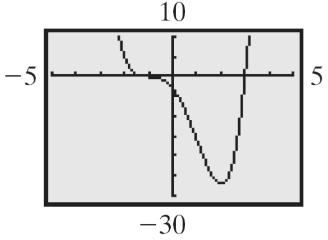
x-intercept: solve $f(x) = 0$

$(x+1)^3(x-3) = 0$

$x = -1$ or $x = 3$

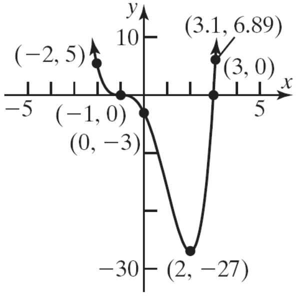
Step 3: Real zeros: -1 with multiplicity three, 3 with multiplicity one. The graph crosses the x-axis at $x = -1$ and $x = 3$.

Step 4: Graphing utility;



Step 5: 1 turning point;
local minimum: (2, –27)

Step 6: Graphing by hand;



Step 7: Domain: $(-\infty, \infty)$; Range: $[-27, \infty)$

Step 8: Increasing on $(2, \infty)$; decreasing on $(-\infty, 2)$

81. $f(x) = x^2(x-3)(x+1)$

Step 1: Degree is 4. The graph of the function resembles $y = x^4$ for large values of $|x|$.

Step 2: y-intercept: $f(0) = (0)^2(0-3)(0+1) = -3$

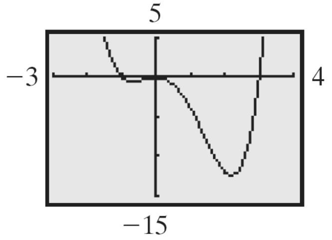
x-intercept: solve $f(x) = 0$

$x^2(x-3)(x+1) = 0$

$x = 0$ or $x = 3$ or $x = -1$

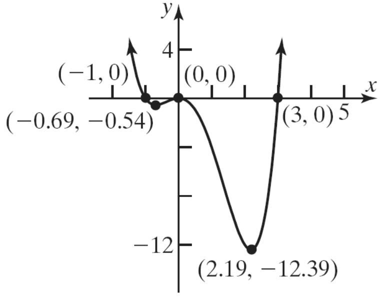
Step 3: Real zeros: 0 with multiplicity two, 3 with multiplicity one, -1 with multiplicity one. The graph touches the x-axis at x = 0, and crosses it at x = 3 and x = –1.

Step 4: Graphing utility:



Step 5: 3 turning points;
local maximum: (0, 0);
local minima: (–0.69, –0.54), (2.19, –12.39)

Step 6: Graphing by hand;



Step 7: Domain: $(-\infty, \infty)$; Range: $[-12.39, \infty)$

Step 8: Increasing on $(-0.69, 0)$ and $(2.19, \infty)$; decreasing on $(-\infty, -0.69)$ and $(0, 2.19)$

82. $f(x) = x^2(x-3)(x-1)$

Step 1: Degree is 4. The graph of the function resembles $y = x^4$ for large values of $|x|$.

Step 2: y-intercept: $f(0) = 0^2(0-3)(0-1) = 0$

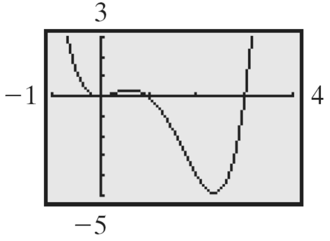
x-intercept: solve $f(x) = 0$

$0 = x^2(x-3)(x-1)$

$x = 0, 3, 1$

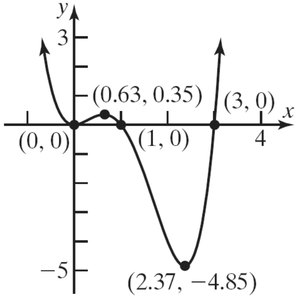
Step 3: Real zeros: 0 with multiplicity two, 3 with multiplicity one, 1 with multiplicity one. The graph touches the x-axis at $x = 0$, and crosses it at $x = 3$ and $x = 1$.

Step 4: Graphing utility:



Step 5: 3 turning points;
local maximum: (0.63, 0.35)
local minimum: (0, 0) and (2.37, –4.85)

Step 6: Graphing by hand;



Step 7: Domain: $(-\infty, \infty)$; Range: $[-4.85, \infty)$

Step 8: Increasing on $(0, 0.63)$ and $(2.37, \infty)$;
decreasing on $(-\infty, 0)$ and $(0.63, 2.37)$

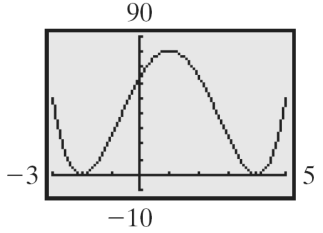
83. $f(x) = (x+2)^2(x-4)^2$

Step 1: Degree is 4. The graph of the function resembles $y = x^4$ for large values of $|x|$.

Step 2: y-intercept: $f(0) = (0+2)^2(0-4)^2 = 64$
x-intercept: solve $f(x) = 0$
$(x+2)^2(x-4)^2 = 0$
$x = -2$ or $x = 4$

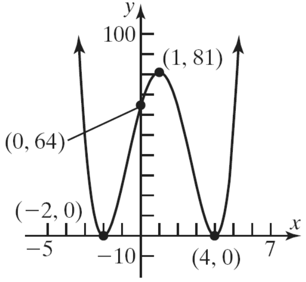
Step 3: Real zeros: -2 with multiplicity two, 4 with multiplicity two. The graph touches the x-axis at $x = -2$ and $x = 4$.

Step 4: Graphing utility:



Step 5: 3 turning points;
local maximum $(1, 81)$;
local minima $(-2, 0)$ and $(4, 0)$

Step 6: Graphing by hand;



Step 7: Domain: $(-\infty, \infty)$; Range: $[0, \infty)$

Step 8: Increasing on $(-2, 1)$ and $(4, \infty)$
decreasing on $(-\infty, -2)$ and $(1, 4)$.

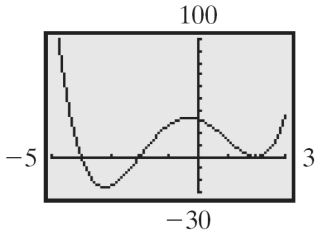
84. $f(x) = (x-2)^2(x+2)(x+4)$

Step 1: Degree is 4. The graph of the function resembles $y = x^4$ for large values of $|x|$.

Step 2: y-intercept: $f(0) = (0-2)^2(0+2)(0+4)$
$= 32$
x-intercept: solve $f(x) = 0$
$(x-2)^2(x+2)(x+4) = 0$
$x = 2$ or $x = -2$ or $x = -4$

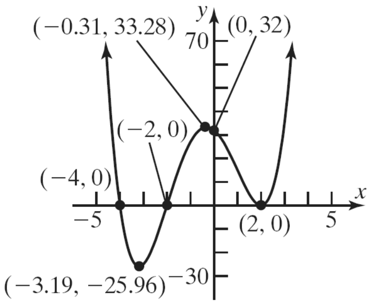
Step 3: Real zeros: 2 with multiplicity two, -2 with multiplicity one, -4 with multiplicity one. The graph touches the x-axis at $x = 2$, and crosses it at $x = -2$ and $x = -4$.

Step 4: Graphing utility:



Step 5: 3 turning points;
local maxima $(-0.31, 33.28)$;
local minima $(-3.19, -25.96)$ and $(2, 0)$

Step 6: Graphing by hand:



Step 7: Domain: $(-\infty, \infty)$; Range: $[-25.96, \infty)$

Step 8: Increasing on $(-3.19, -0.31)$ and $(2, \infty)$;
decreasing on $(-\infty, -3.19)$ and $(-0.31, 2)$

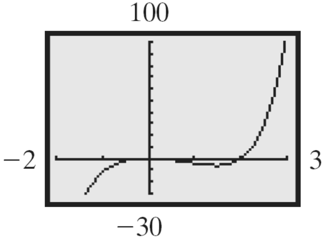
85. $f(x) = x^2(x-2)(x^2+3)$

Step 1: Degree is 5. The graph of the function resembles $y = x^5$ for large values of $|x|$.

Step 2: y-intercept: $f(0) = 0^2(0-2)(0^2+3) = 0$
x-intercept: solve $f(x) = 0$
$x^2(x-2)(x^2+3) = 0$
$x = 0$ or $x = 2$
Note: $x^2 + 3 = 0$ has no real solution.

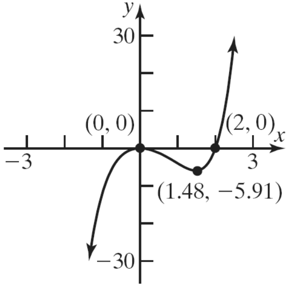
Step 3: Real zeros: 0 with multiplicity two, 2 with multiplicity one. The graph touches the x-axis at $x = 0$ and crosses it $x = 2$.

Step 4: Graphing utility:



Step 5: 2 turning points;
local maximum: (0, 0);
local minimum: (1.48, –5.91)

Step 6: Graphing by hand:



Step 7: Domain: $(-\infty, \infty)$; Range: $(-\infty, \infty)$

Step 8: Increasing on $(-\infty, 0)$ and $(1.48, \infty)$;
decreasing on $(0, 1.48)$

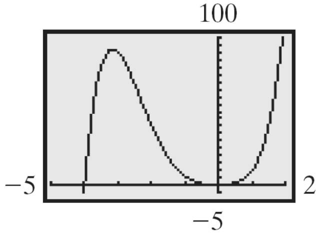
86. $f(x) = x^2(x^2+1)(x+4)$

Step 1: Degree is 5. The graph of the function resembles $y = x^5$ for large values of $|x|$.

Step 2: y-intercept: $f(0) = 0^2(0^2+1)(0+4) = 0$
x-intercept: Solve $f(x) = 0$
$x^2(x^2+1)(x+4) = 0$
$x = 0$ or $x = -4$
Note: $x^2 + 1 = 0$ has no real solution.

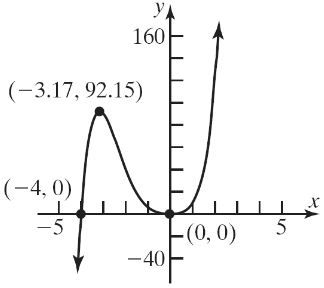
Step 3: Real zeros: 0 with multiplicity two, –4 with multiplicity one. The graph touches the x-axis at $x = 0$ and crosses it at $x = -4$.

Step 4: Graphing utility:



Step 5: 2 turning points;
local maximum: (–3.17, 92.15);
local minimum: (0, 0)

Step 6: Graphing by hand:



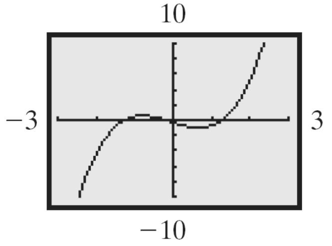
Step 7: Domain: $(-\infty, \infty)$; Range: $(-\infty, \infty)$

Step 8: Increasing on $(-\infty, -3.17)$ and $(0, \infty)$; decreasing on $(-3.17, 0)$

87. $f(x) = x^3 + 0.2x^2 - 1.5876x - 0.31752$

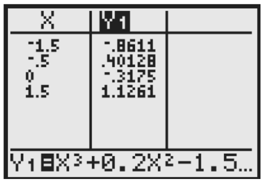
Step 1: Degree = 3; The graph of the function resembles $y = x^3$ for large values of $|x|$.

Step 2: Graphing utility



Step 3: x-intercepts: –1.26, –0.20, 1.26;
y-intercept: –0.31752

Step 4:

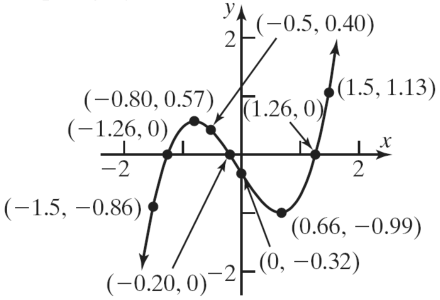


X	Y1
-1.5	-.8611
-.5	.40128
0	-.3175
1.5	1.1261

Y1=X³+0.2X²-1.5…

Step 5: 2 turning points;
local maximum: (–0.80, 0.57);
local minimum: (0.66, –0.99)

Step 6: Graphing by hand



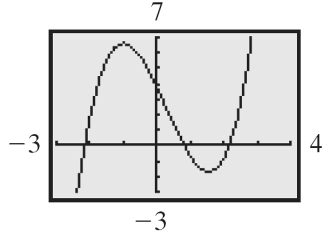
Step 7: Domain: $(-\infty, \infty)$; Range: $(-\infty, \infty)$

Step 8: Increasing on $(-\infty, -0.80)$ and $(0.66, \infty)$; decreasing on $(-0.80, 0.66)$

88. $f(x) = x^3 - 0.8x^2 - 4.6656x + 3.73248$

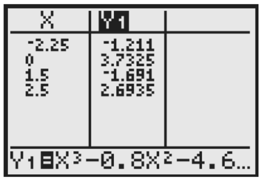
Step 1: Degree = 3; The graph of the function resembles $y = x^3$ for large values of $|x|$.

Step 2: Graphing utility



Step 3: x-intercepts: –2.16, 0.8, 2.16;
y-intercept: 3.73248

Step 4:

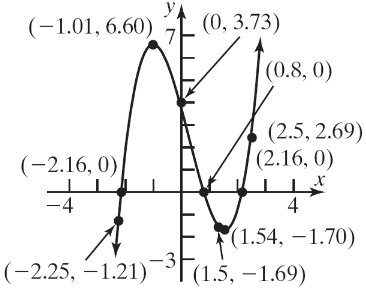


X	Y1
-2.25	-1.211
0	3.7325
1.5	-1.691
2.5	2.6935

Y1=X³-0.8X²-4.6…

Step 5: 2 turning points;
local maximum: (–1.01, 6.60);
local minimum: (1.54, –1.70)

Step 6: Graphing by hand



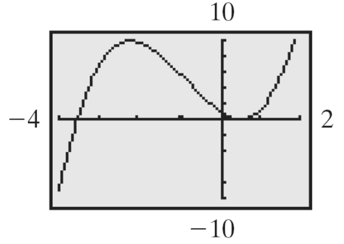
Step 7: Domain: $(-\infty, \infty)$; Range: $(-\infty, \infty)$

Step 8: Increasing on $(-\infty, -1.01)$ and $(1.54, \infty)$; decreasing on $(-1.01, 1.54)$

89. $f(x) = x^3 + 2.56x^2 - 3.31x + 0.89$

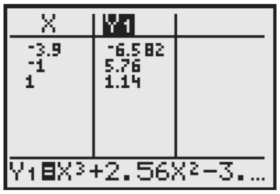
Step 1: Degree = 3; The graph of the function resembles $y = x^3$ for large values of $|x|$.

Step 2: Graphing utility



Step 3: x-intercepts: –3.56, 0.50;
y-intercept: 0.89

Step 4:

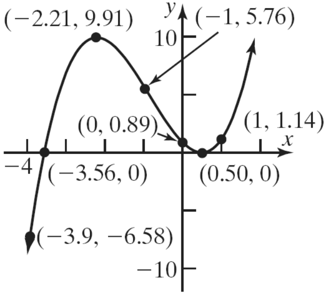


```
  X      Y1
 -3.9   -6.582
 -1      5.76
  1      1.14

Y1■X³+2.56X²-3.…
```

Step 5: 2 turning points;
local maximum: (–2.21, 9.91);
local minimum: (0.50, 0)

Step 6: Graphing by hand



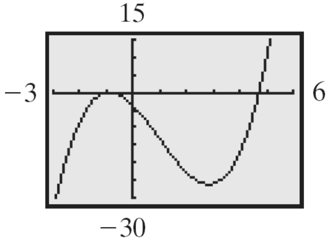
Step 7: Domain: $(-\infty,\infty)$; Range: $(-\infty,\infty)$

Step 8: Increasing on $(-\infty,-2.21)$ and $(0.50,\infty)$;
decreasing on $(-2.21,0.50)$.

90. $f(x)=x^3-2.91x^2-7.668x-3.8151$

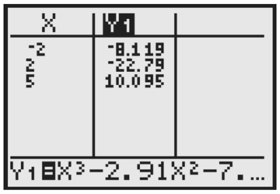
Step 1: Degree = 3; The graph of the function resembles $y=x^3$ for large values of $|x|$.

Step 2: Graphing utility



Step 3: x-intercepts: –0.9, 4.71;
y-intercept: –3.8151

Step 4:

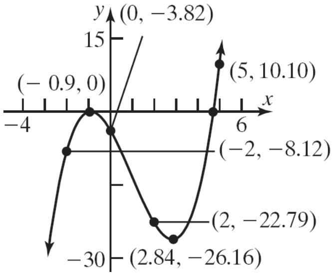


```
  X     Y1
 -2    -8.119
  2    -22.79
  5     10.095

Y1■X³-2.91X²-7.…
```

Step 5: 2 turning points;
local maximum: (–0.9, 0);
local minimum: (2.84, –26.16)

Step 6: Graphing by hand



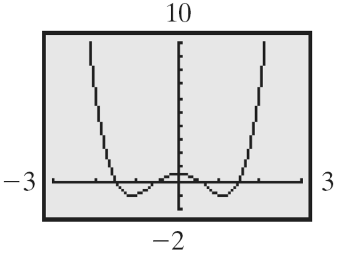
Step 7: Domain: $(-\infty,\infty)$; Range: $(-\infty,\infty)$

Step 8: Increasing on $(-\infty,-0.9)$ and $(2.84,\infty)$;
decreasing on $(-0.9,2.84)$.

91. $f(x)=x^4-2.5x^2+0.5625$

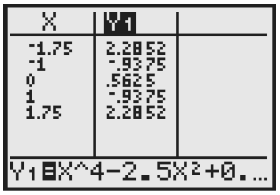
Step 1: Degree = 4; The graph of the function resembles $y=x^4$ for large values of $|x|$.

Step 2: Graphing utility



Step 3: x-intercepts: –1.5, –0.5, 0.5,1.5;
y-intercept: 0.5625

Step 4:

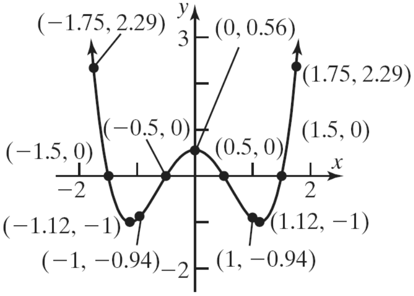


```
  X      Y1
 -1.75   2.2852
 -1     -.9375
  0      .5625
  1     -.9375
  1.75   2.2852

Y1■X^4-2.5X²+0.…
```

Step 5: 3 turning points:
local maximum: (0, 0.5625);
local minima: (–1.12, –1), (1.12, –1)

Step 6: Graphing by hand



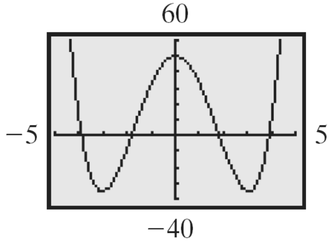
Step 7: Domain: $(-\infty,\infty)$; Range: $[-1,\infty)$

Step 8: Increasing on $(-1.12, 0)$ and $(1.12, \infty)$; decreasing on $(-\infty, -1.12)$ and $(0, 1.12)$

92. $f(x) = x^4 - 18.5x^2 + 50.2619$

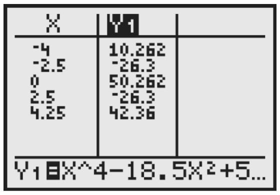
Step 1: Degree = 4; The graph of the function resembles $y = x^4$ for large values of $|x|$.

Step 2: Graphing utility



Step 3: x-intercepts: –3.90, –1.82, 1.82, 3.90; y-intercept: 50.2619

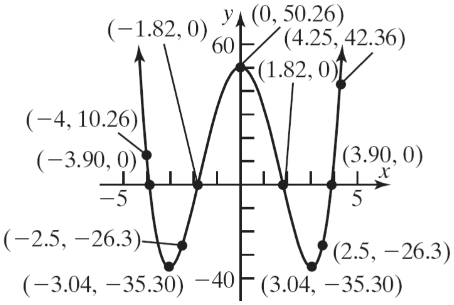
Step 4:



```
  X      Y1
 -4      10.262
 -2.5    -26.3
 0       50.262
 2.5     -26.3
 4.25    42.36
Y1■X^4-18.5X²+5…
```

Step 5: 3 turning points: local maximum: (0, 50.26); local minima: (–3.04, –35.30), (3.04, –35.30)

Step 6: Graphing by hand



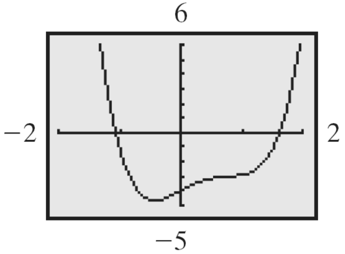
Step 7: Domain: $(-\infty, \infty)$; Range: $[-35.30, \infty)$

Step 8: Increasing on $(-3.04, 0)$ and $(3.04, \infty)$; decreasing on $(-\infty, -3.04)$ and $(0, 3.04)$

93. $f(x) = 2x^4 - \pi\, x^3 + \sqrt{5}x - 4$

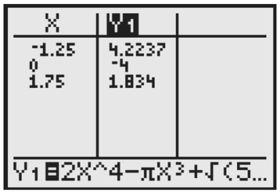
Step 1: Degree = 4; The graph of the function resembles $y = 2x^4$ for large values of $|x|$.

Step 2: Graphing utility:



Step 3: x-intercepts: –1.07, 1.62; y-intercept: –4

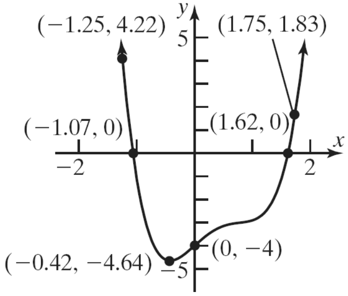
Step 4:



```
  X      Y1
 -1.25   4.2237
 0       -4
 1.75    1.834
Y1■2X^4-πX³+√(5…
```

Step 5: 1 turning point; local minimum: (–0.42, –4.64)

Step 6: Graphing by hand



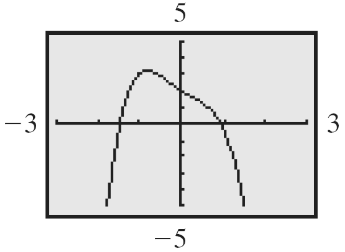
Step 7: Domain: $(-\infty, \infty)$; Range: $[-4.64, \infty)$

Step 8: Increasing on $(-0.42, \infty)$; decreasing on $(-\infty, -0.42)$

94. $f(x) = -1.2x^4 + 0.5x^2 - \sqrt{3}x + 2$

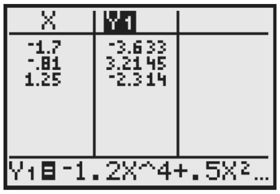
Step 1: Degree = 4; The graph of the function resembles $y = -1.2x^4$ for large values of $|x|$.

Step 2: Graphing utility



Step 3: x-intercepts: –1.47, 0.91; y-intercept: 2

Step 4:

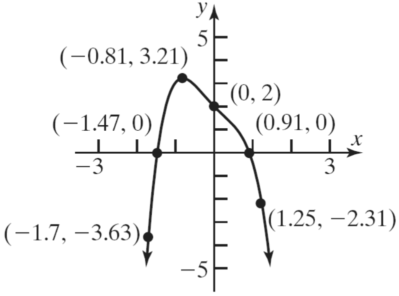


X	Y1
-1.7	-3.633
-.81	3.2145
1.25	-2.314

Y1=-1.2X^4+.5X²…

Step 5: 1 turning point:
local maximum: (–0.81, 3.21)

Step 6: Graphing by hand



Step 7: Domain: $(-\infty, \infty)$; Range: $(-\infty, 3.21]$

Step 8: Increasing on $(-\infty, -0.81)$;
decreasing on $(-0.81, \infty)$

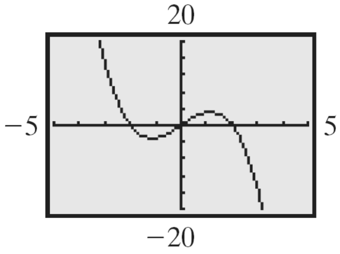
95. $f(x) = 4x - x^3 = -x(x^2 - 4) = -x(x+2)(x-2)$

Step 1: Degree is 3. The function resembles $y = -x^3$ for large values of $|x|$.

Step 2: y-intercept: $f(0) = 4(0) - 0^3 = 0$
x-intercepts: Solve $f(x) = 0$
$0 = -x(x+2)(x-2)$
$x = 0, -2, 2$

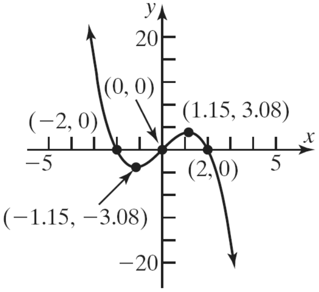
Step 3: Real zeros: 0 with multiplicity one, -2 with multiplicity one, 2 with multiplicity one. The graph crosses the x-axis at $x = 0$, $x = -2$, and $x = 2$.

Step 4: Graphing utility:



Step 5: 2 turning points;
local maximum: (1.15, 3.08);
local minimum: (–1.15, –3.08)

Step 6: Graphing by hand



Step 7: Domain: $(-\infty, \infty)$; Range: $(-\infty, \infty)$

Step 8: Increasing on $(-1.15, 1.15)$;
decreasing on $(-\infty, -1.15)$ and $(1.15, \infty)$.

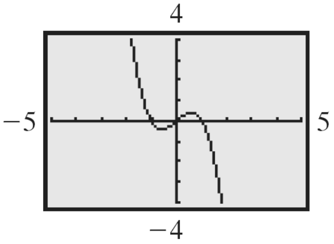
96. $f(x) = x - x^3 = -x(x^2 - 1) = -x(x+1)(x-1)$

Step 1: Degree is 3. The function resembles $y = -x^3$ for large values of $|x|$.

Step 2: y-intercept: $f(0) = 0 - 0^3 = 0$
x-intercepts: Solve $f(x) = 0$
$0 = -x(x+1)(x-1)$
$x = 0, -1, 1$

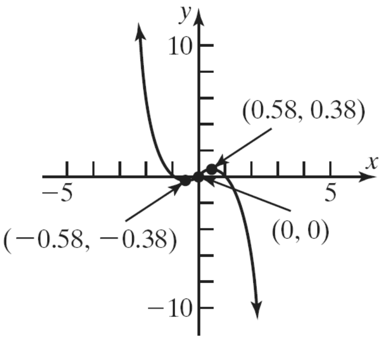
Step 3: Real zeros: 0 with multiplicity one, -1 with multiplicity one, 1 with multiplicity one. The graph crosses the x-axis at $x = 0$, $x = -1$, and $x = 1$.

Step 4: Graphing utility:



Step 5: 2 turning points;
local maximum: (0.58, 0.38);
local minimum: (–0.58, –0.38)

Step 7: Graphing by hand



Step 7: Domain: $(-\infty, \infty)$; Range: $(-\infty, \infty)$

Step 8: Increasing on $(-0.58, 0.58)$; decreasing on $(-\infty, -0.58)$ and $(0.58, \infty)$.

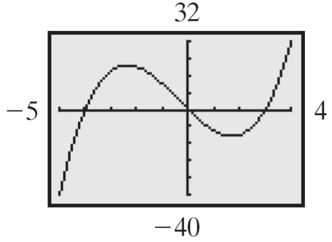
97. $f(x) = x^3 + x^2 - 12x$
$= x(x^2 + x - 12)$
$= x(x+4)(x-3)$

Step 1: Degree is 3. The function resembles $y = -x^3$ for large values of $|x|$.

Step 2: y-intercept: $f(0) = 0^3 + 0^2 - 12(0) = 0$
x-intercepts: Solve $f(x) = 0$
$0 = x(x+4)(x-3)$
$x = 0, -4, 3$

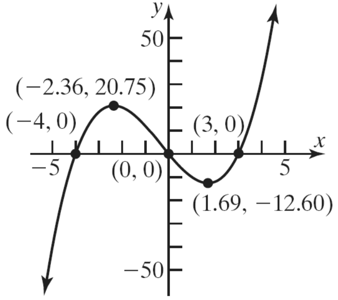
Step 3: Real zeros: 0 with multiplicity one, -4 with multiplicity one, 3 with multiplicity one. The graph crosses the x-axis at $x = 0$, $x = -4$, and $x = 3$.

Step 4: Graphing utility:



Step 5: 2 turning points;
local maximum: $(-2.36, 20.75)$;
local minimum: $(1.69, -12.60)$

Step 6: Graphing by hand



Step 7: Domain: $(-\infty, \infty)$; Range: $(-\infty, \infty)$

Step 8: Increasing on $(-\infty, -2.36)$ and $(1.69, \infty)$; decreasing on $(-2.36, 1.69)$.

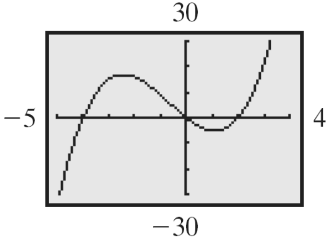
98. $f(x) = x^3 + 2x^2 - 8x$
$= x(x^2 + 2x - 8)$
$= x(x+4)(x-2)$

Step 1: Degree is 3. The function resembles $y = -x^3$ for large values of $|x|$.

Step 2: y-intercept: $f(0) = 0^3 + 2(0)^2 - 8(0) = 0$
x-intercepts: Solve $f(x) = 0$
$0 = x(x+4)(x-2)$
$x = 0, -4, 2$

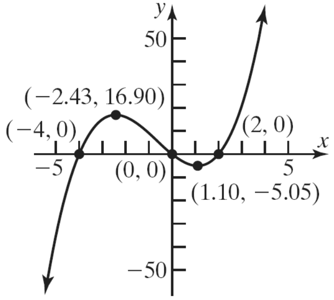
Step 3: Real zeros: 0 with multiplicity one, -4 with multiplicity one, 2 with multiplicity one. The graph crosses the x-axis at $x = 0$, $x = -4$, and $x = 2$.

Step 4: Graphing utility:



Step 5: 2 turning points;
local maximum: $(-2.43, 16.90)$;
local minimum: $(1.09, -5.05)$

Step 6: Graphing by hand



Step 7: Domain: $(-\infty, \infty)$; Range: $(-\infty, \infty)$

Step 8: Increasing on $(-\infty, -2.43)$ and $(1.10, \infty)$; decreasing on $(-2.43, 1.10)$

99. $f(x) = 2x^4 + 12x^3 - 8x^2 - 48x$

$= 2x(x^3 + 6x^2 - 4x - 24)$

$= 2x\left[x^2(x+6) - 4(x+6)\right]$

$= 2x(x+6)(x^2 - 4)$

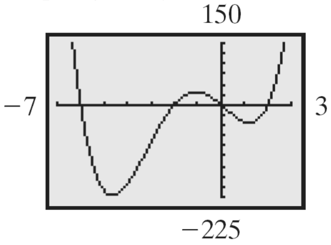
$= 2x(x+6)(x-2)(x+2)$

Step 1: Degree is 3. The function resembles $y = -x^3$ for large values of $|x|$.

Step 2: y-intercept:
$f(0) = 2(0)^4 + 12(0)^3 - 8(0)^2 - 48(0) = 0$
x-intercepts: Solve $f(x) = 0$
$0 = 2x(x+6)(x-2)(x+2)$
$x = 0, -6, 2, -2$

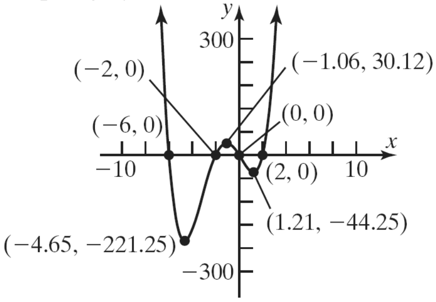
Step 3: Real zeros: 0 with multiplicity one, -6 with multiplicity one, 2 with multiplicity one, -2 with multiplicity one. The graph crosses the x-axis at $x = 0$, $x = -6$, $x = 2$, and $x = -2$.

Step 4: Graphing utility:



Step 5: 2 turning points;
local maximum: $(-1.06, 30.12)$;
local minima: $(-4.65, -221.25)$, $(1.21, -44.25)$

Step 6: Graphing by hand



Step 7: Domain: $(-\infty, \infty)$; Range: $[-221.25, \infty)$

Step 8: Increasing on $(-4.65, -1.06)$ and $(-1.21, \infty)$; decreasing on $(-\infty, -4.65)$ and $(-1.06, 1.21)$.

100. $f(x) = 4x^3 + 10x^2 - 4x - 10$

$= 2(2x^3 + 5x^2 - 2x - 5$

$= 2\left[x^2(2x+5) - 1(2x+5)\right]$

$= 2(2x+5)(x^2 - 1)$

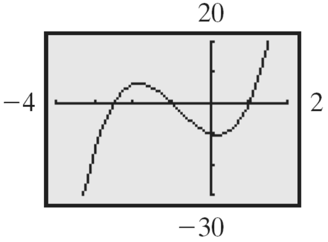
$= 2(2x+5)(x+1)(x-1)$

Step 1: Degree is 3. The function resembles $y = 4x^3$ for large values of $|x|$.

Step 2: y-intercept:
$f(0) = 4(0)^3 + 10(0)^2 - 4(0) - 10 = -10$
x-intercepts: solve $f(x) = 0$
$0 = 2(2x+5)(x+1)(x-1)$
$x = -\frac{5}{2}, -1, 1$

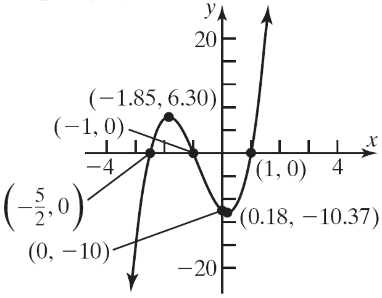
Step 3: Real zeros: $-\frac{5}{2}$ with multiplicity one, -1 with multiplicity one, 1 with multiplicity one. The graph crosses the x-axis at $x = -\frac{5}{2}$, $x = -1$, and $x = 1$.

Step 4: Graphing utility:



Step 5: 2 turning points;
local maximum: $(-1.85, 6.30)$;
local minimum: $(0.18, -10.37)$

Step 6: Graphing by hand



Step 7: Domain: $(-\infty, \infty)$; Range: $(-\infty, \infty)$

Step 8: Increasing on $(-\infty, -1.85)$ and $(0.18, \infty)$; decreasing on $(-1.85, 0.18)$

101. $f(x) = -x^5 - x^4 + x^3 + x^2$

$$\begin{aligned} &= -x^2(x^3 + x^2 - x - 1) \\ &= -x^2\left[x^2(x+1) - 1(x+1)\right] \\ &= -x^2(x+1)(x^2-1) \\ &= -x^2(x+1)(x+1)(x-1) \\ &= -x^2(x+1)^2(x-1) \end{aligned}$$

Step 1: Degree is 5. The graph of the function resembles $y = -x^5$ for large values of $|x|$.

Step 2: y-intercept:

$f(0) = -(0)^5 - (0)^4 + (0)^3 + (0)^2 = 0$

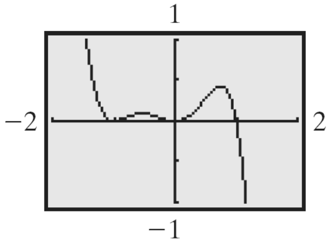
x-intercept: Solve $f(x) = 0$

$-x^2(x+1)^2(x-1) = 0$

$x = 0, -1, 1$

Step 3: Real zeros: 0 with multiplicity two, -1 with multiplicity two, 1 with multiplicity one. The graph touches the x-axis at $x = 0$ and $x = -1$, and crosses it at $x = 1$.

Step 4: Graphing utility:

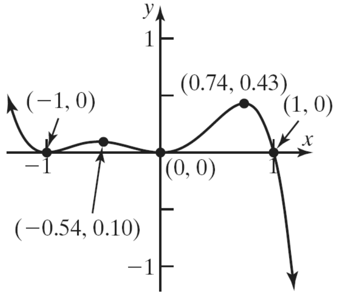


Step 5: 4 turning points;

local maxima: $(-0.54, 0.10), (0.74, 0.43)$;

local minima: $(-1, 0)$, $(0, 0)$

Step 6: Graphing by hand:



Step 7: Domain: $(-\infty, \infty)$; Range: $(-\infty, \infty)$

Step 8: Increasing on $(-1, -0.54)$ and $(0, 0.74)$; decreasing on $(-\infty, -1)$, $(-0.54, 0)$, and $(0.74, \infty)$

102. $f(x) = -x^5 + 5x^4 + 4x^3 - 20x^2$

$$\begin{aligned} &= -x^2(x^3 - 5x^2 - 4x + 20) \\ &= -x^2\left[x^2(x-5) - 4(x-5)\right] \\ &= -x^2(x-5)(x^2-4) \\ &= -x^2(x-5)(x-2)(x+2) \end{aligned}$$

Step 1: Degree is 5. The graph of the function resembles $y = -x^5$ for large values of $|x|$.

Step 2: y-intercept:

$f(0) = -(0)^5 + 5(0)^4 + 4(0)^3 - 20(0)^2 = 0$

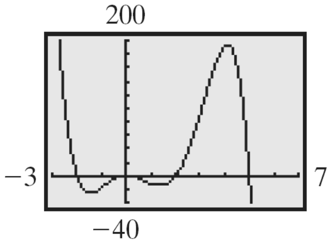
x-intercept: solve $f(x) = 0$

$-x^2(x+2)(x-2)(x-5) = 0$

$x = 0, -2, 2, 5$

Step 3: Real zeros: 0 with multiplicity two, -2 with multiplicity one, 21 with multiplicity one, 5 with multiplicity one. The graph The graph touches the x-axis at $x = 0$ and crosses it at $x = -2$, $x = 2$, and $x = 5$.

Step 4: Graphing utility:

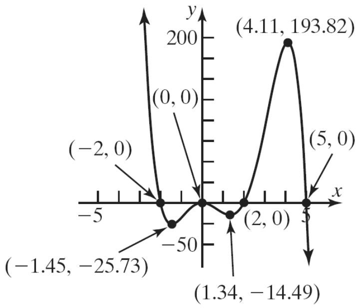


Step 5: 4 turning points;

local maxima: $(0, 0)$, $(4.11, 193.82)$;

local minima: $(-1.45, -25.73)$, $(1.34, -14.49)$

Step 6: Graphing by hand:



Step 7: Domain: $(-\infty, \infty)$; Range: $(-\infty, \infty)$

Step 8: Increasing on $(-1.45, 0)$ and $(1.34, 4.11)$; decreasing on $(-\infty, -1.45)$, $(0, 1.34)$, and $(4.11, \infty)$

103. $f(x)=a(x+3)(x-1)(x-4)$

$36=a(0+3)(0-1)(0-4)$

$36=12a$

$a=3$

$f(x)=3(x+3)(x-1)(x-4)$

104. $f(x)=a(x+4)(x+1)(x-2)$

$16=a(0+4)(0+1)(0-2)$

$16=-8a$

$a=-2$

$f(x)=-2(x+4)(x+1)(x-2)$

105. $f(x)=a(x+5)^2(x-2)(x-4)$

$128=a(3+5)^2(3-2)(3-4)$

$128=-64a$

$a=-2$

$f(x)=-2(x+5)^2(x-2)(x-4)$

106. $f(x)=ax^3(x+4)(x-2)$

$64=a(-2)^3(-2+4)(-2-2)$

$64=64a$

$a=1$

$f(x)=x^3(x+4)(x-2)$

107. **a**. $0=(x+3)^2(x-2)\Rightarrow x=-3, x=2$

b. The graph is shifted to the left 3 units so the x-intercepts would be

$x=-3-3=-6$

and $x=2-3=-1$

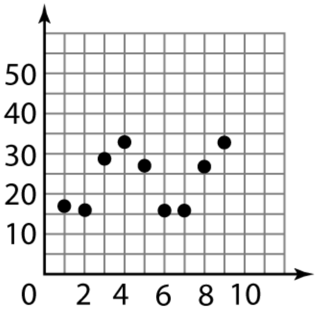
108. **a**. $0=(x+2)(x-4)^3\Rightarrow x=-2, x=4$

b. The graph is shifted to the right 2 units so the x-intercepts would be

$x=-2+2=0$

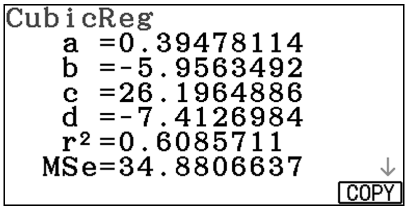
and $x=4+2=6$

109. **a.** Graphing, we see that the graph may be a cubic relation.



b. The cubic function of best fit is $H(x)=$

$0.3948x^3-5.9563x^2+26.1965x-7.4127$



```
CubicReg
   a =0.39478114
   b =-5.9563492
   c =26.1964886
   d =-7.4126984
   r²=0.6085711
  MSe=34.8806637
```

c. For the decade 1961-1970, we have $x=5$.

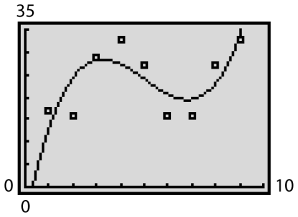
$H(5)=$

$0.3948(5)^3-5.9563(5)^2+26.1695(5)-7.4127$

≈ 24

The model predicts that about 24 major hurricanes struck the Atlantic Basin between 1961 and 1970.

d.



e. For the decade 2011 to 2020 we have $x=10$.

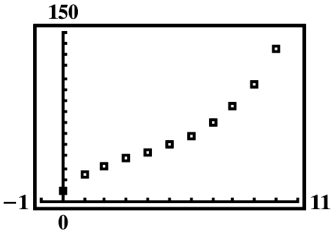
$H(10)=$

$0.3948(10)^3-5.9563(10)^2+26.1695(10)-7.4127$

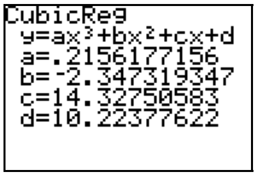
≈ 54

The model predicts that approximately 54 major hurricanes will strike the Atlantic Basin between 2011 and 2020. The prediction does not seem to be reasonable. It appears to be too high.

110. a. Graphing, we see that the graph may be a cubic relation.

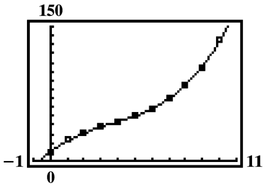


b. The cubic function of best fit is $C(x) =$
$0.2156x^3 - 2.3473x^2 + 14.3275x + 10.2238$



```
CubicReg
 y=ax³+bx²+cx+d
 a=.2156177156
 b=-2.347319347
 c=14.32750583
 d=10.22377622
```

c. Graphing the cubic function of best fit:

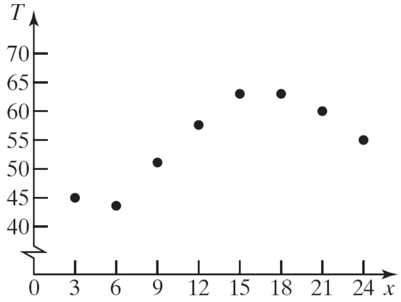


d. $C(11) = 0.2156(11)^3 - 2.3473(11)^2$
$+14.3275(11) + 10.2238$
≈ 170.8

The cost of manufacturing 11 Cobalts hour would be approximately $171,000.

e. The *y*-intercept would indicate the fixed costs before any cars are made and is about $10,200.

111. a. Graphing, we see that the graph may be a cubic relation.



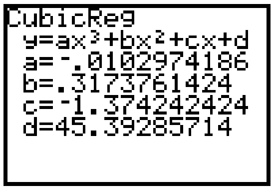
b. $\dfrac{\Delta T}{\Delta x} = \dfrac{T(12) - T(9)}{12 - 9} = \dfrac{57.9 - 51.1}{12 - 9} = \dfrac{6.8}{3} \approx 2.27$

The average rate of change in temperature from 9am to noon was about 2.27°F per hour.

c. $\dfrac{\Delta T}{\Delta x} = \dfrac{T(18) - T(15)}{18 - 15} = \dfrac{63.0 - 63.0}{18 - 15} = 0$

The average rate of change in temperature from 9am to noon was 0°F per hour.

d. The cubic function of best fit is $T(x) =$
$-0.0103x^3 + 0.3174x^2 - 1.3742x + 45.3929$



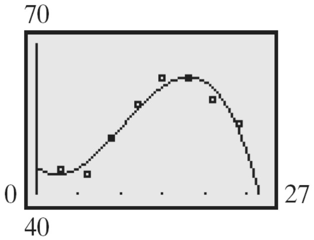
```
CubicReg
 y=ax³+bx²+cx+d
 a=-.0102974186
 b=.3173761424
 c=-1.374242424
 d=45.39285714
```

At 5pm we have $x = 17$.

$T(17) = -0.0103(17)^3 + 0.3174(17)^2$
$-1.3742(17) + 45.3929$
$= 63.1562$

The predicted temperature at 5pm is ≈ 63.2°F.

e.



f. The *y*-intercept is approximately 45.4°F. The model predicts that the midnight temperature was 45.4°F.

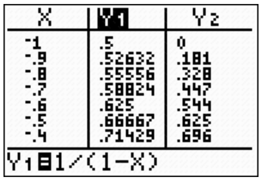
112. a. $T(r) = \underbrace{500(1+r)(1+r)}_{\text{Account value of deposit 1}} + \underbrace{500(1+r)}_{\text{Account value of deposit 2}} + \underbrace{500}_{\text{Deposit 3}}$

$= 500(1 + 2r + r^2) + 500(1 + r) + 500$
$= 500 + 1000r + 500r^2 + 500 + 500r + 500$
$= 500r^2 + 1500r + 1500$

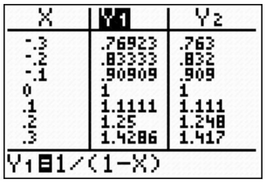
b. $F(0.05)$
$= 500(.05)^3 + 2000(.05)^2 + 3000(.05) + 2000$
≈ 2155.06

The value of the account at the beginning of the fourth year will be $2155.06.

113. a.

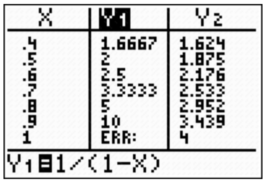


```
   X   | Y1     |  Y2
 -1    | .5     | 0
 -.9   | .52632 | .181
 -.8   | .55556 | .328
 -.7   | .58824 | .447
 -.6   | .625   | .544
 -.5   | .66667 | .625
 -.4   | .71429 | .696
Y1■1/(1-X)
```



X	Y1	Y2
-.3	.76923	.763
-.2	.83333	.832
-.1	.90909	.909
0	1	1
.1	1.1111	1.111
.2	1.25	1.248
.3	1.4286	1.417

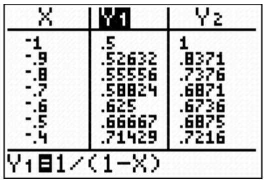
$Y_1 = 1/(1-X)$



X	Y1	Y2
.4	1.6667	1.624
.5	2	1.875
.6	2.5	2.176
.7	3.3333	2.533
.8	5	2.952
.9	10	3.439
1	ERR:	4

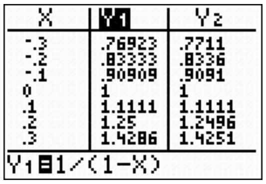
$Y_1 = 1/(1-X)$

b.



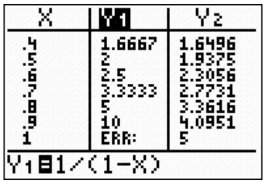
X	Y1	Y2
-1	.5	1
-.9	.52632	.8371
-.8	.55556	.7376
-.7	.58824	.6871
-.6	.625	.6736
-.5	.66667	.6875
-.4	.71429	.7216

$Y_1 = 1/(1-X)$



X	Y1	Y2
-.3	.76923	.7711
-.2	.83333	.8336
-.1	.90909	.9091
0	1	1
.1	1.1111	1.1111
.2	1.25	1.2496
.3	1.4286	1.4251

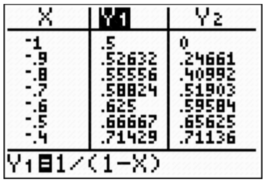
$Y_1 = 1/(1-X)$



X	Y1	Y2
.4	1.6667	1.6496
.5	2	1.9375
.6	2.5	2.3056
.7	3.3333	2.7731
.8	5	3.3616
.9	10	4.0951
1	ERR:	5

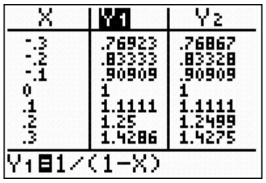
$Y_1 = 1/(1-X)$

c.



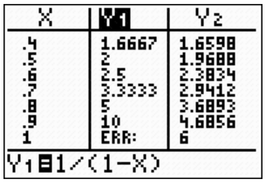
X	Y1	Y2
-1	.5	0
-.9	.52632	.24661
-.8	.55556	.40992
-.7	.58824	.51903
-.6	.625	.59584
-.5	.66667	.65625
-.4	.71429	.71136

$Y_1 = 1/(1-X)$



X	Y1	Y2
-.3	.76923	.76867
-.2	.83333	.83328
-.1	.90909	.90909
0	1	1
.1	1.1111	1.1111
.2	1.25	1.2499
.3	1.4286	1.4275

$Y_1 = 1/(1-X)$



X	Y1	Y2
.4	1.6667	1.6598
.5	2	1.9688
.6	2.5	2.3834
.7	3.3333	2.9412
.8	5	3.6893
.9	10	4.6856
1	ERR:	6

$Y_1 = 1/(1-X)$

d. The values of the polynomial function get closer to the values of *f*. The approximations near 0 are better than those near -1 or 1.

114. The graph of a polynomial function will always have a *y*-intercept since the domain of every polynomial function is the set of real numbers. Therefore $f(0)$ will always produce a *y*-coordinate on the graph. A polynomial function might have no *x*-intercepts. For example, $f(x) = x^2 + 1$ has no *x*-intercepts since the equation $x^2 + 1 = 0$ has no real solutions.

115. Answers will vary.

116. Answers will vary, one such polynomial is $f(x) = x^2(x+1)(4-x)(x-2)^2$

117. Answers will vary , $f(x) = (x+2)(x-1)^2$ and $g(x) = (x+2)^3(x-1)^2$ are two such polynomials.

118. $f(x) = \dfrac{1}{x}$ is smooth but not continuous; $g(x) = |x|$ is continuous but not smooth.

119. $f(x) = x^3 + bx^2 + cx + d$

a. True since every polynomial function has exactly one y-intercept, in this case $(0, d)$.

b. True, a third degree polynomial will have at most 3 x-intercepts since the equation $x^3 + bx^2 + cx + d = 0$ will have at most 3 real solutions.

c. True, a third degree polynomial will have at least one x-intercept since the equation $x^3 + bx^2 + cx + d = 0$ will have at least one real solution.

d. True, since f has degree 3 and the leading coefficient 1.

e. False, since

$$\begin{aligned} f(-x) &= (-x)^3 + b(-x)^2 + c(-x) + d \\ &= -x^3 + bx^2 - cx + d \\ &\neq -f(x). \text{ (unless } b = d = 0) \end{aligned}$$

f. True only if $d = 0$, otherwise the statement is false.

120. **a.** The degree will be even because the ends of the graph go in the same direction.

b. The leading coefficient is positive because both ends go up.

c. The function appears to be symmetric about the *y*-axis. Therefore, it is an even function.

d. The graph touches the *x*-axis at $x = 0$. Therefore, x^n must be a factor, where *n* is even and $n \geq 2$.

e. There are six zeros with odd multiplicity and one with even multiplicity. Therefore, the minimum degree is $6(1) + 1(2) = 8$.

f. Answers will vary.

121. Answers will vary. One possibility:
$$f(x) = -5(x-1)^3(x-2)\left(x - \frac{1}{2}\right)\left(x + \frac{3}{5}\right)$$

122 – 126. Interactive Exercises

Section 4.2

1. $f(-1) = 2(-1)^2 - (-1) = 2 + 1 = 3$

2. $6x^2 + x - 2 = (3x+2)(2x-1)$

3. Using synthetic division:
$$\begin{array}{r|rrrrr} 3 & 3 & -5 & 0 & 7 & -4 \\ & & 9 & 12 & 36 & 129 \\ \hline & 3 & 4 & 12 & 43 & 125 \end{array}$$
Quotient: $3x^3 + 4x^2 + 12x + 43$
Remainder: 125

4. $x^2 + x - 3 = 0$
$$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(-3)}}{2(1)} = \frac{-1 \pm \sqrt{1+12}}{2} = \frac{-1 \pm \sqrt{13}}{2}$$
The solution set is $\left\{\frac{-1-\sqrt{13}}{2}, \frac{-1+\sqrt{13}}{2}\right\}$.

5. Remainder, Dividend

6. $f(c)$

7. –4

8. False; every polynomial function of degree 3 with real coefficients has at most three real zeros.

9. 0.

10. True

11. $f(x) = 4x^3 - 3x^2 - 8x + 4; \quad c = 2$
$$f(2) = 4(2)^3 - 3(2)^2 - 8(2) + 4 = 32 - 12 - 16 + 4 = 8 \neq 0$$
Thus, 2 is not a zero of f and $x-2$ is not a factor of f.

12. $f(x) = -4x^3 + 5x^2 + 8; \quad c = -3$
$$f(-3) = -4(-3)^3 + 5(-3)^2 + 8 = 108 + 45 + 8 = 161 \neq 0$$
Thus, –3 is not a zero of f and $x+3$ is not a factor of f.

13. $f(x) = 3x^4 - 6x^3 - 5x + 10; \quad c = 2$
$$f(2) = 3(2)^4 - 6(2)^3 - 5(2) + 10 = 48 - 48 - 10 + 10 = 0$$
Thus, 2 is a zero of f and $x-2$ is a factor of f.

14. $f(x) = 4x^4 - 15x^2 - 4; \quad c = 2$
$$f(2) = 4(2)^4 - 15(2)^2 - 4 = 64 - 60 - 4 = 0$$
Thus, 2 is a zero of f and $x-2$ is a factor of f.

15. $f(x) = 3x^6 + 82x^3 + 27; \quad c = -3$
$$f(-3) = 3(-3)^6 + 82(-3)^3 + 27 = 2187 - 2214 + 27 = 0$$
Thus, –3 is a zero of f and $x+3$ is a factor of f.

16. $f(x) = 2x^6 - 18x^4 + x^2 - 9; \quad c = -3$
$$f(-3) = 2(-3)^6 - 18(-3)^4 + (-3)^2 - 9 = 1458 - 1458 + 9 - 9 = 0$$
Thus, –3 is a zero of f and $x+3$ is a factor of f.

17. $f(x) = 4x^6 - 64x^4 + x^2 - 15; \quad c = -4$
$$f(-4) = 4(-4)^6 - 64(-4)^4 + (-4)^2 - 15 = 16{,}384 - 16{,}384 + 16 - 15 = 1 \neq 0$$
Thus, –4 is not a zero of f and $x+4$ is not a factor of f.

18. $f(x) = x^6 - 16x^4 + x^2 - 16; \quad c = -4$
$$f(-4) = (-4)^6 - 16(-4)^4 + (-4)^2 - 16 = 4096 - 4096 + 16 - 16 = 0$$

Thus, –4 is a zero of f and $x+4$ is a factor of f .

19. $f(x)=2x^4-x^3+2x-1;\quad c=\dfrac{1}{2}$

$$f\left(\frac{1}{2}\right)=2\left(\frac{1}{2}\right)^4-\left(\frac{1}{2}\right)^3+2\left(\frac{1}{2}\right)-1$$
$$=\frac{1}{8}-\frac{1}{8}+1-1=0$$

Thus, $\dfrac{1}{2}$ is a zero of f and $x-\dfrac{1}{2}$ is a factor of f.

20. $f(x)=3x^4+x^3-3x+1;\quad c=-\dfrac{1}{3}$

$$f\left(-\frac{1}{3}\right)=3\left(-\frac{1}{3}\right)^4+\left(-\frac{1}{3}\right)^3-3\left(-\frac{1}{3}\right)+1$$
$$=\frac{1}{27}-\frac{1}{27}+1+1=2\neq 0$$

Thus, $-\dfrac{1}{3}$ is not a zero of f and $x+\dfrac{1}{3}$ is not a factor of f.

21. $f(x)=3x^4-3x^3+x^2-x+1$

The maximum number of zeros is the degree of the polynomial, which is 4.
p must be a factor of 1: $p=\pm 1$
q must be a factor of 3: $q=\pm 1,\pm 3$
The possible rational zeros are: $\dfrac{p}{q}=\pm 1,\pm\dfrac{1}{3}$

22. $f(x)=x^5-x^4+2x^2+3$

The maximum number of zeros is the degree of the polynomial, which is 5.
p must be a factor of 3: $p=\pm 1,\pm 3$
q must be a factor of 1: $q=\pm 1$
The possible rational zeros are: $\dfrac{p}{q}=\pm 1,\pm 3$

23. $f(x)=x^5-6x^2+9x-3$

The maximum number of zeros is the degree of the polynomial, which is 5.
p must be a factor of –3: $p=\pm 1,\pm 3$
q must be a factor of 1: $q=\pm 1$
The possible rational zeros are: $\dfrac{p}{q}=\pm 1,\pm 3$

24. $f(x)=2x^5-x^4-x^2+1$

The maximum number of zeros is the degree of the polynomial, which is 5.
p must be a factor of 1: $p=\pm 1$
q must be a factor of 2: $q=\pm 1,\pm 2$
The possible rational zeros are: $\dfrac{p}{q}=\pm 1,\pm\dfrac{1}{2}$

25. $f(x)=-4x^3-x^2+x+2$

The maximum number of zeros is the degree of the polynomial, which is 3.
p must be a factor of 2: $p=\pm 1,\pm 2$
q must be a factor of –4: $q=\pm 1,\pm 2,\pm 4$
The possible rational zeros are:
$\dfrac{p}{q}=\pm 1,\pm 2,\pm\dfrac{1}{2},\pm\dfrac{1}{4}$

26. $f(x)=6x^4-x^2+2$

The maximum number of zeros is the degree of the polynomial, which is 4.
p must be a factor of 2: $p=\pm 1,\pm 2$
q must be a factor of 6: $q=\pm 1,\pm 2,\pm 3,\pm 6$
The possible rational zeros are:
$\dfrac{p}{q}=\pm 1,\pm 2,\pm\dfrac{1}{2},\pm\dfrac{1}{3},\pm\dfrac{2}{3},\pm\dfrac{1}{6}$

27. $f(x)=6x^4-x^2+9$

The maximum number of zeros is the degree of the polynomial, which is 4.
p must be a factor of 9: $p=\pm 1,\pm 3,\pm 9$
q must be a factor of 6: $q=\pm 1,\pm 2,\pm 3,\pm 6$
The possible rational zeros are:
$\dfrac{p}{q}=\pm 1,\pm\dfrac{1}{2},\pm\dfrac{1}{3},\pm\dfrac{1}{6},\pm 3,\pm\dfrac{3}{2},\pm 9,\pm\dfrac{9}{2}$

28. $f(x)=-4x^3+x^2+x+6$

The maximum number of zeros is the degree of the polynomial, which is 3.
p must be a factor of 6: $p=\pm 1,\pm 2,\pm 3,\pm 6$
q must be a factor of –4: $q=\pm 1,\pm 2,\pm 4$
The possible rational zeros are:
$\dfrac{p}{q}=\pm 1,\pm 2,\pm\dfrac{1}{2},\pm\dfrac{1}{4},\pm 3,\pm\dfrac{3}{2},\pm\dfrac{3}{4},\pm 6$

29. $f(x)=2x^5-x^3+2x^2+12$

The maximum number of zeros is the degree of the polynomial, which is 5.

p must be a factor of 12:

$p=\pm1,\pm2,\pm3,\pm4,\pm6,\pm12$

q must be a factor of 2: $q=\pm1,\pm2$

The possible rational zeros are:

$$\frac{p}{q}=\pm1,\pm2,\pm4,\pm\frac{1}{2},\pm3,\pm\frac{3}{2},\pm6,\pm12$$

30. $f(x)=3x^5-x^2+2x+18$

The maximum number of zeros is the degree of the polynomial, which is 5.

p must be a factor of 18:

$p=\pm1,\pm2,\pm3,\pm6,\pm9,\pm18$

q must be a factor of 3: $q=\pm1,\pm3$

The possible rational zeros are:

$$\frac{p}{q}=\pm1,\pm\frac{1}{3},\pm2,\pm\frac{2}{3},\pm3,\pm6,\pm9\pm18$$

31. $f(x)=6x^4+2x^3-x^2+20$

The maximum number of zeros is the degree of the polynomial, which is 4.

p must be a factor of 20:

$p=\pm1,\pm2,\pm4,\pm5,\pm10,\pm20$

q must be a factor of 6: $q=\pm1,\pm2,\pm3,\pm6$

The possible rational zeros are:

$$\frac{p}{q}=\pm1,\pm2,\pm\frac{1}{2},\pm\frac{1}{3},\pm\frac{2}{3},\pm\frac{1}{6},\pm4,\pm\frac{4}{3},\pm5,\pm\frac{5}{2},$$
$$\pm\frac{5}{3},\pm\frac{5}{6},\pm10,\pm\frac{10}{3},\pm20,\pm\frac{20}{3}$$

32. $f(x)=-6x^3-x^2+x+10$

The maximum number of zeros is the degree of the polynomial, which is 3.

p must be a factor of 10: $p=\pm1,\pm2,\pm5,\pm10$

q must be a factor of –6: $q=\pm1,\pm2,\pm3,\pm6$

The possible rational zeros are:

$$\frac{p}{q}=\pm1,\pm\frac{1}{2},\pm\frac{1}{3},\pm\frac{1}{6},\pm2,\pm\frac{2}{3},\pm5,\pm\frac{5}{2},$$
$$\pm\frac{5}{3},\pm\frac{5}{6},\pm10,\pm\frac{10}{3}$$

33. $f\left(x\right)=2x^3+x^2-1=2\left(x^3+\frac{1}{2}x^2-\frac{1}{2}\right)$

Note: The leading coefficient must be 1.

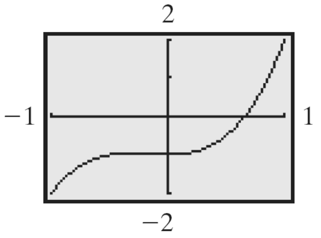
$a_2=\frac{1}{2}, a_1=0, a_0=-\frac{1}{2}$

$$\begin{aligned}Max\left\{1,\left|-\frac{1}{2}\right|+\left|\ 0\ \right|+\left|\ \frac{1}{2}\ \right|\right\}&=Max\left\{1,\frac{1}{2}+0+\frac{1}{2}\right\}\\&=Max\left\{1,1\right\}\\&=1\end{aligned}$$

$$\begin{aligned}1+Max\left\{\left|-\frac{1}{2}\right|,\left|\ 0\ \right|,\left|\ \frac{1}{2}\ \right|\right\}&=1+Max\left\{\frac{1}{2},0,\frac{1}{2}\right\}\\&=1+\frac{1}{2}\\&=\frac{3}{2}\end{aligned}$$

The smaller of the two numbers is 1. Thus, every zero of *f* lies between –1 and 1.

We graph using the bounds and ZOOM-FIT.



34. $f\left(x\right)=3x^3-2x^2+x+4$

$$=3\left(x^3-\frac{2}{3}x^2+\frac{1}{3}x+\frac{4}{3}\right)$$

Note: The leading coefficient must be 1.

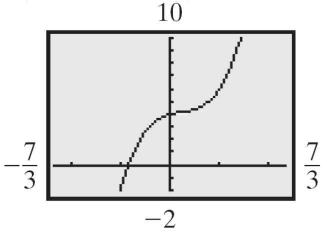
$a_2=-\frac{2}{3}, a_1=\frac{1}{3}, a_0=\frac{4}{3}$

$$\begin{aligned}Max\left\{1,\left|-\frac{2}{3}\right|+\left|\ \frac{1}{3}\ \right|+\left|\ \frac{4}{3}\ \right|\right\}&=Max\left\{1,\frac{2}{3}+\frac{1}{3}+\frac{4}{3}\right\}\\&=Max\left\{1,\frac{7}{3}\right\}\\&=\frac{7}{3}\end{aligned}$$

$$\begin{aligned}1+Max\left\{\left|-\frac{2}{3}\right|,\left|\ \frac{1}{3}\ \right|,\left|\ \frac{4}{3}\ \right|\right\}&=1+Max\left\{\frac{2}{3},\frac{1}{3},\frac{4}{3}\right\}\\&=1+\frac{4}{3}\\&=\frac{7}{3}\end{aligned}$$

The smaller of the two numbers is $\frac{7}{3}$. Thus, every zero of *f* lies between $-\frac{7}{3}$ and $\frac{7}{3}$.

We graph using the bounds and ZOOM-FIT, and adjust the viewing window to improve the graph.



35. $f(x) = x^3 - 5x^2 - 11x + 11$

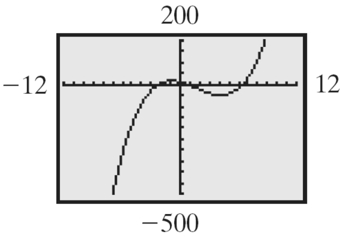
$a_2 = -5, a_1 = -11, a_0 = 1$

$$\begin{aligned} Max\{1, |-5| + |-11| + |\ 11\ |\} &= Max\{1, 5 + 11 + 11\} \\ &= Max\{1, 27\} \\ &= 27 \end{aligned}$$

$$\begin{aligned} 1 + Max\{|-5|, |-11|, |\ 11\ |\} &= 1 + Max\{1, 5, 11, 11\} \\ &= 1 + 11 \\ &= 12 \end{aligned}$$

The smaller of the two numbers is 12. Thus, every zero of *f* lies between –12 and 12.

We graph using the bounds and ZOOM-FIT, and adjust the viewing window to improve the graph.



36. $f(x) = 2x^3 - x^2 - 11x - 6$

$$= 2\left(x^3 - \frac{1}{2}x^2 - \frac{11}{2}x - 3\right)$$

Note: The leading coefficient must be 1.

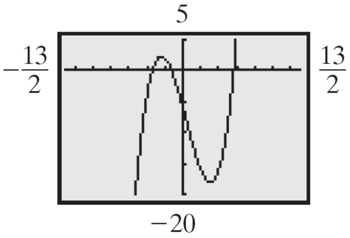
$a_2 = -\frac{1}{2}, a_1 = -\frac{11}{2}, a_0 = -3$

$$\begin{aligned} Max\left\{1, \left|-\frac{1}{2}\right| + \left|\ -\frac{11}{2}\ \right| + |\ -3\ |\right\} &= Max\left\{1, \frac{1}{2} + \frac{11}{2} + 3\right\} \\ &= Max\{1, 9\} \\ &= 9 \end{aligned}$$

$$\begin{aligned} 1 + Max\left\{\left|-\frac{1}{2}\right|, \left|\ -\frac{11}{2}\ \right|, |\ -3\ |\right\} &= 1 + Max\left\{\frac{1}{2}, \frac{11}{2}, 3\right\} \\ &= 1 + \frac{11}{2} \\ &= 6.5 \end{aligned}$$

The smaller of the two numbers is 6.5. Thus, every zero of *f* lies between -6.5 and 6.5.

We graph using the bounds and ZOOM-FIT, and adjust the viewing window to improve the graph.



37. $f(x) = x^4 + 3x^3 - 5x^2 + 9$

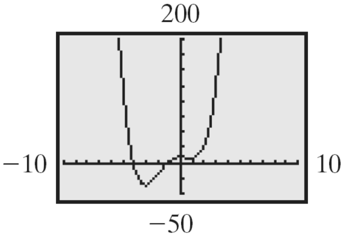
$a_3 = 3, a_2 = -5, a_1 = 0, a_0 = 9$

$$\begin{aligned} &Max\{1, |\ 9\ | + |\ 0\ | + |-5| + |\ 3\ |\} \\ &= Max\{1, 9 + 0 + 5 + 3\} \\ &= Max\{1, 17\} \\ &= 17 \end{aligned}$$

$$\begin{aligned} &1 + Max\{|\ 9\ |, |\ 0\ |, |-5|, |\ 3\ |\} \\ &= 1 + Max\{9, 0, 5, 3\} \\ &= 1 + 9 \\ &= 10 \end{aligned}$$

The smaller of the two numbers is 10. Thus, every zero of *f* lies between – 10 and 10.

We graph using the bounds and ZOOM-FIT, and adjust the viewing window to improve the graph.



38. $f(x) = 4x^4 - 12x^3 + 27x^2 - 54x + 81$

$$= 4\left(x^4 - 3x^3 + \frac{27}{4}x^2 - \frac{27}{2}x + \frac{81}{4}\right)$$

Note: The leading coefficient must be 1.

$a_3 = -3, a_2 = \frac{27}{4}, a_1 = -\frac{27}{2}, a_0 = \frac{81}{4}$

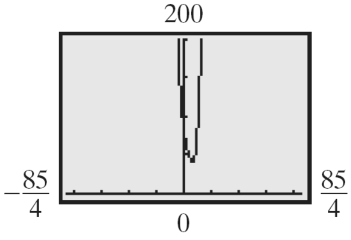
$$\begin{aligned} &Max\left\{1, \left|\frac{81}{4}\right| + \left|-\frac{27}{2}\right| + \left|\ \frac{27}{4}\ \right| + |\ -3\ |\right\} \\ &= Max\left\{1, \frac{81}{4} + \frac{27}{2} + \frac{27}{4} + 3\right\} \\ &= Max\left\{1, \frac{87}{2}\right\} = \frac{87}{2} = 43.5 \end{aligned}$$

$$1 + Max\left\{\left|\frac{81}{4}\right|, \left|-\frac{27}{2}\right|, \left|\ \frac{27}{4}\ \right|, \left|\ -3\ \right|\right\}$$

$$= 1 + Max\left\{\frac{81}{4}, \frac{27}{2}, \frac{27}{4}, 3\right\}$$

$$= 1 + \frac{81}{4} = \frac{85}{4} = 21.25$$

The smaller of the two numbers is 21.25. Thus, every zero of *f* lies between – 21.25 and 21.25. We graph using the bounds and ZOOM-FIT, and adjust the viewing window to improve the graph.



39. $f(x) = x^3 + 2x^2 - 5x - 6$

Step 1: $f(x)$ has at most 3 real zeros.

Step 2: Possible rational zeros:
$p = \pm 1, \pm 2, \pm 3, \pm 6; \quad q = \pm 1;$

$\frac{p}{q} = \pm 1, \pm 2, \pm 3, \pm 6$

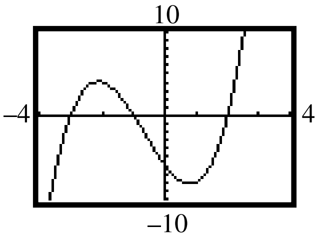
Step 3: Using the Bounds on Zeros Theorem:
$a_2 = 2, \quad a_1 = -5, \quad a_0 = -6$

$$\text{Max } \{1, |-6| + |-5| + |2|\}$$

$$= \text{Max } \{1, 13\} = 13$$

$$1 + \text{Max } \{|-6|, |-5|, |2|\} = 1 + 6 = 7$$

The smaller of the two numbers is 7. Thus, every zero of f lies between –7 and 7.



Step 4: From the graph it appears that there are *x*-intercepts at –3, –1, and 2.
Using synthetic division:

$$\begin{array}{r|rrrr} -3 & 1 & 2 & -5 & -6 \\ & & -3 & 3 & 6 \\ \hline & 1 & -1 & -2 & 0 \end{array}$$

Since the remainder is 0, $x - (-3) = x + 3$ is a factor. Thus,

$$f(x) = (x+3)\left(x^2 - x - 2\right)$$

$$= (x+3)(x+1)(x-2)$$

The zeros are –3, –1, and 2.

40. $f(x) = x^3 + 8x^2 + 11x - 20$

Step 1: $f(x)$ has at most 3 real zeros.

Step 2: Possible rational zeros:
$p = \pm 1, \pm 2, \pm 4, \pm 5, \pm 10, \pm 20; \quad q = \pm 1;$

$\frac{p}{q} = \pm 1, \pm 2, \pm 4, \pm 5, \pm 10, \pm 20$

Step 3: Using the Bounds on Zeros Theorem:
$a_2 = 8, \quad a_1 = 11, \quad a_0 = -20$

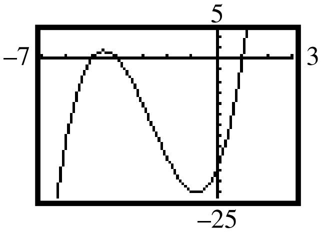
$$\text{Max } \{1, |-20| + |11| + |8|\}$$

$$= \text{Max } \{1, 39\} = 39$$

$$1 + \text{Max } \{|-20|, |11|, |8|\}$$

$$= 1 + 20 = 21$$

The smaller of the two numbers is 21. Thus, every zero of f lies between -21 and 21. We graph using the bounds and ZOOM-FIT, and adjust the viewing window to improve the graph.



Step 4: From the graph it appears that there are *x*-intercepts at –5, –4, and 1.
Using synthetic division:

$$\begin{array}{r|rrrr} -5 & 1 & 8 & 11 & -20 \\ & & -5 & -15 & 20 \\ \hline & 1 & 3 & -4 & 0 \end{array}$$

Since the remainder is 0, $x - (-5) = x + 5$ is a factor. Thus,

$$f(x) = (x+5)\left(x^2 + 3x - 4\right)$$

$$= (x+5)(x+4)(x-1)$$

The zeros are –5, –4, and 1.

41. $f(x)=2x^3-13x^2+24x-9$

Step 1: $f(x)$ has at most 3 real zeros.

Step 2: Possible rational zeros:
$p=\pm 1, \pm 3, \pm 9;\quad q=\pm 1, \pm 2;$

$$\frac{p}{q}=\pm 1, \pm 3, \pm 9, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{9}{2}$$

Step 3: Using the Bounds on Zeros Theorem:

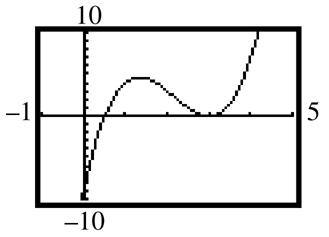
$$f(x)=2\left(x^3-6.5x^2+12x-4.5\right)$$

$$a_2=-6.5,\ a_1=12,\ a_0=-4.5$$

$$\text{Max }\{1, |-4.5|+|12|+|-6.5|\}$$
$$=\text{Max }\{1, 23\}=23$$

$$1+\text{Max }\{|-4.5|, |12|, |-6.5|\}$$
$$=1+12=13$$

The smaller of the two numbers is 13. Thus, every zero of *f* lies between −13 and 13. We graph using the bounds and ZOOM-FIT, and adjust the viewing window to improve the graph.



Step 4: From the graph it appears that there are x-intercepts at 0.5 and 3 .

Using synthetic division:

$$\begin{array}{r|rrrr} 3 & 2 & -13 & 24 & -9 \\ & & 6 & -21 & 9 \\ \hline & 2 & -7 & 3 & 0 \end{array}$$

Since the remainder is 0, $x-3$ is a factor. Thus,

$$f(x)=(x-3)\left(2x^2-7x+3\right)$$
$$=(x-3)(2x-1)(x-3)$$
$$=(2x-1)(x-3)^2$$

The zeros are $\frac{1}{2}$ and 3 (multiplicity 2).

42. $f(x)=2x^3-5x^2-4x+12$

Step 1: $f(x)$ has at most 3 real zeros.

Step 2: Possible rational zeros:
$p=\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12;$
$q=\pm 1, \pm 2;$

$$\frac{p}{q}=\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12, \pm\frac{1}{2}, \pm\frac{3}{2}$$

Step 3: Using the Bounds on Zeros Theorem:

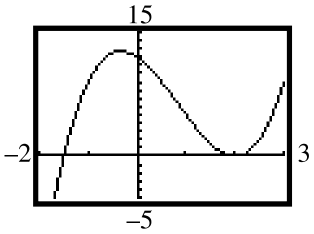
$$f(x)=2\left(x^3-2.5x^2-2x+6\right)$$

$$a_2=-2.5,\quad a_1=-2,\quad a_0=6$$

$$\text{Max }\{1, |6|+|-2|+|-2.5|\}$$
$$=\text{Max }\{1, 10.5\}=10.5$$

$$1+\text{Max }\{|6|, |-2|, |-2.5|\}$$
$$=1+6=7$$

The smaller of the two numbers is 7. Thus, every zero of *f* lies between −7 and 7. We graph using the bounds and ZOOM-FIT, and adjust the viewing window to improve the graph.



Step 4: From the graph it appears that there are x-intercepts at $-\frac{3}{2}$ and 2 .

Using synthetic division:

$$\begin{array}{r|rrrr} 2 & 2 & -5 & -4 & 12 \\ & & 4 & -2 & -12 \\ \hline & 2 & -1 & -6 & 0 \end{array}$$

Since the remainder is 0, $x-2$ is a factor. Thus,

$$f(x)=(x-2)\left(2x^2-x-6\right)$$
$$=(x-2)(2x+3)(x-2)$$

The zeros are $-\frac{3}{2}$ and 2 (multiplicity 2).

43. $f(x) = 3x^3 + 4x^2 + 4x + 1$

Step 1: $f(x)$ has at most 3 real zeros..

Step 2: Possible rational zeros:

$$p = \pm 1; \quad q = \pm 1, \pm 3; \quad \frac{p}{q} = \pm 1, \pm \frac{1}{3}$$

Step 3: Using the Bounds on Zeros Theorem:

$$f(x) = 3\left(x^3 + \frac{4}{3}x^2 + \frac{4}{3}x + \frac{1}{3}\right)$$

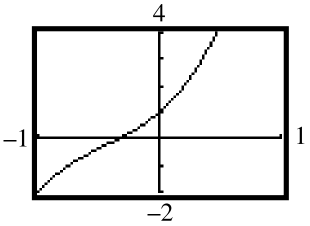
$$a_2 = \frac{4}{3}, \quad a_1 = \frac{4}{3}, \quad a_0 = \frac{1}{3}$$

$$\text{Max}\left\{1, \left|\frac{1}{3}\right| + \left|\frac{4}{3}\right| + \left|\frac{4}{3}\right|\right\} = \text{Max}\{1, 3\} = 3$$

$$1 + \text{Max}\left\{\left|\frac{1}{3}\right|, \left|\frac{4}{3}\right|, \left|\frac{4}{3}\right|\right\} = 1 + \frac{4}{3} = \frac{7}{3}$$

The smaller of the two numbers is $\frac{7}{3}$.

Thus, every zero of *f* lies between $-\frac{7}{3}$ and $\frac{7}{3}$. We graph using the bounds and ZOOM-FIT, and adjust the viewing window to improve the graph.



Step 4: From the graph it appears that there is an *x*-intercepts at $-\frac{1}{3}$.

Using synthetic division:

$$\begin{array}{r|rrrr} -\frac{1}{3} & 3 & 4 & 4 & 1 \\ & & -1 & -1 & -1 \\ \hline & 3 & 3 & 3 & 0 \end{array}$$

The remainder is 0, so $x - \left(-\frac{1}{3}\right) = x + \frac{1}{3}$

Thus, $f(x) = \left(x + \frac{1}{3}\right)(3x^2 + 3x + 3)$.

$$= 3\left(x + \frac{1}{3}\right)(x^2 + x + 1)$$

$$= (3x + 1)(x^2 + x + 1)$$

Note that $x^2 + x + 1 = 0$ has no real solution. The only real zero is $-\frac{1}{3}$.

44. $f(x) = 3x^3 - 7x^2 + 12x - 28$

Step 1: $f(x)$ has at most 3 real zeros.

Step 2: Possible rational zeros:

$$p = \pm 1, \pm 2, \pm 4, \pm 7, \pm 14, \pm 28;$$

$$q = \pm 1, \pm 3;$$

$$\frac{p}{q} = \pm 1, \pm 2, \pm 4, \pm 7, \pm 14, \pm 28, \pm \frac{1}{3},$$

$$\pm \frac{2}{3}, \pm \frac{4}{3}, \pm \frac{7}{3}, \pm \frac{14}{3}, \pm \frac{28}{3}$$

Step 3: Using the Bounds on Zeros Theorem:

$$f(x) = 3\left(x^3 - \frac{7}{3}x^2 + 4x - \frac{28}{3}\right)$$

$$a_2 = -\frac{7}{3}, \quad a_1 = 4, \quad a_0 = -\frac{28}{3}$$

$$\text{Max}\left\{1, \left|-\frac{28}{3}\right| + |4| + \left|-\frac{7}{3}\right|\right\}$$

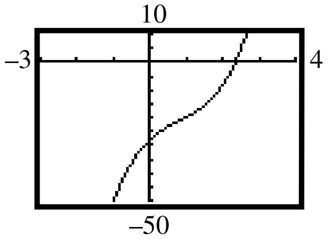
$$= \text{Max}\left\{1, \frac{47}{3}\right\} = \frac{47}{3}$$

$$1 + \text{Max}\left\{\left|-\frac{28}{3}\right|, |4|, \left|-\frac{7}{3}\right|\right\}$$

$$= 1 + \frac{28}{3} = \frac{31}{3}$$

The smaller of the two numbers is $\frac{31}{3}$.

Thus, every zero of *f* lies between $-\frac{31}{3}$ and $\frac{31}{3}$. We graph using the bounds and ZOOM-FIT, and adjust the viewing window to improve the graph.



Step 4: From the graph it appears that there is an *x*-intercept at $\frac{7}{3}$.

Using synthetic division:

$$\begin{array}{r|rrrr} \frac{7}{3} & 3 & -7 & 12 & -28 \\ & & 7 & 0 & 28 \\ \hline & 3 & 0 & 12 & 0 \end{array}$$

Since the remainder is 0, $x-\frac{7}{3}$ is a factor. Thus,

$$f(x)=\left(x-\frac{7}{3}\right)\left(3x^2+12\right).$$
$$=3\left(x-\frac{7}{3}\right)\left(x^2+4\right)$$
$$=\left(3x-7\right)\left(x^2+4\right)$$

Note that $x^2+4=0$ has no real solution.

The only real zero is $\frac{7}{3}$.

45. $f(x)=x^3-8x^2+17x-6$

Step 1: $f(x)$ has at most 3 real zeros.

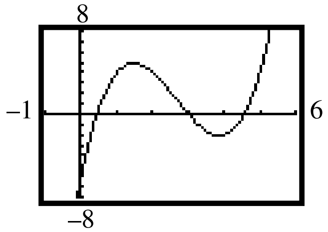
Step 2: Possible rational zeros:
$p=\pm 1, \pm 2, \pm 3, \pm 6;\quad q=\pm 1;$
$\frac{p}{q}=\pm 1, \pm 2, \pm 3, \pm 6$

Step 3: Using the Bounds on Zeros Theorem:
$a_2=-8,\quad a_1=17,\quad a_0=-6$

$\text{Max }\{1, |-6|+|17|+|-8|\}$
$=\text{Max }\{1, 31\}=31$

$1+\text{Max }\{|-6|, |17|, |-8|\}$
$=1+17=18$

The smaller of the two numbers is 18. Thus, every zero of f lies between -18 and 18. We graph using the bounds and ZOOM-FIT, and adjust the viewing window to improve the graph.



Step 4: From the graph it appears that there are x-intercepts at 0.5, 3, and 4.5.
Using synthetic division:

$$\begin{array}{r|rrrr} 3 & 1 & -8 & 17 & -6 \\ & & 3 & -15 & 6 \\ \hline & 1 & -5 & 2 & 0 \end{array}$$

Since the remainder is 0, $x-3$ is a factor. Thus,

$$f(x)=(x-3)\left(x^2-5x+2\right).$$

Using the quadratic formula to find the solutions of the depressed equation $x^2-5x+2=0$:

$$x=\frac{-(-5)\pm\sqrt{(-5)^2-4(1)(2)}}{2(1)}=\frac{5\pm\sqrt{17}}{2}$$

Thus,

$$f(x)=(x-3)\left(x-\left(\frac{5+\sqrt{17}}{2}\right)\right)\left(x-\left(\frac{5-\sqrt{17}}{2}\right)\right).$$

The zeros are 3, $\frac{5+\sqrt{17}}{2}$, and $\frac{5-\sqrt{17}}{2}$.

46. $f(x)=x^3+6x^2+6x-4$

Step 1: $f(x)$ has at most 3 real zeros.

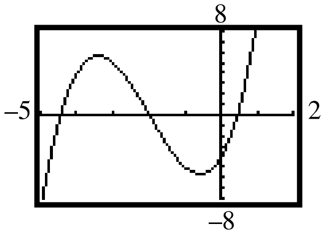
Step 2: Possible rational zeros:
$p=\pm 1, \pm 2, \pm 4;\quad q=\pm 1;$
$\frac{p}{q}=\pm 1, \pm 2, \pm 4$

Step 3: Using the Bounds on Zeros Theorem:
$a_2=6,\quad a_1=6,\quad a_0=-4$

$\text{Max }\{1, |-4|+|6|+|6|\}$
$=\text{Max }\{1, 16\}=16$

$1+\text{Max }\{|-4|, |6|, |6|\}=1+6=7$

The smaller of the two numbers is 7. Thus, every zero of *f* lies between –7 and 7. We graph using the bounds and ZOOM-FIT, and adjust the viewing window to improve the graph.



Step 4: From the graph it appears that there are x-intercepts at –4.5, –2, and 0.5.
Using synthetic division:

$$\begin{array}{r|rrrr} -2 & 1 & 6 & 6 & -4 \\ & & -2 & -8 & 4 \\ \hline & 1 & 4 & -2 & 0 \end{array}$$

Since the remainder is 0, $x-(-2)=x+2$ is a factor. Thus,

$f(x) = (x+2)\left(x^2+4x-2\right)$. Using the quadratic formula to find the solutions of the depressed equation $x^2+4x-2=0$:

$$x = \frac{-4 \pm \sqrt{4^2-4(1)(-2)}}{2(1)} = \frac{-4 \pm \sqrt{24}}{2}$$

$$= \frac{-4 \pm 2\sqrt{6}}{2} = -2 \pm \sqrt{6}$$

Thus,

$$f(x) = (x+2)\left(x-\left(-2+\sqrt{6}\right)\right)\left(x-\left(-2-\sqrt{6}\right)\right)$$

$$= \left(x+2\right)\left(x+2-\sqrt{6}\right)\left(x+2+\sqrt{6}\right)$$

The zeros are $-2, -2-\sqrt{6}$, and $-2+\sqrt{6}$.

47. $f(x) = x^4+x^3-3x^2-x+2$

Step 1: $f(x)$ has at most 4 real zeros.

Step 2: Possible rational zeros:

$p = \pm 1, \pm 2; \quad q = \pm 1; \quad \dfrac{p}{q} = \pm 1, \pm 2$

Step 3: Using the Bounds on Zeros Theorem:

$a_3 = 1, \ a_2 = -3, \quad a_1 = -1, \quad a_0 = 2$

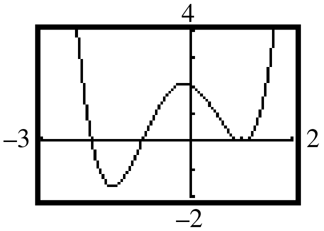
$\text{Max}\ \{1, |2|+|-1|+|-3|+|1|\}$

$= \text{Max}\ \{1, 7\} = 7$

$1 + \text{Max}\ \{|2|, |-1|, |-3|, |1|\}$

$= 1+3 = 4$

The smaller of the two numbers is 4. Thus, every zero of *f* lies between –4 and 4. We graph using the bounds and ZOOM-FIT, and adjust the viewing window to improve the graph.



Step 4: From the graph it appears that there are *x*-intercepts at –2, –1, and 1.

Using synthetic division:

$$\begin{array}{r|rrrrr} -2 & 1 & 1 & -3 & -1 & 2 \\ & & -2 & 2 & 2 & -2 \\ \hline & 1 & -1 & -1 & 1 & 0 \end{array}$$

Since the remainder is 0, $x+2$ is a factor. Using synthetic division again:

$$\begin{array}{r|rrrr} -1 & 1 & -1 & -1 & 1 \\ & & -1 & 2 & -1 \\ \hline & 1 & -2 & 1 & 0 \end{array}$$

Since the remainder is 0, $x+1$ is also a factor. Thus,

$$f(x) = (x+2)(x+1)(x^2-2x+1)$$

$$= (x+2)(x+1)(x-1)^2$$

The zeros are –2, –1, and 1 (multiplicity 2).

48. $f(x) = x^4-x^3-6x^2+4x+8$

Step 1: $f(x)$ has at most 4 real zeros.

Step 2: Possible rational zeros:

$p = \pm 1, \pm 2, \pm 4, \pm 8; \quad q = \pm 1;$

$\dfrac{p}{q} = \pm 1, \pm 2, \pm 4, \pm 8$

Step 3: Using the Bounds on Zeros Theorem:

$a_3 = -1, \ a_2 = -6, \quad a_1 = 4, \quad a_0 = 8$

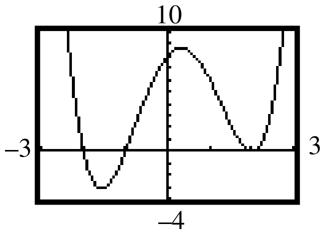
$\text{Max}\ \{1, |8|+|4|+|-6|+|-1|\}$

$= \text{Max}\ \{1, 19\} = 19$

$1 + \text{Max}\ \{|8|, |4|, |-6|, |-1|\}$

$= 1+8 = 9$

The smaller of the two numbers is 9. Thus, every zero of *f* lies between –9 and 9. We graph using the bounds and ZOOM-FIT, and adjust the viewing window to improve the graph.



Step 4: From the graph it appears that there are *x*-intercepts at –2, –1, and 2.

Using synthetic division:

$$\begin{array}{r|rrrrr} -2 & 1 & -1 & -6 & 4 & 8 \\ & & -2 & 6 & 0 & -8 \\ \hline & 1 & -3 & 0 & 4 & 0 \end{array}$$

Since the remainder is 0, $x+2$ is a factor. Using synthetic division again:

$$\begin{array}{r|rrrr} -1 & 1 & -3 & 0 & 4 \\ & & -1 & 4 & -4 \\ \hline & 1 & -4 & 4 & 0 \end{array}$$

Since the remainder is 0, $x+1$ is also a factor. Thus,

$$f(x) = (x+2)(x+1)(x^2-4x+4)$$
$$= (x+2)(x+1)(x-2)^2$$

The zeros are –2, –1, and 2 (multiplicity 2).

49. $f(x) = 2x^4 + 17x^3 + 35x^2 - 9x - 45$

Step 1: $f(x)$ has at most 4 real zeros.

Step 2: Possible rational zeros:

$p = \pm 1, \pm 3, \pm 5, \pm 9, \pm 15, \pm 45;$

$q = \pm 1, \pm 2;$

$$\frac{p}{q} = \pm 1, \pm 3, \pm 5, \pm 9, \pm 15, \pm 45, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \pm\frac{9}{2}, \pm\frac{15}{2}, \pm\frac{45}{2}$$

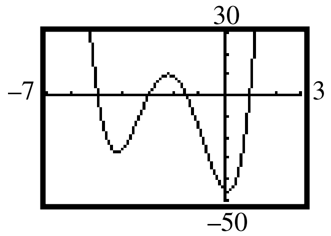
Step 3: Using the Bounds on Zeros Theorem:

$$f(x) = 2\left(x^4 + 8.5x^3 + 17.5x^2 - 4.5x - 22.5\right)$$

$a_3 = 8.5,\ a_2 = 17.5,\ a_1 = -4.5,\ a_0 = -22.5$

$$\text{Max }\{1, |-22.5| + |-4.5| + |17.5| + |8.5|\}$$
$$= \text{Max }\{1, 53\} = 53$$
$$1 + \text{Max }\{|-22.5|, |-4.5|, |17.5|, |8.5|\}$$
$$= 1 + 22.5 = 23.5$$

The smaller of the two numbers is 23.5. Thus, every zero of *f* lies between –23.5 and 23.5. We graph using the bounds and ZOOM-FIT, and adjust the viewing window to improve the graph.



Step 4: From the graph it appears that there are *x*-intercepts at –5, –3, –1.5, and 1. Using synthetic division:

$$\begin{array}{r|rrrrr} -5 & 2 & 17 & 35 & -9 & -45 \\ & & -10 & -35 & 0 & 45 \\ \hline & 2 & 7 & 0 & -9 & 0 \end{array}$$

Since the remainder is 0, $x+5$ is a factor. Using synthetic division again:

$$\begin{array}{r|rrrr} -3 & 2 & 7 & 0 & -9 \\ & & -6 & -3 & 9 \\ \hline & 2 & 1 & -3 & 0 \end{array}$$

Since the remainder is 0, $x+3$ is also a factor. $2x^2 + x - 3$. Thus,

$$f(x) = (x+5)(x+3)(2x^2 + x - 3)$$
$$= (x+5)(x+3)(2x+3)(x-1)$$

The zeros are $-5, -3, -\frac{3}{2}$, and 1.

50. $f(x) = 4x^4 - 15x^3 - 8x^2 + 15x + 4$

Step 1: $f(x)$ has at most 4 real zeros.

Step 2: Possible rational zeros:

$p = \pm 1, \pm 2, \pm 4;\quad q = \pm 1, \pm 2, \pm 4;$

$$\frac{p}{q} = \pm 1, \pm 2, \pm 4, \pm\frac{1}{2}, \pm\frac{1}{4}$$

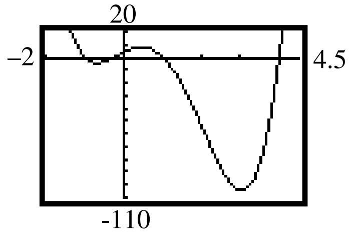
Step 3: Using the Bounds on Zeros Theorem:

$$f(x) = 4\left(x^4 - 3.75x^3 - 2x^2 + 3.75x + 1\right)$$

$a_3 = -3.75,\ a_2 = -2,\ a_1 = 3.75,\ a_0 = 1$

$$\text{Max }\{1, |1| + |3.75| + |-2| + |-3.75|\}$$
$$= \text{Max }\{1, 10.5\} = 10.5$$
$$1 + \text{Max }\{|1|, |3.75|, |-2|, |-3.75|\}$$
$$= 1 + 3.75 = 4.75$$

The smaller of the two numbers is 4.75. Thus, every zero of *f* lies between –4.75 and 4.75. We graph using the bounds and ZOOM-FIT, and adjust the viewing window to improve the graph.



Step 4: From the graph it appears that there are *x*-intercepts at –1, –0.25, 1, and 4. Using synthetic division:

$$\begin{array}{r|rrrrr} -1 & 4 & -15 & -8 & 15 & 4 \\ & & -4 & 19 & -11 & -4 \\ \hline & 4 & -19 & 11 & 4 & 0 \end{array}$$

Since the remainder is 0, $x+1$ is a factor. Using synthetic division again:

$$\begin{array}{r|rrrr} 4 & 4 & -19 & 11 & 4 \\ & & 16 & -12 & -4 \\ \hline & 4 & -3 & -1 & 0 \end{array}$$

Since the remainder is 0, $x-4$ is also a factor. Thus,

$$\begin{aligned} f(x) &= (x+1)(x-4)(4x^2-3x-1) \\ &= (x+1)(x-4)(4x+1)(x-1) \end{aligned}$$

The zeros are –1, 4, $-\frac{1}{4}$, and 1.

51. $f(x)=2x^4-3x^3-21x^2-2x+24$

Step 1: $f(x)$ has at most 4 real zeros.

Step 2: Possible rational zeros:

$p=\pm1,\pm2,\pm3,\pm4,\pm6,\pm8,\pm12,\pm24;$

$q=\pm1,\pm2;$

$$\frac{p}{q}=\pm1,\pm2,\pm3,\pm4,\pm6,\pm8,\pm12, \pm24,\pm\frac{1}{2},\pm\frac{3}{2}$$

Step 3: Using the Bounds on Zeros Theorem:

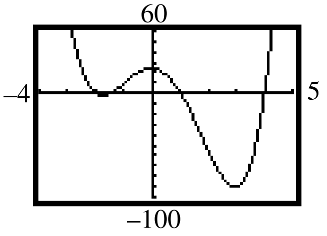
$f(x)=2\left(x^4-1.5x^3-10.5x^2-x+12\right)$

$a_3=-1.5,\ a_2=-10.5,\ a_1=-1,\ a_0=12$

$$\begin{aligned} &\text{Max}\ \{1, |12|+|-1|+|-10.5|+|-1.5|\} \\ &\quad= \text{Max}\ \{1, 25\} = 25 \end{aligned}$$

$$\begin{aligned} &1+\text{Max}\ \{|12|, |-1|, |-10.5|, |-1.5|\} \\ &\quad= 1+12 = 13 \end{aligned}$$

The smaller of the two numbers is 13. Thus, every zero of *f* lies between -13 and 13. We graph using the bounds and ZOOM-FIT, and adjust the viewing window to improve the graph.



Step 4: From the graph it appears that there are *x*-intercepts at –2, –1.5, 1, and 4. Using synthetic division:

$$\begin{array}{r|rrrrr} -2 & 2 & -3 & -21 & -2 & 24 \\ & & -4 & 14 & 14 & -24 \\ \hline & 2 & -7 & -7 & 12 & 0 \end{array}$$

Since the remainder is 0, $x+2$ is a factor. Using synthetic division again:

$$\begin{array}{r|rrrr} 4 & 2 & -7 & -7 & 12 \\ & & 8 & 4 & -12 \\ \hline & 2 & 1 & -3 & 0 \end{array}$$

Since the remainder is 0, $x-4$ is also a factor. Thus,

$$\begin{aligned} f(x) &= (x+2)(x-4)(2x^2+x-3) \\ &= (x+2)(x-4)(2x+3)(x-1) \end{aligned}$$

The zeros are –2, 4, $-\frac{3}{2}$, and 1.

52. $f(x)=2x^4+11x^3-5x^2-43x+35$

Step 1: $f(x)$ has at most 4 real zeros.

Step 2: Possible rational zeros:

$p=\pm1,\pm5,\pm7,\pm35;\quad q=\pm1,\pm2;$

$$\frac{p}{q}=\pm1,\pm5,\pm7,\pm35,\pm\frac{1}{2},\pm\frac{5}{2},\pm\frac{7}{2},\pm\frac{35}{2}$$

Step 3: Using the Bounds on Zeros Theorem:

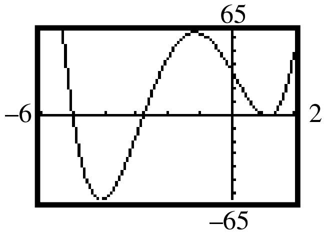
$f(x)=2\left(x^4+5.5x^3-2.5x^2-21.5x+17.5\right)$

$a_3=5.5,\ a_2=-2.5,\ a_1=-21.5,\ a_0=17.5$

$$\begin{aligned} &\text{Max}\ \{1, |17.5|+|-21.5|+|-2.5|+|5.5|\} \\ &\quad= \text{Max}\ \{1, 47\} = 47 \end{aligned}$$

$$\begin{aligned} &1+\text{Max}\ \{|17.5|, |-21.5|, |-2.5|, |5.5|\} \\ &\quad= 1+21.5\ = 22.5 \end{aligned}$$

The smaller of the two numbers is 22.5. Thus, every zero of *f* lies between -22.5 and 22.5. We graph using the bounds and ZOOM-FIT, and adjust the viewing window to improve the graph.



Step 4: From the graph it appears that there are *x*-intercepts at –5, –2.8, 1, and 1.2. Using synthetic division:

$$\begin{array}{r|rrrrr} -5 & 2 & 11 & -5 & -43 & 35 \\ & & -10 & -5 & 50 & -35 \\ \hline & 2 & 1 & -10 & 7 & 0 \end{array}$$

Since the remainder is 0, $x+5$ is a factor. Using synthetic division again:

$$\begin{array}{r|rrrr} 1 & 2 & 1 & -10 & 7 \\ & & 2 & 3 & -7 \\ \hline & 2 & 3 & -7 & 0 \end{array}$$

Since the remainder is 0, $x-1$ is also a factor. Thus,

$f(x)=(x+5)(x-1)\left(2x^2+3x-7\right)$.

Using the quadratic formula to find the solutions of the depressed equation $2x^2+3x-7=0$:

$$x=\frac{-3\pm\sqrt{3^2-4(2)(-7)}}{2(2)}=\frac{-3\pm\sqrt{65}}{4}$$

Thus, $f(x)=$

$$2(x+5)(x-1)\left(x+\frac{3-\sqrt{65}}{4}\right)\left(x+\frac{3+\sqrt{65}}{4}\right).$$

The zeros are $-5, 1, \dfrac{-3-\sqrt{65}}{4}$, and $\dfrac{-3+\sqrt{65}}{4}$.

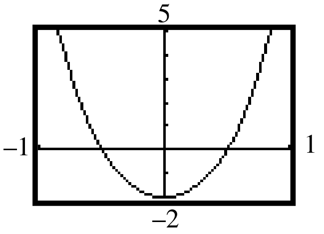
53. $f(x)=4x^4+7x^2-2$

Step 1: $f(x)$ has at most 4 real zeros.

Step 2: Possible rational zeros:
$p=\pm1, \pm2;\quad q=\pm1,\pm2,\pm4;$
$\dfrac{p}{q}=\pm1,\pm2,\pm\dfrac{1}{2},\pm\dfrac{1}{4}$

Step 3: Using the Bounds on Zeros Theorem:
$f(x)=4\left(x^4+1.75x^2-0.5\right)$
$a_3=0,\ a_2=1.75,\quad a_1=0,\quad a_0=-0.5$
$\text{Max}\ \{1,|-0.5|+|0|+|1.75|+|0|\}$
$=\text{Max}\ \{1, 2.25\}=2.25$
$1+\text{Max}\ \{|-0.5|,|0|,|1.75|,|0|\}$
$=1+1.75=2.75$

The smaller of the two numbers is 2.25. Thus, every zero of *f* lies between -2.25 and 2.25. We graph using the bounds and ZOOM-FIT, and adjust the viewing window to improve the graph.



Step 4: From the graph it appears that there are x-intercepts at $-\dfrac{1}{2}$ and $\dfrac{1}{2}$.
Using synthetic division:

$$\begin{array}{r|rrrrr} -\frac{1}{2} & 4 & 0 & 7 & 0 & -2 \\ & & -2 & 1 & -4 & 2 \\ \hline & 4 & -2 & 8 & -4 & 0 \end{array}$$

Since the remainder is 0, $x+\dfrac{1}{2}$ is a factor. Using synthetic division again:

$$\begin{array}{r|rrrr} \frac{1}{2} & 4 & -2 & 8 & -4 \\ & & 2 & 0 & 4 \\ \hline & 4 & 0 & 8 & 0 \end{array}$$

Since the remainder is 0, $x-\dfrac{1}{2}$ is also a factor. Thus,

$$\begin{aligned} f(x)&=\left(x+\frac{1}{2}\right)\left(x-\frac{1}{2}\right)\left(4x^2+8\right)\\ &=4\left(x+\frac{1}{2}\right)\left(x-\frac{1}{2}\right)\left(x^2+2\right)\\ &=(2x+1)(2x-1)\left(x^2+2\right)\end{aligned}$$

The depressed equation has no real zeros. The real zeros are $-\dfrac{1}{2}$ and $\dfrac{1}{2}$.

54. $f(x)=4x^4+15x^2-4$

Step 1: $f(x)$ has at most 4 real zeros.

Step 2: Possible rational zeros:
$p=\pm1,\pm2,\pm4;\quad q=\pm1,\pm2,\pm4;$
$\dfrac{p}{q}=\pm1,\pm2,\pm4,\pm\dfrac{1}{2},\pm\dfrac{1}{4}$

Step 3: Using the Bounds on Zeros Theorem:
$f(x)=4\left(x^4+3.75x^2-1\right)$
$a_3=0,\ a_2=3.75,\quad a_1=0,\quad a_0=-1$

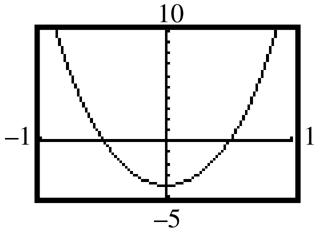
$\text{Max}\ \{1, |-1|+|0|+|3.75|+|0|\}$

$= \text{Max}\ \{1, 4.75\} = 4.75$

$1+\text{Max}\ \{|-1|, |0|, |3.75|, |0|\}$

$= 1+3.75 = 4.75$

The smaller of the two numbers is 4.75. Thus, every zero of *f* lies between -4.75 and 4.75. We graph using the bounds and ZOOM-FIT, and adjust the viewing window to improve the graph.



Step 4: From the graph it appears that there are *x*-intercepts at $-\frac{1}{2}$ and $\frac{1}{2}$..

Using synthetic division:

$$\begin{array}{r|rrrrr} -\frac{1}{2} & 4 & 0 & 15 & 0 & -4 \\ & & -2 & 1 & -8 & 4 \\ \hline & 4 & -2 & 16 & -8 & 0 \end{array}$$

Since the remainder is 0, $x+\frac{1}{2}$ is a factor. Using synthetic division again:

$$\begin{array}{r|rrrr} \frac{1}{2} & 4 & -2 & 16 & -8 \\ & & 2 & 0 & 8 \\ \hline & 4 & 0 & 16 & 0 \end{array}$$

Since the remainder is 0, $x-\frac{1}{2}$ is also a factor. Thus,

$$\begin{aligned} f(x) &= \left(x+\frac{1}{2}\right)\left(x-\frac{1}{2}\right)(4x^2+16) \\ &= 4\left(x+\frac{1}{2}\right)\left(x-\frac{1}{2}\right)(x^2+4) \\ &= (2x+1)(2x-1)(x^2+4) \end{aligned}$$

The depressed equation has no real zeros. The real zeros are $-\frac{1}{2}$ and $\frac{1}{2}$.

55. $f(x) = 4x^5 - 8x^4 - x + 2$

Step 1: $f(x)$ has at most 5 real zeros.

Step 2: Possible rational zeros:
$p = \pm 1, \pm 2;\quad q = \pm 1, \pm 2, \pm 4;$
$\frac{p}{q} = \pm 1, \pm 2, \pm\frac{1}{2}, \pm\frac{1}{4}$

Step 3: Using the Bounds on Zeros Theorem:
$f(x) = 4\left(x^5 - 2x^4 - 0.25x + 0.5\right)$
$a_4 = -2, a_3 = 0, a_2 = 0, a_1 = -0.25,$
$a_0 = 0.5$

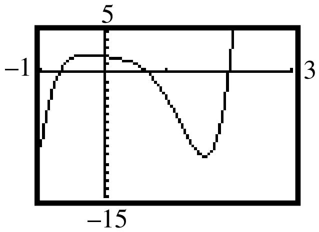
$\text{Max}\{1, |0.5|+|-0.25|+|0|+|0|+|-2|\}$

$= \text{Max}\{1, 2.75\} = 2.75$

$1+\text{Max}\{|0.5|, |-0.25|, |0|, |0|, |-2|\}$

$= 1+2 = 3$

The smaller of the two numbers is 2.75. Thus, every zero of *f* lies between -2.75 and 2.75. We graph using the bounds and ZOOM-FIT, and adjust the viewing window to improve the graph.



Step 4: From the graph it appears that there are *x*-intercepts at –0.7, 0.7 and 2.

Using synthetic division:

$$\begin{array}{r|rrrrrr} 2 & 4 & -8 & 0 & 0 & -1 & 2 \\ & & 8 & 0 & 0 & 0 & -2 \\ \hline & 4 & 0 & 0 & 0 & -1 & 0 \end{array}$$

Since the remainder is 0, $x-2$ is a factor. Thus,

$$\begin{aligned} f(x) &= (x-2)(4x^4-1) \\ &= (x-2)(2x^2-1)(2x^2+1) \\ &= (x-2)\left(\sqrt{2}x-1\right)\left(\sqrt{2}x+1\right)\left(2x^2+1\right) \end{aligned}$$

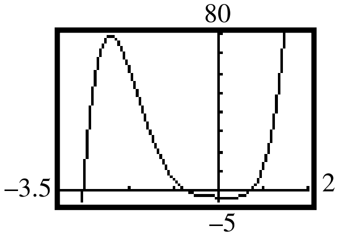
Note that the depressed equation $2x^2+1=0$ has no real solution. The real zeros are $-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}$, and 2.

56. $f(x)=4x^5+12x^4-x-3$

Step 1: $f(x)$ has at most 5 real zeros.

Step 2: Possible rational zeros:
$p=\pm1,\pm3;\quad q=\pm1,\pm2,\pm4;$
$\frac{p}{q}=\pm1,\pm3,\pm\frac{1}{2},\pm\frac{3}{2},\pm\frac{1}{4},\pm\frac{3}{4}$

Step 3: Using the Bounds on Zeros Theorem:
$f(x)=4\left(x^5+3x^4-0.25x-0.75\right)$
$a_4=3, a_3=0, a_2=0,\ a_1=-0.25,$
$a_0=-0.75$
$\text{Max}\{1,|-0.75|+|-0.25|+|0|+|0|+|3|\}$
$=\text{Max }\{1,4\}=4$
$1+\text{Max}\{|-0.75|,|-0.25|,|0|,|0|,|3|\}$
$=1+3=4$
The smaller of the two numbers is 4. Thus, every zero of *f* lies between –4 and 4. We graph using the bounds and ZOOM-FIT, and adjust the viewing window to improve the graph.



Step 4: From the graph it appears that there are *x*-intercepts at –3, –0.7, and 0.7.
Using synthetic division:

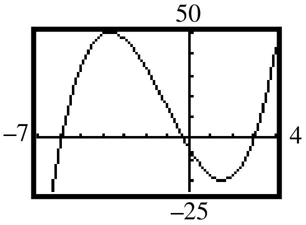
$$\begin{array}{r|rrrrrr} -3 & 4 & 12 & 0 & 0 & -1 & -3 \\ & & -12 & 0 & 0 & 0 & 3 \\ \hline & 4 & 0 & 0 & 0 & -1 & 0 \end{array}$$

Since the remainder is 0, $x+3$ is a factor. Thus,
$f(x)=(x+3)(4x^4-1)$
$=(x+3)(2x^2-1)(2x^2+1)$
$=(x+3)\left(\sqrt{2}x-1\right)\left(\sqrt{2}x+1\right)\left(2x^2+1\right)$
Note that the depressed equation $2x^2+1=0$ has no real solution. The real zeros are $-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}$, and -3.

57. $f(x)=x^3+3.2x^2-16.83x-5.31$

$f(x)$ has at most 3 real zeros.

Solving by graphing (using ZERO):

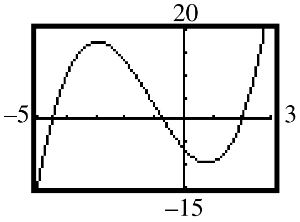


The zeros are approximately –5.9, –0.3, and 3.

58. $f(x)=x^3+3.2x^2-7.25x-6.3$

$f(x)$ has at most 3 real zeros.

Solving by graphing (using ZERO):

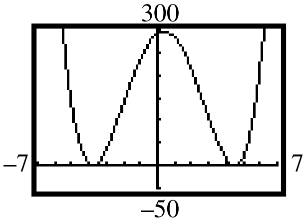


The zeros are approximately –4.5, –0.7, and 2.

59. $f(x)=x^4-1.4x^3-33.71x^2+23.94x+292.41$

$f(x)$ has at most 4 real zeros.

Solving by graphing (using ZERO):

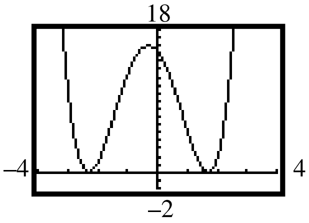


The zeros are approximately –3.8 and 4.5. These zeros are each of multiplicity 2.

60. $f(x)=x^4+1.2x^3-7.46x^2-4.692x+15.2881$

$f(x)$ has at most 4 real zeros.

Solving by graphing (using ZERO):

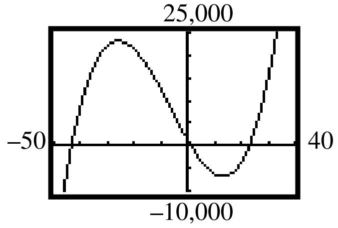


The zeros are approximately –2.3 and 1.7. These zeros are each of multiplicity 2.

61. $f(x)=x^3+19.5x^2-1021x+1000.5$

$f(x)$ has at most 3 real zeros.

Solving by graphing (using ZERO):

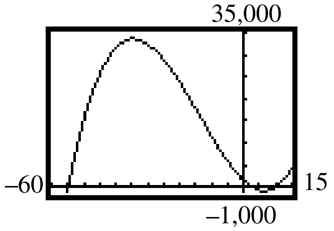


The zeros are approximately –43.5, 1, and 23.

62. $f(x)=x^3+42.2x^2-664.8x+1490.4$

$f(x)$ has at most 3 real zeros.

Solving by graphing (using ZERO):



The zeros are approximately –54.82, 2.76, and 9.87.

63. $x^4-x^3+2x^2-4x-8=0$

The solutions of the equation are the zeros of $f(x)=x^4-x^3+2x^2-4x-8$.

Step 1: $f(x)$ has at most 4 real zeros.

Step 2: Possible rational zeros:

$p=\pm 1,\pm 2,\pm 4,\pm 8;\quad q=\pm 1;$

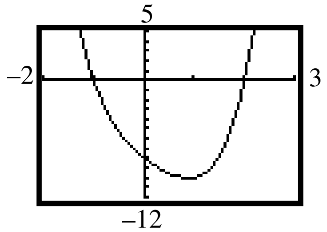
$$\frac{p}{q}=\pm 1,\pm 2,\pm 4,\pm 8$$

Step 3: Using the Bounds on Zeros Theorem:

$a_3=-1,\ a_2=2,\quad a_1=-4,\quad a_0=-8$

$$\text{Max }\{1,|-8|+|-4|+|2|+|-1|\}$$
$$=\text{Max }\{1,15\}=15$$
$$1+\text{Max }\{|-8|,|-4|,|2|,|-1|\}$$
$$=1+8=9$$

The smaller of the two numbers is 9. Thus, every zero of *f* lies between –9 and 9. We graph using the bounds and ZOOM-FIT, and adjust the viewing window to improve the graph.



Step 4: From the graph it appears that there are *x*-intercepts at –1 and 2.

Using synthetic division:

$$\begin{array}{r|rrrrr} -1 & 1 & -1 & 2 & -4 & -8 \\ & & -1 & 2 & -4 & 8 \\ \hline & 1 & -2 & 4 & -8 & 0 \end{array}$$

Since the remainder is 0, $x+1$ is a factor. Using synthetic division again:

$$\begin{array}{r|rrrr} 2 & 1 & -2 & 4 & -8 \\ & & 2 & 0 & 8 \\ \hline & 1 & 0 & 4 & 0 \end{array}$$

Since the remainder is 0, $x-2$ is also a factor. Thus,

$f(x)=(x+1)(x-2)(x^2+4)$.

The real solutions of the equation are -1 and 2. (Note that $x^2+4=0$ has no real solutions.)

64. $2x^3+3x^2+2x+3=0$

Solve by factoring:

$$x^2(2x+3)+(2x+3)=0$$
$$(2x+3)\left(x^2+1\right)=0$$
$$x=-\frac{3}{2}$$

The only real zero is $-\frac{3}{2}$. (Note that $x^2+1=0$ has no real solutions.)

65. $3x^3+4x^2-7x+2=0$

The solutions of the equation are the zeros of $f(x)=3x^3+4x^2-7x+2$.

Step 1: $f(x)$ has at most 3 real zeros.

Step 2: Possible rational zeros:

$p=\pm 1,\pm 2;\quad q=\pm 1,\pm 3;$

$$\frac{p}{q}=\pm 1,\pm 2,\pm\frac{1}{3},\pm\frac{2}{3}$$

Step 3: Using the Bounds on Zeros Theorem:

$$f(x)=3\left(x^3+\frac{4}{3}x^2-\frac{7}{3}x+\frac{2}{3}\right)$$

$$a_2=\frac{4}{3},\quad a_1=-\frac{7}{3},\quad a_0=\frac{2}{3}$$

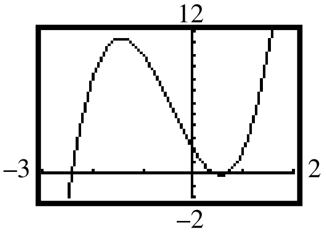
$$\text{Max }\left\{1,\left|\frac{2}{3}\right|+\left|-\frac{7}{3}\right|+\left|\frac{4}{3}\right|\right\}$$

$$=\text{Max }\left\{1,\frac{13}{3}\right\}=\frac{13}{3}\approx 4.333$$

$$1+\text{Max }\left\{\left|\frac{2}{3}\right|,\left|-\frac{7}{3}\right|,\left|\frac{4}{3}\right|\right\}$$

$$=1+\frac{7}{3}=\frac{10}{3}\approx 3.333$$

The smaller of the two numbers is 3.33. Thus, every zero of *f* lies between -3.33 and 3.33. We graph using the bounds and ZOOM-FIT, and adjust the viewing window to improve the graph.



Step 4: From the graph it appears that there are x-intercepts at $\frac{1}{3},\frac{2}{3},$ and -2.4.

Using synthetic division:

$$\begin{array}{r|rrrr} \frac{2}{3} & 3 & 4 & -7 & 2 \\ & & 2 & 4 & -2 \\ \hline & 3 & 6 & -3 & 0 \end{array}$$

Since the remainder is 0, $x-\frac{2}{3}$ is a factor. The other factor is the quotient: $3x^2+6x-3$.

$$f(x)=\left(x-\frac{2}{3}\right)\left(3x^2+6x-3\right)$$

$$=3\left(x-\frac{2}{3}\right)\left(x^2+2x-1\right)$$

Using the quadratic formula to solve $x^2+2x-1=0$:

$$x=\frac{-2\pm\sqrt{4-4(1)(-1)}}{2(1)}=\frac{-2\pm\sqrt{8}}{2}$$

$$=\frac{-2\pm 2\sqrt{2}}{2}=-1\pm\sqrt{2}$$

The real solutions of the equation are $\frac{2}{3}, -1+\sqrt{2},$ and $-1-\sqrt{2}$.

66. $2x^3-3x^2-3x-5=0$

The solutions of the equation are the zeros of $f(x)=2x^3-3x^2-3x-5$.

Step 1: $f(x)$ has at most 3 real zeros.

Step 2: Possible rational zeros:

$$p=\pm 1,\pm 5;\quad q=\pm 1,\pm 2;$$

$$\frac{p}{q}=\pm 1,\pm 5,\pm\frac{1}{2},\pm\frac{5}{2}$$

Step 3: Using the Bounds on Zeros Theorem:

$$f(x)=2\left(x^3-1.5x^2-1.5x-2.5\right)$$

$$a_2=-1.5,\quad a_1=-1.5,\quad a_0=-2.5$$

$$\text{Max }\{1,|-2.5|+|-1.5|+|-1.5|\}$$

$$=\text{Max }\{1, 5.5\}$$

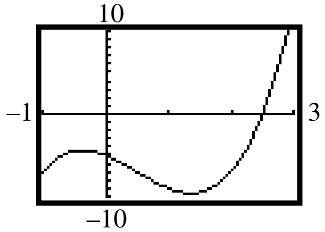
$$=5.5$$

$$1+\text{Max }\{|-2.5|,|-1.5|,|-1.5|\}$$

$$=1+2.5$$

$$=3.5$$

The smaller of the two numbers is 3.5. Thus, every zero of *f* lies between -3.5 and 3.5. We graph using the bounds and ZOOM-FIT, and adjust the viewing window to improve the graph.



Step 4: From the graph it appears that there is an x-intercept at $\frac{5}{2}$.

Using synthetic division:

$$\begin{array}{r|rrrr} \frac{5}{2} & 2 & -3 & -3 & -5 \\ & & 5 & 5 & 5 \\ \hline & 2 & 2 & 2 & 0 \end{array}$$

Since the remainder is 0, $x-\frac{5}{2}$ is a factor. Thus,

$$\begin{aligned} f(x) &= \left(x-\frac{5}{2}\right)\left(2x^2+2x+2\right) \\ &= 2\left(x-\frac{5}{2}\right)\left(x^2+x+1\right) \\ &= (2x-5)\left(x^2+x+1\right) \end{aligned}$$

Note that $x^2+x+1=0$ has no real zeros. The only real solutions to the equation is $\frac{5}{2}$.

67. $3x^3-x^2-15x+5=0$

Solving by factoring:

$$\begin{aligned} x^2(3x-1)-5(3x-1) &= 0 \\ (3x-1)(x^2-5) &= 0 \\ (3x-1)\left(x-\sqrt{5}\right)\left(x+\sqrt{5}\right) &= 0 \end{aligned}$$

The solutions of the equation are $\frac{1}{3}, \sqrt{5},$ and $-\sqrt{5}$.

68. $2x^3-11x^2+10x+8=0$

The solutions of the equation are the zeros of $f(x)=2x^3-11x^2+10x+8$.

Step 1: $f(x)$ has at most 3 real zeros.

Step 2: Possible rational zeros:

$p=\pm1, \pm2, \pm4, \pm8; \quad q=\pm1, \pm2;$

$\frac{p}{q}=\pm1, \pm2, \pm4, \pm8, \pm\frac{1}{2}$

Step 3: Using the Bounds on Zeros Theorem:

$f(x)=2\left(x^3-5.5x^2+5x+4\right)$

$a_2=-5.5, \quad a_1=5, \quad a_0=4$

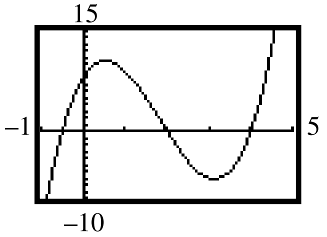
$\text{Max}\ \{1, |4|+|3|+|-5.5|\}$

$=\text{Max}\ \{1, 12.5\}=12.5$

$1+\text{Max}\ \{|4|, |3|, |-5.5|\}$

$=1+5.5=6.5$

The smaller of the two numbers is 6.5. Thus, every zero of *f* lies between –6.5 and 6.5. We graph using the bounds and ZOOM-FIT, and adjust the viewing window to improve the graph.



Step 4: From the graph it appears that there are *x*-intercepts at –0.5, 2, and 4.

Using synthetic division:

$$\begin{array}{r|rrrr} 4 & 2 & -11 & 10 & 8 \\ & & 8 & -12 & -8 \\ \hline & 2 & -3 & -2 & 0 \end{array}$$

Since the remainder is 0, $x-4$ is a factor. Thus,

$$\begin{aligned} f(x) &= 2x^3-11x^2+10x+8 \\ &= (x-4)(2x^2-3x-2) \\ &= (x-4)(2x+1)(x-2) \end{aligned}$$

The solutions of the equation are $-\frac{1}{2}$, 2, and 4.

69. $x^4+4x^3+2x^2-x+6=0$

The solutions of the equation are the zeros of $f(x)=x^4+4x^3+2x^2-x+6$.

Step 1: $f(x)$ has at most 4 real zeros.

Step 2: Possible rational zeros:

$p=\pm1, \pm2, \pm3, \pm6; \quad q=\pm1;$

$\frac{p}{q}=\pm1, \pm2, \pm3, \pm6$

Step 3: Using the Bounds on Zeros Theorem:

$a_3=4, \ a_2=2, \quad a_1=-1, \quad a_0=6$

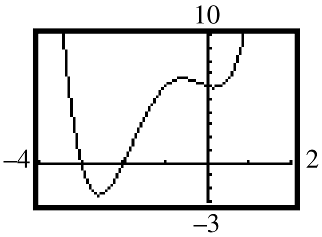
$\text{Max}\ \{1, |6|+|-1|+|2|+|4|\}$

$=\text{Max}\ \{1, 13\}=13$

$1+\text{Max}\ \{|6|, |-1|, |2|, |4|\}$

$=1+6=7$

The smaller of the two numbers is 7. Thus, every zero of *f* lies between –7 and 7. We graph using the bounds and ZOOM-FIT, and adjust the viewing window to improve the graph.



Step 4: From the graph it appears that there are x-intercepts at –3 and –2.
Using synthetic division:

$$\begin{array}{r|rrrrr} -3 & 1 & 4 & 2 & -1 & 6 \\ & & -3 & -3 & 3 & -6 \\ \hline & 1 & 1 & -1 & 2 & 0 \end{array}$$

Since the remainder is 0, $x+3$ is a factor. Using synthetic division again:

$$\begin{array}{r|rrrr} -2 & 1 & 1 & -1 & 2 \\ & & -2 & 2 & -2 \\ \hline & 1 & -1 & 1 & 0 \end{array}$$

Since the remainder is 0, $x+2$ is also a factor. Thus,

$$f(x) = x^4 + 4x^3 + 2x^2 - x + 6$$
$$= (x+2)(x+3)(x^2 - x + 1)$$

The real solutions of the equation are –3 and –2. (Note that $x^2 - x + 1 = 0$ has no real solutions.)

70. $x^4 - 2x^3 + 10x^2 - 18x + 9 = 0$

The solutions of the equation are the zeros of $f(x) = x^4 - 2x^3 + 10x^2 - 18x + 9$.

Step 1: $f(x)$ has at most 4 real zeros.

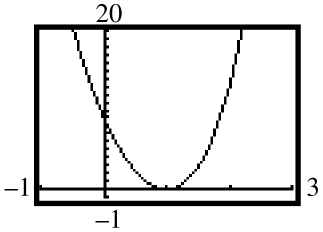
Step 2: Possible rational zeros:

$$p = \pm 1, \pm 3, \pm 9;\ q = \pm 1;\ \frac{p}{q} = \pm 1, \pm 3, \pm 9$$

Step 3: Using the Bounds on Zeros Theorem:
$a_3 = -2,\ a_2 = 10,\ \ a_1 = -18,\ a_0 = 9$

$$\text{Max } \{1, |9| + |-18| + |10| + |-2|\}$$
$$= \text{Max } \{1, 39\} = 39$$
$$1 + \text{Max } \{|9|, |-18|, |10|, |-2|\}$$
$$= 1 + 18 = 19$$

The smaller of the two numbers is 19. Thus, every zero of *f* lies between –19 and 19. We graph using the bounds and ZOOM-FIT, and adjust the viewing window to improve the graph.



Step 4: From the graph it appears that there is an x-intercept at 1. Using synthetic division:

$$\begin{array}{r|rrrrr} 1 & 1 & -2 & 10 & -18 & 9 \\ & & 1 & -1 & 9 & -9 \\ \hline & 1 & -1 & 9 & -9 & 0 \end{array}$$

Since the remainder is 0, $x-1$ is a factor. Using synthetic division again:

$$\begin{array}{r|rrrr} 1 & 1 & -1 & 9 & -9 \\ & & 1 & 0 & 9 \\ \hline & 1 & 0 & 9 & 0 \end{array}$$

Since the remainder is 0, $x-1$ is a factor again. Thus,

$$f(x) = x^4 - 2x^3 + 10x^2 - 18x + 9$$
$$= (x-1)^2(x^2 + 9)$$

The real solution to the equation is 1 (multiplicity 2). (Note that $x^2 + 9 = 0$ has no real solutions.)

71. $x^3 - \frac{2}{3}x^2 + \frac{8}{3}x + 1 = 0$

The solutions of the equation are the zeros of

$$f(x) = x^3 - \frac{2}{3}x^2 + \frac{8}{3}x + 1.$$

Step 1: $f(x)$ has at most 3 real zeros.

Step 2: Use the equivalent equation $3x^3 - 2x^2 + 8x + 3 = 0$ to find the possible rational zeros:
$p = \pm 1, \pm 3;\quad q = \pm 1, \pm 3;$

$$\frac{p}{q} = \pm 1, \pm 3, \pm \frac{1}{3}$$

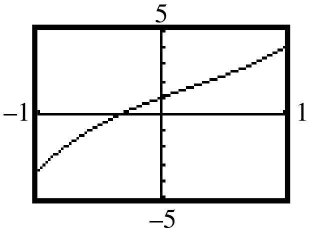
Step 3: Using the Bounds on Zeros Theorem:

$$a_2 = -\frac{2}{3},\quad a_1 = \frac{8}{3},\quad a_0 = 1$$
$$\text{Max } \left\{1, |1| + \left|\frac{8}{3}\right| + \left|-\frac{2}{3}\right|\right\}$$
$$= \text{Max } \left\{1, \frac{13}{3}\right\} = \frac{13}{3}$$

$$1+\text{Max}\left\{|1|,\left|\frac{8}{3}\right|,\left|-\frac{2}{3}\right|\right\}=1+\frac{8}{3}=\frac{11}{3}$$

The smaller of the two numbers is $\frac{11}{3}$.

Thus, every zero of *f* lies between $-\frac{11}{3}$ and $\frac{11}{3}$. We graph using the bounds and ZOOM-FIT, and adjust the viewing window to improve the graph.



Step 4: From the graph it appears that there is an *x*-intercept at $-\frac{1}{3}$.

Using synthetic division:

$$\begin{array}{r|rrrr} -\frac{1}{3} & 1 & -\frac{2}{3} & \frac{8}{2} & 1 \\ & & -\frac{1}{3} & \frac{1}{3} & -1 \\ \hline & 1 & -1 & 3 & 0 \end{array}$$

Since the remainder is 0, $x+\frac{1}{3}$ is a factor. Thus,

$$f(x)=x^3-\frac{2}{3}x^2+\frac{8}{3}x+1$$
$$=\left(x+\frac{1}{3}\right)(x^2-x+3)$$

The only real solution to the equation is $-\frac{1}{3}$. (Note that $x^2-x+3=0$ has no real solutions.)

72. $x^3-\frac{2}{3}x^2+3x-2=0$

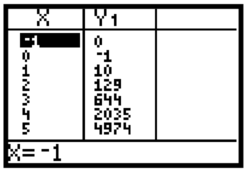
Solving by factoring:

$$x^2\left(x-\frac{2}{3}\right)+3\left(x-\frac{2}{3}\right)=0$$
$$\left(x-\frac{2}{3}\right)(x^2+3)=0$$
$$x=\frac{2}{3}$$

The only real zero is $\frac{2}{3}$. (Note that $x^2+3=0$ has no real solutions.)

73. Using the TABLE feature to show that there is a zero in the interval:

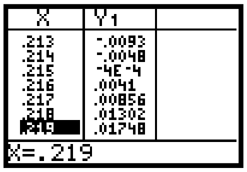
$f(x)=8x^4-2x^2+5x-1;\quad [0,1]$



```
   X    | Y1   |
 -1     | 0    |
 0      | -1   |
 1      | 10   |
 2      | 129  |
 3      | 644  |
 4      | 2035 |
 5      | 4974 |
X=-1
```

$f(0)=-1<0$ and $f(1)=10>0$

Since one is positive and one is negative, there is a zero in the interval. Using the TABLE feature to approximate the zero to two decimal places:

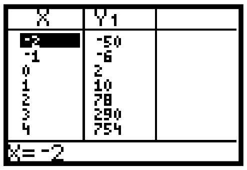


```
   X    | Y1     |
 .213   | -.0093 |
 .214   | -.0048 |
 .215   | -4E-4  |
 .216   | .0041  |
 .217   | .00856 |
 .218   | .01302 |
 .219   | .01748 |
X=.219
```

The zero is approximately 0.22.

74. Use the TABLE feature to show that there is a zero in the interval:

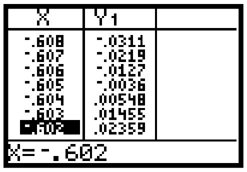
$f(x)=x^4+8x^3-x^2+2;\quad [-1,0]$



```
   X    | Y1   |
 -2     | -50  |
 -1     | -6   |
 0      | 2    |
 1      | 10   |
 2      | 78   |
 3      | 290  |
 4      | 754  |
X=-2
```

$f(-1)=-6<0$ and $f(0)=2>0$

Since one is positive and one is negative, there is a zero in the interval. Using the TABLE feature to approximate the zero to two decimal places:

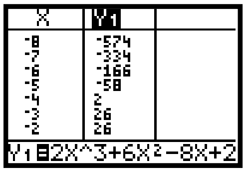


```
   X    | Y1     |
 -.608  | -.0311 |
 -.607  | -.0219 |
 -.606  | -.0127 |
 -.605  | -.0036 |
 -.604  | .00548 |
 -.603  | .01455 |
 -.602  | .02359 |
X=-.602
```

The zero is approximately –0.61.

75. Using the TABLE feature to show that there is a zero in the interval:

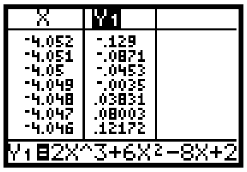
$f(x)=2x^3+6x^2-8x+2;\quad [-5,-4]$



```
   X    | Y1   |
 -8     | -574 |
 -7     | -334 |
 -6     | -166 |
 -5     | -58  |
 -4     | 2    |
 -3     | 26   |
 -2     | 26   |
Y1=2X^3+6X²-8X+2
```

$f(-5) = -58 < 0$ and $f(-4) = 2 > 0$

Since one is positive and one is negative, there is a zero in the interval. Using the TABLE feature to approximate the zero to two decimal places:

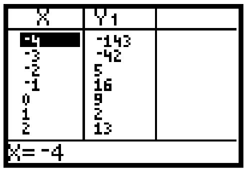


```
   X      Y1
-4.052  -.129
-4.051  -.0871
-4.05   -.0453
-4.049  -.0035
-4.048  .03831
-4.047  .08003
-4.046  .12172
Y1=2X^3+6X²-8X+2
```

The zero is approximately –4.05.

76. Using the TABLE feature to show that there is a zero in the interval:

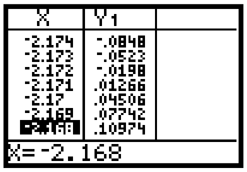
$f(x) = 3x^3 - 10x + 9; \quad [-3, -2]$



```
   X      Y1
 -4      -143
 -3      -42
 -2      5
 -1      16
 0       9
 1       2
 2       13
X=-4
```

$f(-3) = -42 < 0$ and $f(-2) = 5 > 0$

Since one is positive and one is negative, there is a zero in the interval. Using the TABLE feature to approximate the zero to two decimal places:

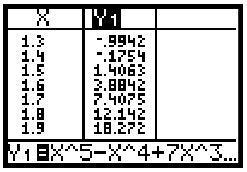


```
   X      Y1
-2.174  -.0848
-2.173  -.0523
-2.172  -.0198
-2.171  .01266
-2.17   .04506
-2.169  .07742
-2.168  .10974
X=-2.168
```

The zero is approximately –2.17.

77. Using the TABLE feature to show that there is a zero in the interval:

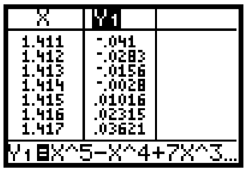
$f(x) = x^5 - x^4 + 7x^3 - 7x^2 - 18x + 18; \quad [1.4, 1.5]$



```
   X      Y1
 1.3     -.9942
 1.4     -.1754
 1.5     1.4063
 1.6     3.8842
 1.7     7.4075
 1.8     12.142
 1.9     18.272
Y1=X^5-X^4+7X^3…
```

$f(1.4) = -0.1754 < 0$ and $f(1.5) = 1.4063 > 0$

Since one is positive and one is negative, there is a zero in the interval. Using the TABLE feature to approximate the zero to two decimal places:

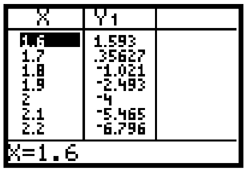


```
   X      Y1
 1.411   -.041
 1.412   -.0283
 1.413   -.0156
 1.414   -.0028
 1.415   .01016
 1.416   .02315
 1.417   .03621
Y1=X^5-X^4+7X^3…
```

The zero is approximately 1.41.

78. Using the TABLE feature to show that there is a zero in the interval:

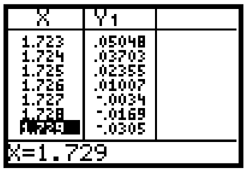
$f(x) = x^5 - 3x^4 - 2x^3 + 6x^2 + x + 2; \quad [1.7, 1.8]$



```
   X      Y1
 1.6     1.593
 1.7     .35627
 1.8     -1.021
 1.9     -2.493
 2       -4
 2.1     -5.465
 2.2     -6.796
X=1.6
```

$f(1.7) = 0.35627 > 0$ and $f(1.8) = -1.021 < 0$

Since one is positive and one is negative, there is a zero in the interval. Using the TABLE feature to approximate the zero to two decimal places:



```
   X      Y1
 1.723   .05048
 1.724   .03703
 1.725   .02355
 1.726   .01007
 1.727   -.0034
 1.728   -.0169
 1.729   -.0305
X=1.729
```

The zero is approximately 1.73.

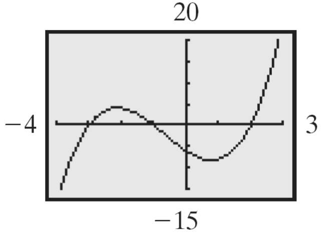
79. $f(x) = x^3 + 2x^2 - 5x - 6$

Step 1: Degree = 3; The graph of the function resembles $y = x^3$ for large values of $|x|$.

Step 2: y-intercept:
$f(0) = (0)^3 + 2(0)^2 - 5(0) - 6 = -6$
x-intercepts: Solve $f(x) = 0$.
From Problem 39, we found
$x = -3, -1$, and 2

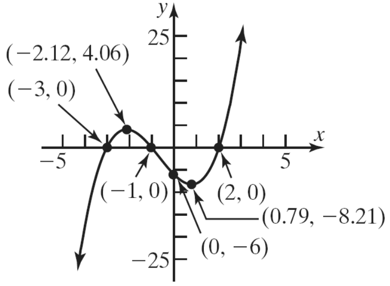
Step 3: Real zeros: -3 with multiplicity one, -1 with multiplicity one, 2 with multiplicity one. The graph crosses the x-axis at $x = -3, -1, 2$.

Step 4: Graphing utility:



Step 5: 2 turning points;
local maximum: (–2.12, 4.06);
local minimum: (0.79, –8.21)

Step 6: Graphing by hand



Step 7: Domain: $(-\infty,\infty)$; Range: $(-\infty,\infty)$

Step 8: Increasing on $(-\infty,-2.12)$ and $(0.79,\infty)$; decreasing on $(-2.12, 0.79)$

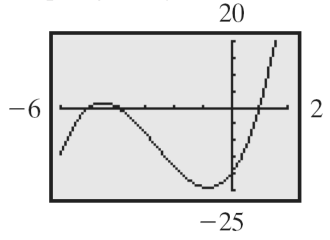
80. $f(x)=x^3+8x^2+11x-20$

Step 1: Degree = 3; The graph of the function resembles $y=x^3$ for large values of $|x|$.

Step 2: y-intercept:
$f(0)=(0)^3+8(0)^2+11(0)-20=-20$
x-intercepts: Solve $f(x)=0$.
From Problem 40, we found
$x=-5, -4$, and 1

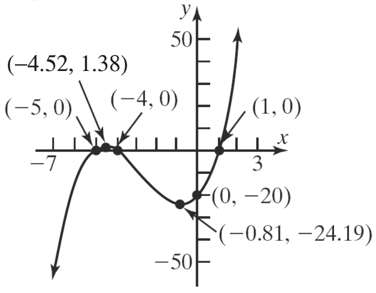
Step 3: Real zeros: -5 with multiplicity one, -4 with multiplicity one, 1 with multiplicity one. The graph crosses the x-axis at $x=-5,-4,1$.

Step 4: Graphing utility:



Step 5: 2 turning points;
local maximum: (–4.52, 1.38);
local minimum: (–0.81, –24.19)

Step 6: Graphing by hand



Step 7: Domain: $(-\infty,\infty)$; Range: $(-\infty,\infty)$

Step 8: Increasing on $(-\infty,-4.52)$ and $(-0.81,\infty)$; decreasing on $(-4.52,-0.81)$

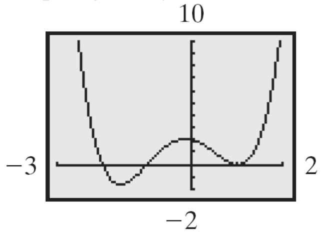
81. $f(x)=x^4+x^3-3x^2-x+2$

Step 1: Degree = 4; The graph of the function resembles $y=x^4$ for large values of $|x|$.

Step 2: y-intercept:
$f(0)=(0)^4+(0)^3-3(0)^2-0+2=2$
x-intercepts: Solve $f(x)=0$.
From Problem 47, we found
$x=-2, -1$, and 1

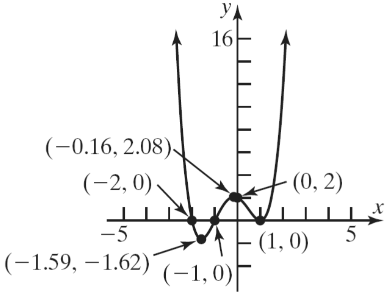
Step 3: Real zeros: -2 with multiplicity one, -1 with multiplicity one, 1 with multiplicity two. The graph crosses the x-axis at $x=-2$ and $x=-1$, and touches it at $x=1$.

Step 4: Graphing utility:



Step 5: 2 turning points;
local maximum: (–0.16, 2.08);
local minima: (–1.59, –1.62) and (1, 0)

Step 6: Graphing by hand



Step 7: Domain: $(-\infty,\infty)$; Range: $[-1.62,\infty)$

Step 8: Increasing on $(-1.59,-0.16)$ and $(1,\infty)$; decreasing on $(-\infty,-1.59)$ and $(-0.16,1)$

82. $f(x)=x^4-x^3-6x^2+4x+8$

Step 1: Degree = 4; The graph of the function resembles $y=x^4$ for large values of $|x|$.

Step 2: y-intercept:

$f(0)=(0)^4-(0)^3-6(0)^2+4(0)+8=8$

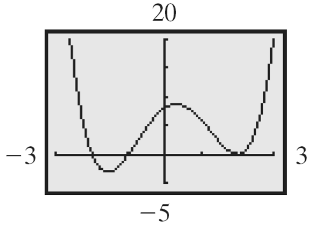
x-intercepts: Solve $f(x)=0$.

From Problem 48, we found

$x=-2, -1,$ and 2

Step 3: Real zeros: -2 with multiplicity one, -1 with multiplicity one, 2 with multiplicity two. The graph crosses the x-axis at $x=-2$ and $x=-1$, and touches it at $x=2$.

Step 4: Graphing utility:

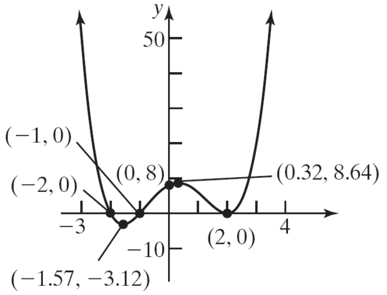


Step 5: 2 turning points;

local maximum: (0.32, 8.64);

local minima: (–1.57, –3.12) and (2, 0)

Step 6: Graphing by hand



Step 7: Domain: $(-\infty,\infty)$; Range: $[-3.12,\infty)$

Step 8: Increasing on $(-1.57, 0.32)$ and $(2,\infty)$; decreasing on $(-\infty,-1.57)$ and $(0.32, 2)$

83. $f(x)=4x^5-8x^4-x+2$

Step 1: Degree = 5; The graph of the function resembles $y=4x^5$ for large values of $|x|$.

Step 2: y-intercept:

$f(0)=4(0)^5-8(0)^4-0+2=2$

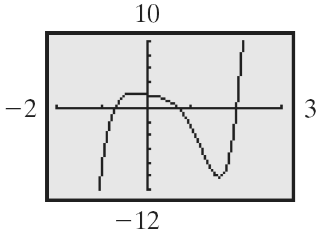
x-intercepts: Solve $f(x)=0$.

From Problem 55, we found

$x=-\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2},$ and 2.

Step 3: Real zeros: $-\dfrac{\sqrt{2}}{2}$ with multiplicity one, $\dfrac{\sqrt{2}}{2}$ with multiplicity one, 2 with multiplicity one. The graph crosses the x-axis at $x=-\dfrac{\sqrt{2}}{2}$, $x=\dfrac{\sqrt{2}}{2}$, and $x=2$.

Step 4: Graphing utility:

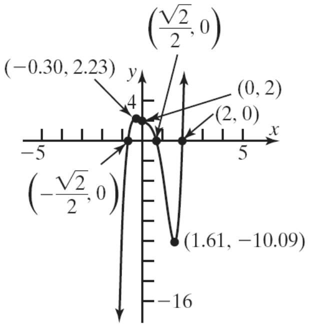


Step 5: 2 turning points;

local maximum: (–0.30, 2.23);

local minimum: (1.61, –10.09)

Step 6: Graphing by hand



Step 7: Domain: $(-\infty,\infty)$; Range: $(-\infty,\infty)$

Step 8: Increasing on $(-\infty,-0.30)$ and $(1.61,\infty)$; decreasing on $(-0.30, 1.61)$

84. $f(x)=4x^5+12x^4-x-3$

Step 1: Degree = 5; The graph of the function resembles $y=4x^5$ for large values of $|x|$.

Step 2: y-intercept:

$f(0)=4(0)^5+12(0)^4-0-3=-3$

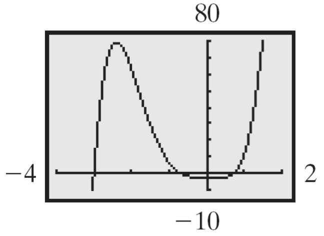
x-intercepts: Solve $f(x)=0$.

From Problem 56, we found

$x=-3,-\dfrac{\sqrt{2}}{2},$ and $\dfrac{\sqrt{2}}{2}$.

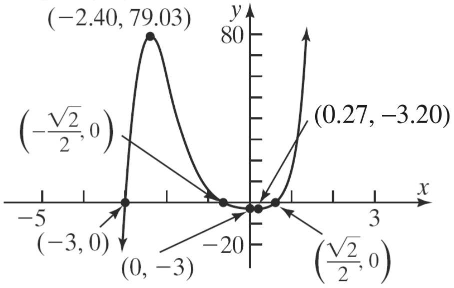
Step 3: Real zeros: -3 with multiplicity one, $-\dfrac{\sqrt{2}}{2}$ with multiplicity one, $\dfrac{\sqrt{2}}{2}$ with multiplicity one. The graph crosses the x-axis at $x=-3$, $x=-\frac{\sqrt{2}}{2}$, and $x=\frac{\sqrt{2}}{2}$.

Step 4: Graphing utility:



Step 5: 2 turning points;
local maximum: (–2.40, 79.03);
local minimum: (0.27, –3.20)

Step 6: Graphing by hand



Step 7: Domain: $(-\infty,\infty)$; Range: $(-\infty,\infty)$

Step 8: Increasing on $(-\infty,-2.40)$ and $(0.27,\infty)$; decreasing on $(-2.40, 0.27)$

85. $f(x)=6x^4-37x^3+58x^2+3x-18$

Step 1: Degree = 4; The graph of the function resembles $y=6x^4$ for large values of $|x|$.

Step 2: y-intercept:

$$f(0)=6(0)^4-37(0)^3+58(0)^2+3(0)-18$$
$$=-18$$

x-intercepts: Solve $f(x)=0$.

The possible rational zeroes are:

$\pm 1, \pm 2, \pm 3, \pm 6, \pm 9, \pm 18,$

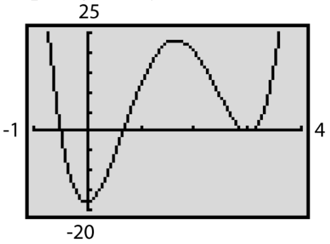
$\pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{3}{2}, \pm\frac{9}{2}, \pm\frac{2}{3}$

We find the factoring to be:

$f(x)=(x-3)^2(2x+1)(3x-2)$ so the x intercepts are: $3, -\frac{1}{2}, \frac{2}{3}$

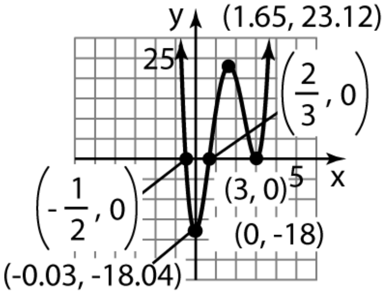
Step 3: Real zeros: $-\frac{1}{2}$ and $\frac{2}{3}$ with multiplicity one, 3 with multiplicity 2. The graph crosses the x-axis at $-\frac{1}{2}$ and $\frac{2}{3}$ and touches at $x=3$.

Step 4: Graphing utility:



Step 5: 3 turning points;
local minimum: (–0.03, -18.04),(3,0);
local maximum: (1.65, 23.12)

Step 6: Graphing by hand



Step 7: Domain: $(-\infty,\infty)$; Range: $(-18.04,\infty)$

Step 8: Increasing on $(-0.03, 1.65)$ and $(3,\infty)$; decreasing on $(-\infty,-0.03),(1.65,3)$

86. $f(x)=20x^4+73x^3+46x^2-52x-24$

Step 1: Degree = 5; The graph of the function resembles $y=20x^4$ for large values of $|x|$.

Step 2: y-intercept:

$$f(0)=20(0)^4+73(0)^3+46(0)^2-52(0)-24$$
$$=-24$$

x-intercepts: Solve $f(x)=0$.

The possible rational zeroes are:

$\pm1, \pm2, \pm3, \pm4, \pm6, \pm8, \pm12, \pm24, \pm\frac{1}{2},$

$\pm\frac{3}{2}, \pm\frac{1}{4}, \pm\frac{3}{4}, \pm\frac{1}{5}, \pm\frac{2}{5}, \pm\frac{3}{5}, \pm\frac{4}{5}, \pm\frac{6}{5},$

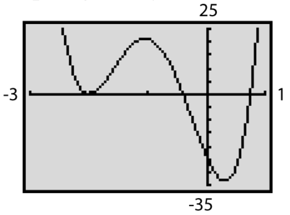
$\pm\frac{8}{5}, \pm\frac{12}{5}, \pm\frac{24}{5}, \pm\frac{1}{10}, \pm\frac{3}{10}, \pm\frac{1}{20}, \pm\frac{3}{20}$

We find the factoring to be:

$f(x) = (x+2)^2(5x+2)(4x-3)$ so the x intercepts are: $-2, -\frac{2}{5}, \frac{3}{4}$

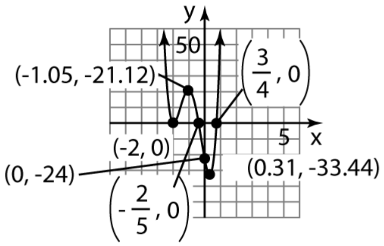
Step 3: Real zeros: $-\frac{2}{5}$ and $\frac{3}{4}$ with multiplicity one, -2 with multiplicity 2. The graph crosses the *x*-axis at $-\frac{2}{5}$ and $\frac{3}{4}$ and touches at $x = -2$.

Step 4: Graphing utility:



Step 5: 3 turning points;
local minimum: (0.31, -33.34),(-2,0);
local maximum: (-1.05, 21.12)

Step 6: Graphing by hand



Step 7: Domain: $(-\infty, \infty)$; Range: $(-33.34, \infty)$

Step 8: Increasing on $(-2, -1.05)$ and $(0.31, \infty)$; decreasing on $(-\infty, -2), (\ 1.05, 0.31)$

87. From the Remainder and Factor Theorems, $x-2$ is a factor of *f* if $f(2) = 0$.

$$\begin{aligned}(2)^3 - k(2)^2 + k(2) + 2 &= 0\\ 8 - 4k + 2k + 2 &= 0\\ -2k + 10 &= 0\\ -2k &= -10\\ k &= 5\end{aligned}$$

88. From the Remainder and Factor Theorems, $x+2$ is a factor of *f* if $f(-2) = 0$.

$$\begin{aligned}(-2)^4 - k(-2)^3 + k(-2)^2 + 1 &= 0\\ 16 + 8k + 4k + 1 &= 0\\ 12k + 17 &= 0\\ 12k &= -17\\ k &= -\frac{17}{12}\end{aligned}$$

89. From the Remainder Theorem, we know that the remainder is

$f(1) = 2(1)^{20} - 8(1)^{10} + 1 - 2 = 2 - 8 + 1 - 2 = -7$

The remainder is –7.

90. From the Remainder Theorem, we know that the remainder is

$f(-1) = -3(-1)^{17} + (-1)^9 - (-1)^5 + 2(-1) = 1$

The remainder is 1.

91. We want to prove that $x-c$ is a factor of $x^n - c^n$, for any positive integer *n*. By the Factor Theorem, $x-c$ will be a factor of $f(x)$ provided $f(c) = 0$. Here, $f(x) = x^n - c^n$, so that $f(c) = c^n - c^n = 0$. Therefore, $x-c$ is a factor of $x^n - c^n$.

92. We want to prove that $x+c$ is a factor of $x^n + c^n$, if $n \ge 1$ is an odd integer. By the Factor Theorem, $x+c$ will be a factor of $f(x)$ provided $f(-c) = 0$. Here, $f(x) = x^n + c^n$, so that $f(-c) = (-c)^n + c^n = -c^n + c^n = 0$ if $n \ge 1$ is an odd integer. Therefore, $x+c$ is a factor of $x^n + c^n$ if $n \ge 1$ is an odd integer.

93. $x^3 - 8x^2 + 16x - 3 = 0$ has solution $x = 3$, so $x-3$ is a factor of $f(x) = x^3 - 8x^2 + 16x - 3$. Using synthetic division

$$\begin{array}{r|rrrr} 3 & 1 & -8 & 16 & -3 \\ & & 3 & -15 & 3 \\ \hline & 1 & -5 & 1 & 0 \end{array}$$

Thus,

$f(x) = x^3 - 8x^2 + 16x - 3 = (x-3)\left(x^2 - 5x + 1\right)$.

Solving $x^2 - 5x + 1 = 0$

$$x = \frac{5 \pm \sqrt{25-4}}{2} = \frac{5 \pm \sqrt{21}}{2}$$

The sum of these two roots is

$$\frac{5+\sqrt{21}}{2} + \frac{5-\sqrt{21}}{2} = \frac{10}{2} = 5.$$

94. $x^3 + 5x^2 + 5x - 2 = 0$ has solution $x = -2$, so $x + 2$ is a factor of $f(x) = x^3 + 5x^2 + 5x - 2$.

Using synthetic division

$$\begin{array}{r|rrrr} -2 & 1 & 5 & 5 & -2 \\ & & -2 & -6 & 2 \\ \hline & 1 & 3 & -1 & 0 \end{array}$$

Thus,

$f(x) = x^3 + 5x^2 + 5x - 2 = (x+2)\left(x^2 + 3x - 1\right)$.

Solving $x^2 + 3x - 1 = 0$,

$$x = \frac{-3 \pm \sqrt{9+4}}{2} = \frac{-3 \pm \sqrt{13}}{2}.$$

The sum of these two roots is

$$\frac{-3+\sqrt{13}}{2} + \frac{-3-\sqrt{13}}{2} = \frac{-6}{2} = -3.$$

95. Let *x* be the length of a side of the original cube. After removing the 1-inch slice, one dimension will be $x - 1$. The volume of the new solid will be: $(x-1) \cdot x \cdot x$.

Solve the volume equation:

$$(x-1) \cdot x \cdot x = 294$$
$$x^3 - x^2 = 294$$
$$x^3 - x^2 - 294 = 0$$

The solutions to this equation are the same as the real zeros of $f(x) = x^3 - x^2 - 294$.

By Descartes' Rule of Signs, we know that there is one positive real zero.

$p = \pm 1, \pm 2, \pm 3, \pm 6, \pm 7, \pm 14, \pm 21, \pm 42, \pm 49, \pm 98,$
$\pm 147, \pm 294$

$q = \pm 1$

The possible rational zeros are the same as the values for *p*.

$\frac{p}{q} = \pm 1, \pm 2, \pm 3, \pm 6, \pm 7, \pm 14, \pm 21, \pm 42, \pm 49, \pm 98,$
$\pm 147, \pm 294$

Using synthetic division:

$$\begin{array}{r|rrrr} 7 & 1 & -1 & 0 & -294 \\ & & 7 & 42 & 294 \\ \hline & 1 & 6 & 42 & 0 \end{array}$$

7 is a zero, so the length of the edge of the original cube was 7 inches.

96. Let *x* be the length of a side of the original cube. The volume is x^3. The dimensions are changed to $x+6$, $x+12$, and $x-4$. The volume of the new solid will be $(x+6)(x+12)(x-4)$.

Solve the volume equation:

$$(x+6)(x+12)(x-4) = 2x^3$$
$$\left(x^2 + 18x + 72\right)(x-4) = 2x^3$$
$$x^3 + 14x^2 - 288 = 2x^3$$
$$x^3 - 14x^2 + 288 = 0$$

The solutions to this equation are the same as the real zeros of $f(x) = x^3 - 14x^2 + 288$.

By Descartes' Rule of Signs, we know that there are two positive real zeros or no positive real zeros.

$p = \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 9, \pm 12, \pm 16, \pm 18, \pm 24,$
$\pm 32, \pm 36, \pm 48, \pm 72, \pm 96, \pm 144, \pm 288$

$q = \pm 1$

The possible rational zeros are the same as the values for *p*:

$\frac{p}{q} = \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 9, \pm 12, \pm 16, \pm 18, \pm 24,$
$\pm 32, \pm 36, \pm 48, \pm 72, \pm 96, \pm 144, \pm 288$

Using synthetic division:

$$\begin{array}{r|rrrr} 6 & 1 & -14 & 0 & 288 \\ & & 6 & -48 & -288 \\ \hline & 1 & -8 & -48 & 0 \end{array}$$

Therefore, 6 is a zero; the other factor is $x^2 - 8x - 48 = (x-12)(x+4)$. The other zeros are 12 and –4. The length of the edge of the original cube was 6 cm or 12 cm.

97. $f(x) = x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ... + a_1 x + a_0$; where $a_{n-1}, a_{n-2}, ... a_1, a_0$ are integers. If *r* is a real zero of f, then *r* is either rational or irrational. We know that the rational roots of *f* must be of the form $\frac{p}{q}$ where p is a divisor of a_0 and q is a divisor of 1. This means that $q = \pm 1$. So if r is rational, then $r = \frac{p}{q} = \pm p$. Therefore, r is an integer or r is irrational.

---

98. Let $\frac{p}{q}$ be a root of the polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ... + a_1 x + a_0$ where $a_n, a_{n-1}, a_{n-2}, ... a_1, a_0$ are integers . Suppose also that p and q have no common factors other than 1 and –1. Then

$$f\left(\frac{p}{q}\right) = a_n\left(\frac{p}{q}\right)^n + a_{n-1}\left(\frac{p}{q}\right)^{n-1} + a_{n-2}\left(\frac{p}{q}\right)^{n-2} + ... + a_1\left(\frac{p}{q}\right) + a_0 = 0$$

$$\Rightarrow \frac{1}{q^n}\left(a_n p^n + a_{n-1}p^{n-1}q + a_{n-2}p^{n-2}q^2 + ... + a_1 pq^{n-1} + a_0 q^n\right) = 0$$

$$\Rightarrow a_n p^n + a_{n-1}p^{n-1}q + a_{n-2}p^{n-2}q^2 + ... + a_1 pq^{n-1} + a_0 q^n = 0$$

$$\Rightarrow a_n p^n + a_{n-1}p^{n-1}q + a_{n-2}p^{n-2}q^2 + ... + a_1 pq^{n-1} = -a_0 q^n$$

Because p is a factor of the left side of this equation, p must also be a factor of $a_0 q^n$. Since p is not a factor of q (remember p and q have no common factors other than 1 and –1), p must be a factor of a_0. Similarly, q must be a factor of a_n

99. $f(x) = 8x^4 - 2x^2 + 5x - 1 \quad 0 \le r \le 1$

We begin with the interval [0,1].

$f(0) = -1; \quad f(1) = 10$

Let m_i = the midpoint of the interval being considered.

So $m_1 = 0.5$

n	m_n	$f(m_n)$	New interval
1	0.5	$f(0.5) = 1.5 > 0$	[0,0.5]
2	0.25	$f(0.25) = 0.15625 > 0$	[0,0.25]
3	0.125	$f(0.125) \approx -0.4043 < 0$	[0.125,0.25]
4	0.1875	$f(0.1875) \approx -0.1229 < 0$	[0.1875,0.25]
5	0.21875	$f(0.21875) \approx 0.0164 > 0$	[0.1875,0.21875]
6	0.203125	$f(0.203125) \approx -0.0533 < 0$	[0.203125,0.21875]
7	0.2109375	$f(0.2109375) \approx -0.0185 < 0$	[0.2109375,0.21875]
8	0.21484375	$f(0.21484375) \approx -0.0011 < 0$	[0.21484375,0.21875]
9	0.216796875	$f(0.216796875) \approx 0.0077 > 0$	[0.21484375,0.216796875]
10	0.2158203125	$f(0.2158203125) \approx 0.0033 > 0$	[0.21484375,0.2158203125]
11	0.2153320313	$f(0.2153320313) \approx 0.0011 > 0$	[0.21484375,0.2153320313]

If rounded to three decimal places, both endpoints of the new interval at Step 11 agree at 0.215. Therefore, $r \approx 0.215$, correct to three decimal places.

---

100. $f(x) = 2x^3 + 3x^2 - 6x + 7$

By the Rational Zero Theorem, the only possible rational zeros are: $\frac{p}{q} = \pm 1, \pm 7, \pm\frac{1}{2}, \pm\frac{7}{2}$.

Since $\frac{1}{3}$ is not in the list of possible rational zeros, it is not a zero of *f*.

101. $f(x) = 4x^3 - 5x^2 - 3x + 1$

By the Rational Zero Theorem, the only possible rational zeros are: $\frac{p}{q} = \pm 1, \pm\frac{1}{2}, \pm\frac{1}{4}$.

Since $\frac{1}{3}$ is not in the list of possible rational zeros, it is not a zero of *f*.

102. $f(x) = 2x^6 - 5x^4 + x^3 - x + 1$

By the Rational Zero Theorem, the only possible rational zeros are: $\frac{p}{q} = \pm 1, \pm\frac{1}{2}$.

Since $\frac{3}{5}$ is not in the list of possible rational zeros, it is not a zero of *f*.

103. $f(x) = x^7 + 6x^5 - x^4 + x + 2$

By the Rational Zero Theorem, the only possible rational zeros are: $\frac{p}{q} = \pm 1, \pm 2$.

Since $\frac{2}{3}$ is not in the list of possible rational zeros, it is not a zero of *f*.

Section 4.3

1. $(3-2i)+(-3+5i) = 3-3-2i+5i$
$= 3i$

$(3-2i)(-3+5i) = -9+15i+6i-10i^2$
$= -9+21i-10(-1)$
$= 1+21i$

2. The zeros of $f(x)$ are the solutions to the equation $x^2+2x+2=0$.

$x^2+2x+2=0$

$a=1, b=2, c=2$

$$x = \frac{-2\pm\sqrt{2^2-4(1)(2)}}{2(1)} = \frac{-2\pm\sqrt{-4}}{2} = \frac{-2\pm 2i}{2} = -1\pm i$$

The solution set is $\{-1-i, -1+i\}$.

3. one

4. $3-4i$

5. True

6. False; would also need $-3-5i$

7. Since complex zeros appear in conjugate pairs, $4+i$, the conjugate of $4-i$, is the remaining zero of f.

8. Since complex zeros appear in conjugate pairs, $3-i$, the conjugate of $3+i$, is the remaining zero of f.

9. Since complex zeros appear in conjugate pairs, $-i$, the conjugate of i, and $1-i$, the conjugate of $1+i$, are the remaining zeros of f.

10. Since complex zeros appear in conjugate pairs, $2-i$, the conjugate of $2+i$, is the remaining zero of f.

11. Since complex zeros appear in conjugate pairs, $-i$, the conjugate of i, and $-2i$, the conjugate of $2i$, are the remaining zeros of f.

12. Since complex zeros appear in conjugate pairs, $-i$, the conjugate of i, is the remaining zero.

13. Since complex zeros appear in conjugate pairs, $-i$, the conjugate of i, is the remaining zero.

14. Since complex zeros appear in conjugate pairs, $2+i$, the conjugate of $2-i$, and i, the conjugate of $-i$, are the remaining zeros of f.

15. Since complex zeros appear in conjugate pairs, $2-i$, the conjugate of $2+i$, and $-3+i$, the conjugate of $-3-i$, are the remaining zeros.

16. Since complex zeros appear in conjugate pairs, $-i$, the conjugate of i, $3+2i$, the conjugate of $3-2i$, and $-2-i$, the conjugate of $-2+i$, are the remaining zeros of f.

For 17–22, we will use $a_n = 1$ as the lead coefficient of the polynomial. Also note that

$$\begin{aligned}(x-(a+bi))(x-(a-bi)) &= ((x-a)-bi)((x-a)+bi)\\ &= (x-a)^2-(bi)^2\end{aligned}$$

17. Since $3+2i$ is a zero, its conjugate $3-2i$ is also a zero of f.

$$\begin{aligned}f(x) &= (x-4)(x-4)\left(x-(3+2i)\right)\left(x-(3-2i)\right)\\ &= \left(x^2-8x+16\right)\left((x-3)-2i\right)\left((x-3)+2i\right)\\ &= \left(x^2-8x+16\right)\left(x^2-6x+9-4i^2\right)\\ &= \left(x^2-8x+16\right)\left(x^2-6x+13\right)\\ &= x^4-6x^3+13x^2-8x^3+48x^2\\ &\qquad -104x+16x^2-96x+208\\ &= x^4-14x^3+77x^2-200x+208\end{aligned}$$

18. Since $1+2i$ and i are zeros, their conjugates $1-2i$ and $-i$ are also zeros of f.

$$\begin{aligned}f(x) &= (x-i)(x-(-i))\left(x-(1+2i)\right)\left(x-(1-2i)\right)\\ &= (x-i)(x+i)\left((x-1)-2i\right)\left((x-1)+2i\right)\\ &= \left(x^2-i^2\right)\left(x^2-2x+1-4i^2\right)\\ &= \left(x^2+1\right)\left(x^2-2x+5\right)\\ &= x^4-2x^3+5x^2+1x^2-2x+5\\ &= x^4-2x^3+6x^2-2x+5\end{aligned}$$

19. Since $-i$ is a zero, its conjugate i is also a zero, and since $1+i$ is a zero, its conjugate $1-i$ is also a zero of f.

$$\begin{aligned}f(x) &= (x-2)(x+i)(x-i)\left(x-(1+i)\right)\left(x-(1-i)\right)\\ &= (x-2)\left(x^2-i^2\right)\left((x-1)-i\right)\left((x-1)+i\right)\\ &= (x-2)\left(x^2+1\right)\left(x^2-2x+1-i^2\right)\\ &= \left(x^3-2x^2+x-2\right)\left(x^2-2x+2\right)\\ &= x^5-2x^4+2x^3-2x^4+4x^3-4x^2\\ &\qquad +x^3-2x^2+2x-2x^2+4x-4\\ &= x^5-4x^4+7x^3-8x^2+6x-4\end{aligned}$$

20. Since i is a zero, its conjugate $-i$ is also a zero; since $4-i$ is a zero, its conjugate $4+i$ is also a zero; and since $2+i$ is a zero, its conjugate $2-i$ is also a zero of f.

$$\begin{aligned}f(x) &= (x+i)(x-i)\left(x-(4+i)\right)\left(x-(4-i)\right)\left(x-(2+i)\right)\left(x-(2-i)\right)\\ &= \left(x^2-i^2\right)\left((x-4)-i\right)\left((x-4)+i\right)\left((x-2)-i\right)\left((x-2)+i\right)\\ &= \left(x^2+1\right)\left(x^2-8x+16-i^2\right)\left(x^2-4x+4-i^2\right)\\ &= \left(x^2+1\right)\left(x^2-8x+17\right)\left(x^2-4x+5\right)\\ &= \left(x^4-8x^3+17x^2+x^2-8x+17\right)\left(x^2-4x+5\right)\\ &= \left(x^4-8x^3+18x^2-8x+17\right)\left(x^2-4x+5\right)\\ &= x^6-4x^5+5x^4-8x^5+32x^4-40x^3+18x^4-72x^3+90x^2-8x^3+32x^2-40x+17x^2-68x+85\\ &= x^6-12x^5+55x^4-120x^3+139x^2-108x+85\end{aligned}$$

21. Since $-i$ is a zero, its conjugate i is also a zero.

$$\begin{aligned} f(x) &= (x-3)(x-3)(x+i)(x-i) \\ &= \left(x^2-6x+9\right)\left(x^2-i^2\right) \\ &= \left(x^2-6x+9\right)\left(x^2+1\right) \\ &= x^4+x^2-6x^3-6x+9x^2+9 \\ &= x^4-6x^3+10x^2-6x+9 \end{aligned}$$

22. Since $1+i$ is a zero, its conjugate $1-i$ is also a zero of f.

$$\begin{aligned} f(x) &= (x-1)^3\left(x-(1+i)\right)\left(x-(1-i)\right) \\ &= \left(x^3-3x^2+3x-1\right)\left((x-1)-i\right)\left((x-1)+i\right) \\ &= \left(x^3-3x^2+3x-1\right)\left(x^2-2x+1-i^2\right) \\ &= \left(x^3-3x^2+3x-1\right)\left(x^2-2x+2\right) \\ &= x^5-2x^4+2x^3-3x^4+6x^3-6x^2 \\ &\qquad +3x^3-6x^2+6x-x^2+2x-2 \\ &= x^5-5x^4+11x^3-13x^2+8x-2 \end{aligned}$$

23. Since $2i$ is a zero, its conjugate $-2i$ is also a zero of f. $x-2i$ and $x+2i$ are factors of f. Thus, $(x-2i)(x+2i)=x^2+4$ is a factor of f. Using division to find the other factor:

$$\begin{array}{r} x-4 \\ x^2+4\overline{)x^3-4x^2+4x-16} \\ \underline{x^3 \qquad +4x} \\ -4x^2 \qquad -16 \\ \underline{-4x^2 \qquad -16} \end{array}$$

$x-4$ is a factor, so the remaining zero is 4. The zeros of f are $4, 2i, -2i$.

24. Since $-5i$ is a zero, its conjugate $5i$ is also a zero of g. $x+5i$ and $x-5i$ are factors of g. Thus, $(x+5i)(x-5i)=x^2+25$ is a factor of g. Using division to find the other factor:

$$\begin{array}{r} x+3 \\ x^2+25\overline{)x^3+3x^2+25x+75} \\ \underline{x^3 \qquad +25x} \\ 3x^2 \qquad +75 \\ \underline{3x^2 \qquad +75} \end{array}$$

$x+3$ is a factor, so the remaining zero is –3. The zeros of g are $-3, 5i, -5i$.

25. Since $-2i$ is a zero, its conjugate $2i$ is also a zero of f. $x-2i$ and $x+2i$ are factors of f. Thus, $(x-2i)(x+2i)=x^2+4$ is a factor of f. Using division to find the other factor:

$$\begin{array}{r} 2x^2+5x-3 \\ x^2+4\overline{)2x^4+5x^3+5x^2+20x-12} \\ \underline{2x^4 \qquad +8x^2} \\ 5x^3-3x^2+20x \\ \underline{5x^3 \qquad +20x} \\ -3x^2 \qquad -12 \\ \underline{-3x^2 \qquad -12} \end{array}$$

$$2x^2+5x-3=(2x-1)(x+3)$$

The remaining zeros are $\frac{1}{2}$ and -3.

The zeros of f are $2i, -2i, -3, \frac{1}{2}$.

26. Since $3i$ is a zero, its conjugate $-3i$ is also a zero of h. $x-3i$ and $x+3i$ are factors of h. Thus, $(x-3i)(x+3i)=x^2+9$ is a factor of h. Using division to find the other factor:

$$\begin{array}{r} 3x^2+5x-2 \\ x^2+9\overline{)3x^4+5x^3+25x^2+45x-18} \\ \underline{3x^4 \qquad +27x^2} \\ 5x^3-\ 2x^2+45x \\ \underline{5x^3 \qquad +45x} \\ -2x^2 \qquad -18 \\ \underline{-2x^2 \qquad -18} \end{array}$$

$$3x^2+5x-2=(3x-1)(x+2)$$

The remaining zeros are $\frac{1}{3}$ and -2.

The zeros of h are $3i, -3i, -2, \frac{1}{3}$.

27. Since $3-2i$ is a zero, its conjugate $3+2i$ is also a zero of h. $x-(3-2i)$ and $x-(3+2i)$ are factors of h.

Thus,

$$\begin{aligned}(x-(3-2i))(x-(3+2i)) &= ((x-3)+2i)((x-3)-2i)\\ &= x^2-6x+9-4i^2\\ &= x^2-6x+13\end{aligned}$$

is a factor of h.

Using division to find the other factor:

$$\begin{array}{r}
x^2-3x-10 \\
x^2-6x+13\overline{\big)x^4-9x^3+21x^2+21x-130} \\
\underline{x^4-6x^3+13x^2} \\
-3x^3+\ 8x^2+21x \\
\underline{-3x^3+18x^2-39x} \\
-10x^2+60x-130 \\
\underline{-10x^2+60x-130}
\end{array}$$

$x^2-3x-10=(x+2)(x-5)$

The remaining zeros are –2 and 5.

The zeros of h are $3-2i, 3+2i, -2, 5$.

28. Since $1+3i$ is a zero, its conjugate $1-3i$ is also a zero of f. $x-(1+3i)$ and $x-(1-3i)$ are factors of f. Thus,

$(x-(1+3i))(x-(1-3i))=((x-1)-3i)((x-1)+3i)$ is a factor of f.

$$\begin{aligned}((x-1)-3i)((x-1)+3i) &= x^2-2x+1-9i^2\\ &= x^2-2x+10\end{aligned}$$

Using division to find the other factor:

$$\begin{array}{r}
x^2-5x-6 \\
x^2-2x+10\overline{\big)x^4-7x^3+14x^2-38x-60} \\
\underline{x^4-2x^3+10x^2} \\
-5x^3+\ 4x^2-38x \\
\underline{-5x^3+10x^2-50x} \\
-6x^2+\ 12x-60 \\
\underline{-6x^2+\ 12x-60}
\end{array}$$

$x^2-5x-6=(x+1)(x-6)$

The remaining zeros are –1 and 6.

The zeros of f are $1+3i, 1-3i, -1, 6$.

29. Since $-4i$ is a zero, its conjugate $4i$ is also a zero of h. $x-4i$ and $x+4i$ are factors of h. Thus, $(x-4i)(x+4i)=x^2+16$ is a factor of h. Using division to find the other factor:

$$\begin{array}{r}
3x^3+2x^2-33x-22 \\
x^2+16\overline{\big)3x^5+2x^4+15x^3+10x^2-528x-352} \\
\underline{3x^5+48x^3} \\
2x^4-33x^3+10x^2 \\
\underline{2x^4+32x^2} \\
-33x^3-22x^2-528x \\
\underline{-33x^3-528x} \\
-22x^2-352 \\
\underline{-22x^2-352}
\end{array}$$

$$\begin{aligned}3x^3+2x^2-33x-22 &= x^2(3x+2)-11(3x+2)\\ &= (3x+2)(x^2-11)\end{aligned}$$

$$=(3x+2)\left(x-\sqrt{11}\right)\left(x+\sqrt{11}\right)$$

The remaining zeros are $-\frac{2}{3}, \sqrt{11}$, and $-\sqrt{11}$.

The zeros of h are $4i, -4i, -\sqrt{11}, \sqrt{11}, -\frac{2}{3}$.

30. Since $3i$ is a zero, its conjugate $-3i$ is also a zero of g. $x-3i$ and $x+3i$ are factors of g. Thus, $(x-3i)(x+3i)=x^2+9$ is a factor of g. Using division to find the other factor:

$$\begin{array}{r}
2x^3-3x^2-23x+12 \\
x^2+9\overline{\big)2x^5-3x^4-\ 5x^3-15x^2-207x+108} \\
\underline{2x^5+18x^3} \\
-3x^4-23x^3-15x^2 \\
\underline{-3x^4-27x^2} \\
-23x^3+12x^2-207x \\
\underline{-23x^3-207x} \\
12x^2+108 \\
\underline{12x^2+108} \\
0
\end{array}$$

Using the Rational Root theorem, we see that -3 is a potential rational zero.

$$\begin{array}{r|rrrr} -3 & 2 & -3 & -23 & 12 \\ & & -6 & 27 & -12 \\ \hline & 2 & -9 & 4 & 0 \end{array}$$

$x+3$ is a factor. The remaining factor is $2x^2-9x+4=(2x-1)(x-4)$.

The zeros of g are $3i, -3i, -3, \frac{1}{2}, 4$.

31. $f(x)=x^3-1=(x-1)\left(x^2+x+1\right)$ The solutions of $x^2+x+1=0$ are:

$$x=\frac{-1\pm\sqrt{1^2-4(1)(1)}}{2(1)}=\frac{-1\pm\sqrt{-3}}{2}$$
$$=-\frac{1}{2}+\frac{\sqrt{3}}{2}i \text{ and } -\frac{1}{2}-\frac{\sqrt{3}}{2}i$$

The zeros are: $1, -\frac{1}{2}+\frac{\sqrt{3}}{2}i, -\frac{1}{2}-\frac{\sqrt{3}}{2}i$.

$$f(x)=(x-1)\left(x+\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)\left(x+\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)$$

32. $f(x)=x^4-1=\left(x^2-1\right)\left(x^2+1\right)$
$$=(x-1)(x+1)\left(x^2+1\right)$$

The solutions of $x^2+1=0$ are $x=\pm i$.
The zeros are: $-1, 1, -i, i$.

$$f(x)=(x+1)(x-1)(x+i)(x-i)$$

33. $f(x)=x^3-8x^2+25x-26$

Step 1: $f(x)$ has 3 complex zeros.

Step 2: Possible rational zeros:

$p=\pm1, \pm2, \pm13, \pm26; \quad q=\pm1;$

$\frac{p}{q}=\pm1, \pm2, \pm13, \pm26$

Using synthetic division:

We try $x-2$:

$$\begin{array}{r|rrrr} 2 & 1 & -8 & 25 & -26 \\ & & 2 & -12 & 26 \\ \hline & 1 & -6 & 13 & 0 \end{array}$$

$x-2$ is a factor. The other factor is the quotient: $x^2-6x+13$.

The solutions of $x^2-6x+13=0$ are:

$$x=\frac{-(-6)\pm\sqrt{(-6)^2-4(1)(13)}}{2(1)}.$$
$$=\frac{6\pm\sqrt{-16}}{2}=\frac{6\pm4i}{2}=3\pm2i$$

The zeros are $2, 3-2i, 3+2i$.

$$f(x)=(x-2)(x-3+2i)(x-3-2i)$$

34. $f(x)=x^3+13x^2+57x+85$

Step 1: $f(x)$ has 3 complex zeros.

Step 2: Possible rational zeros:

$p=\pm1, \pm5, \pm17, \pm85; \quad q=\pm1;$

$\frac{p}{q}=\pm1, \pm5, \pm17, \pm85$

Using synthetic division:

We try $x+5$:

$$\begin{array}{r|rrrr} -5 & 1 & 13 & 57 & 85 \\ & & -5 & -40 & -85 \\ \hline & 1 & 8 & 17 & 0 \end{array}$$

$x+5$ is a factor. The other factor is the quotient: $x^2+8x+17$.

The solutions of $x^2+8x+17=0$ are:

$$x=\frac{-8\pm\sqrt{8^2-4(1)(17)}}{2(1)}=\frac{-8\pm\sqrt{-4}}{2}$$
$$=\frac{-8\pm2i}{2}=-4\pm i$$

The zeros are $-5, -4-i, -4+i$.

$$f(x)=(x+5)(x+4+i)(x+4-i)$$

35. $f(x)=x^4+5x^2+4=\left(x^2+4\right)\left(x^2+1\right)$
$$=(x+2i)(x-2i)(x+i)(x-i)$$

The zeros are: $-2i, -i, i, 2i$.

36. $f(x)=x^4+13x^2+36=\left(x^2+4\right)\left(x^2+9\right)$

$=(x+2i)(x-2i)(x+3i)(x-3i)$

The zeros are: $-3i,\ -2i,\ 2i,\ 3i$.

$f\left(x\right)=\left(x+3i\right)\left(x+2i\right)\left(x-2i\right)\left(x-3i\right)$

37. $f(x)=x^4+2x^3+22x^2+50x-75$

Step 1: $f(x)$ has 4 complex zeros.

Step 2: Possible rational zeros:

$p=\pm 1, \pm 3, \pm 5, \pm 15, \pm 25, \pm 75; \quad q=\pm 1;$

$\dfrac{p}{q}=\pm 1, \pm 3, \pm 5, \pm 15, \pm 25, \pm 75$

Using synthetic division:

We try $x+3$:

$$\begin{array}{r|rrrrr} -3 & 1 & 2 & 22 & 50 & -75 \\ & & -3 & 3 & -75 & 75 \\ \hline & 1 & -1 & 25 & -25 & 0 \end{array}$$

$x+3$ is a factor. The other factor is the quotient: $x^3-x^2+25x-25$.

$$\begin{aligned} x^3-x^2+25x-25 &= x^2(x-1)+25(x-1) \\ &= (x-1)\left(x^2+25\right) \\ &= (x-1)(x+5i)(x-5i) \end{aligned}$$

The zeros are $-3,\ 1,\ -5i,\ 5i$.

$f\left(x\right)=\left(x+3\right)\left(x-1\right)\left(x+5i\right)\left(x-5i\right)$

38. $f(x)=x^4+3x^3-19x^2+27x-252$

Step 1: $f(x)$ has 4 complex zeros.

Step 2: Possible rational zeros:

$p=\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 7, \pm 9,$
$\pm 12, \pm 14, \pm 18, \pm 21, \pm 28, \pm 36,$
$\pm 42, \pm 63, \pm 84, \pm 126, \pm 252;$

$q=\pm 1;$

The possible rational zeros are the same as the values of p.

Using synthetic division:

We try $x+7$:

$$\begin{array}{r|rrrrr} -7 & 1 & 3 & -19 & 27 & -252 \\ & & -7 & 28 & -63 & 252 \\ \hline & 1 & -4 & 9 & -36 & 0 \end{array}$$

$x+7$ is a factor. The other factor is the quotient:

$$\begin{aligned} x^3-4x^2+9x-36 &= x^2(x-4)+9(x-4) \quad . \\ &= (x-4)\left(x^2+9\right) \\ &= (x-4)(x+3i)(x-3i) \end{aligned}$$

The zeros are $-7,\ 4,\ -3i,\ 3i$.

$f\left(x\right)=\left(x+7\right)\left(x-4\right)\left(x+3i\right)\left(x-3i\right)$

39. $f(x)=3x^4-x^3-9x^2+159x-52$

Step 1: $f(x)$ has 4 complex zeros.

Step 2: Possible rational zeros:

$p=\pm 1, \pm 2, \pm 4, \pm 13, \pm 26, \pm 52;$

$q=\pm 1, \pm 3;$

$$\frac{p}{q}=\pm 1, \pm 2, \pm 4, \pm 13, \pm 26, \pm 52, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{4}{3}, \pm\frac{13}{3}, \pm\frac{26}{3}, \pm\frac{52}{3}$$

Using synthetic division:

We try $x+4$:

$$\begin{array}{r|rrrrr} -4 & 3 & -1 & -9 & 159 & -52 \\ & & -12 & 52 & -172 & 52 \\ \hline & 3 & -13 & 43 & -13 & 0 \end{array}$$

$x+4$ is a factor and the quotient is $3x^3-13x^2+43x-13$.

We try $x-\dfrac{1}{3}$ on $3x^3-13x^2+43x-13$:

$$\begin{array}{r|rrrr} \frac{1}{3} & 3 & -13 & 43 & -13 \\ & & 1 & -4 & 13 \\ \hline & 3 & -12 & 39 & 0 \end{array}$$

$x-\frac{1}{3}$ is a factor and the quotient is $3x^2-12x+39$.

$$3x^2-12x+39=3\left(x^2-4x+13\right)$$

The solutions of $x^2-4x+13=0$ are:

$$x=\frac{-(-4)\pm\sqrt{(-4)^2-4(1)(13)}}{2(1)}.$$
$$=\frac{4\pm\sqrt{-36}}{2}=\frac{4\pm 6i}{2}=2\pm 3i$$

The zeros are $-4,\ \frac{1}{3},\ 2-3i,\ 2+3i$.

$$f\left(x\right)=3\left(x+4\right)\left(x-\frac{1}{3}\right)\left(x-2+3i\right)\left(x-2-3i\right)$$

40. $f(x)=2x^4+x^3-35x^2-113x+65$

Step 1: $f(x)$ has 4 complex zeros.

Step 2: Possible rational zeros:

$p=\pm 1,\pm 5,\pm 13,\pm 65;\quad q=\pm 1,\pm 2;$

$\frac{p}{q}=\pm 1,\pm 5,\pm 13,\pm 65,\pm\frac{1}{2},\pm\frac{5}{2},\pm\frac{13}{2},\pm\frac{65}{2}$

Using synthetic division:

We try $x-5$:

$$\begin{array}{r|rrrrr} 5 & 2 & 1 & -35 & -113 & 65 \\ & & 10 & 55 & 100 & -65 \\ \hline & 2 & 11 & 20 & -13 & 0 \end{array}$$

$x-5$ is a factor and the quotient is $2x^3+11x^2+20x-13$

We try $x-\frac{1}{2}$ on $2x^3+11x^2+20x-13$:

$$\begin{array}{r|rrrr} \frac{1}{2} & 2 & 11 & 20 & -13 \\ & & 1 & 6 & 13 \\ \hline & 2 & 12 & 26 & 0 \end{array}$$

$x-\frac{1}{2}$ is a factor and the quotient is $2x^2+12x+26$.

$$2x^2+12x+26=2\left(x^2+6x+13\right)$$

The solutions of $x^2+6x+13=0$ are:

$$x=\frac{-6\pm\sqrt{6^2-4(1)(13)}}{2(1)}.$$
$$=\frac{-6\pm\sqrt{-16}}{2}=\frac{-6\pm 4i}{2}=-3\pm 2i$$

The zeros are $5,\ \frac{1}{2},\ -3-2i,\ -3+2i$.

$$f\left(x\right)=2\left(x-5\right)\left(x-\frac{1}{2}\right)\left(x+3+2i\right)\left(x+3-2i\right)$$

41. If the coefficients are real numbers and $2+i$ is a zero, then $2-i$ would also be a zero. This would then require a polynomial of degree 4.

42. Three zeros are given. If the coefficients are real numbers, then the complex zeros would also have their conjugates as zeros. This would mean that there are 5 zeros, which would require a polynomial of degree 5.

43. If the coefficients are real numbers, then complex zeros must appear in conjugate pairs. We have a conjugate pair and one real zero. Thus, there is only one remaining zero, and it must be real because a complex zero would require a pair of complex conjugates.

44. One of the remaining zeros must be $4+i$, the conjugate of $4-i$. The third zero is a real number. Thus, the fourth zero must also be a real number in order to have a degree 4 polynomial.

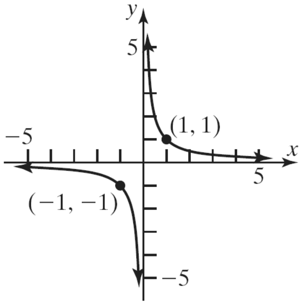
Section 4.4

1. True

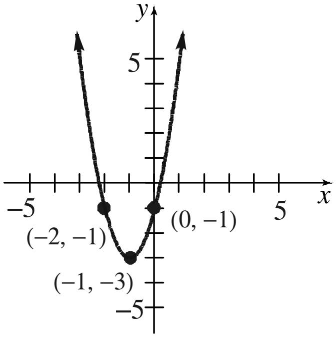
2. Quotient: $3x+3$; Remainder: $2x^2-3x-3$

$$\begin{array}{r} 3x+3 \\ x^3-x^2+1\overline{\big)3x^4+0x^3-x^2+0x+0} \\ -(3x^4-3x^3 \qquad +3x) \\ \hline 3x^3-x^2-3x \\ -(3x^3-3x^2 \qquad +3) \\ \hline 2x^2-3x-3 \end{array}$$

3. $y=\dfrac{1}{x}$



4. Using the graph of $y=x^2$, stretch vertically by a factor of 2, then shift left 1 unit, then shift down 3 units.



5. False

6. horizontal asymptote

7. vertical asymptote

8. proper

9. True

10. False, a graph cannot intersect a vertical asymptote.

11. $y=0$

12. True

13. In $R(x)=\dfrac{4x}{x-3}$, the denominator, $q(x)=x-3$, has a zero at 3. Thus, the domain of $R(x)$ is all real numbers except 3. $\{x \mid x \neq 3\}$

14. In $R(x)=\dfrac{5x^2}{3+x}$, the denominator, $q(x)=3+x$, has a zero at –3. Thus, the domain of $R(x)$ is all real numbers except –3. $\{x \mid x \neq -3\}$

15. In $H(x)=\dfrac{-4x^2}{(x-2)(x+4)}$, the denominator, $q(x)=(x-2)(x+4)$, has zeros at 2 and –4. Thus, the domain of $H(x)$ is all real numbers except –4 and 2. $\{x \mid x \neq -4, x \neq 2\}$

16. In $G(x)=\dfrac{6}{(x+3)(4-x)}$, the denominator, $q(x)=(x+3)(4-x)$, has zeros at –3 and 4. Thus, the domain of $G(x)$ is all real numbers except –3 and 4. $\{x \mid x \neq -3, x \neq 4\}$

17. In $F(x)=\dfrac{3x(x-1)}{2x^2-5x-3}$, the denominator, $q(x)=2x^2-5x-3=(2x+1)(x-3)$, has zeros at $-\dfrac{1}{2}$ and 3. Thus, the domain of $F(x)$ is all real numbers except $-\dfrac{1}{2}$ and 3. $\left\{x \mid x \neq -\dfrac{1}{2}, x \neq 3\right\}$

18. In $Q(x)=\dfrac{-x(1-x)}{3x^2+5x-2}$, the denominator, $q(x)=3x^2+5x-2=(3x-1)(x+2)$, has zeros at $\dfrac{1}{3}$ and -2. Thus, the domain of $Q(x)$ is all real numbers except –2 and $\dfrac{1}{3}$. $\left\{x \mid x \neq -2, x \neq \dfrac{1}{3}\right\}$

19. In $R(x)=\dfrac{x}{x^3-8}$, the denominator, $q(x)=x^3-8=(x-2)(x^2+2x+4)$, has a zero at 2 (x^2+2x+4 has no real zeros). Thus, the domain of $R(x)$ is all real numbers except 2. $\{x \mid x \neq 2\}$

20. In $R(x) = \dfrac{x}{x^4 - 1}$, the denominator, $q(x) = x^4 - 1 = (x-1)(x+1)(x^2+1)$, has zeros at –1 and 1 (x^2+1 has no real zeros). Thus, the domain of $R(x)$ is all real numbers except –1 and 1. $\{x \mid x \neq -1, x \neq 1\}$

21. In $H(x) = \dfrac{3x^2 + x}{x^2 + 4}$, the denominator, $q(x) = x^2 + 4$, has no real zeros. Thus, the domain of $H(x)$ is all real numbers.

22. In $G(x) = \dfrac{x-3}{x^4+1}$, the denominator, $q(x) = x^4 + 1$, has no real zeros. Thus, the domain of $G(x)$ is all real numbers.

23. In $R(x) = \dfrac{3(x^2 - x - 6)}{4(x^2 - 9)}$, the denominator, $q(x) = 4(x^2 - 9) = 4(x-3)(x+3)$, has zeros at 3 and –3. Thus, the domain of $R(x)$ is all real numbers except –3 and 3. $\{x \mid x \neq -3, x \neq 3\}$

24. In $F(x) = \dfrac{-2(x^2 - 4)}{3(x^2 + 4x + 4)}$, the denominator, $q(x) = 3(x^2 + 4x + 4) = 3(x+2)^2$, has a zero at –2. Thus, the domain of $F(x)$ is all real numbers except –2. $\{x \mid x \neq -2\}$

25. **a.** Domain: $\{x \mid x \neq 2\}$; Range: $\{y \mid y \neq 1\}$

b. Intercept: (0, 0)

c. Horizontal Asymptote: $y = 1$

d. Vertical Asymptote: $x = 2$

e. Oblique Asymptote: none

26. **a.** Domain: $\{x \mid x \neq -1\}$; Range: $\{y \mid y > 0\}$

b. Intercept: (0, 2)

c. Horizontal Asymptote: $y = 0$

d. Vertical Asymptote: $x = -1$

e. Oblique Asymptote: none

27. **a.** Domain: $\{x \mid x \neq 0\}$;
Range: all real numbers

b. Intercepts: (–1, 0) and (1, 0)

c. Horizontal Asymptote: none

d. Vertical Asymptote: $x = 0$

e. Oblique Asymptote: $y = 2x$

28. **a.** Domain: $\{x \mid x \neq 0\}$;
Range: $\{ y \mid y \leq -2 \text{ or } y \geq 2 \}$

b. Intercepts: none

c. Horizontal Asymptote: none

d. Vertical Asymptote: $x = 0$

e. Oblique Asymptote: $y = -x$

29. **a.** Domain: $\{x \mid x \neq -2, x \neq 2\}$;
Range: $\{y \mid y \leq 0 \text{ or } y > 1\}$

b. Intercept: (0, 0)

c. Horizontal Asymptote: $y = 1$

d. Vertical Asymptotes: $x = -2, x = 2$

e. Oblique Asymptote: none

30. **a.** Domain: $\{x \mid x \neq -1, x \neq 1\}$;
Range: all real numbers

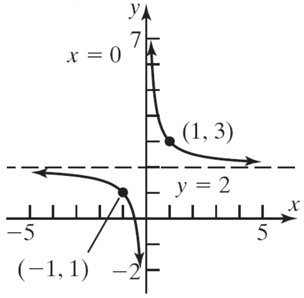
b. Intercept: (0, 0)

c. Horizontal Asymptote: $y = 0$

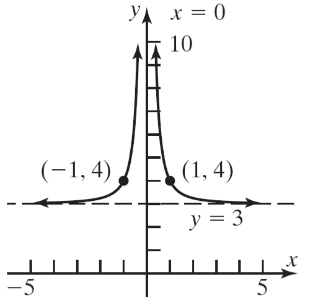
d. Vertical Asymptotes: $x = -1, x = 1$

e. Oblique Asymptote: none

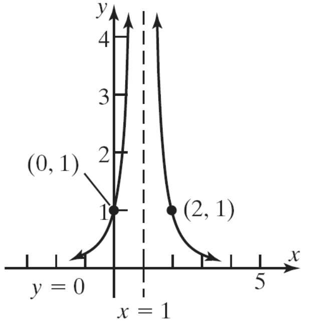
31. $F(x) = 2 + \dfrac{1}{x}$; Using the function, $y = \dfrac{1}{x}$, shift the graph vertically 2 units up.



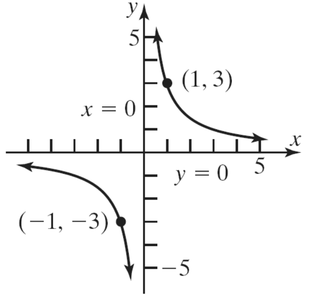
32. $Q(x)=3+\frac{1}{x^2}$; Using the function $y=\frac{1}{x^2}$, shift the graph vertically 3 units up.



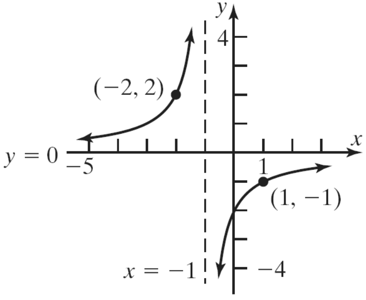
33. $R(x)=\frac{1}{(x-1)^2}$; Using the function, $y=\frac{1}{x^2}$, shift the graph horizontally 1 unit to the right.



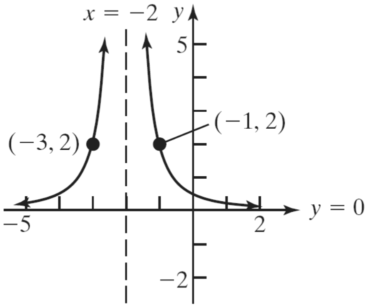
34. $Q(x)=\frac{3}{x}=3\left(\frac{1}{x}\right)$; Using the function $y=\frac{1}{x}$, stretch the graph vertically by a factor of 3.



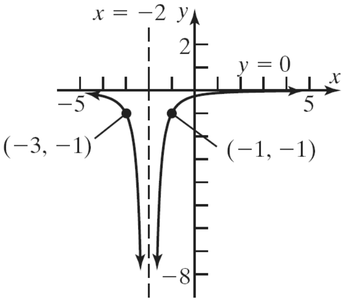
35. $H(x)=\frac{-2}{x+1}=-2\left(\frac{1}{x+1}\right)$; Using the function $y=\frac{1}{x}$, shift the graph horizontally 1 unit to the left, reflect about the x-axis, and stretch vertically by a factor of 2.



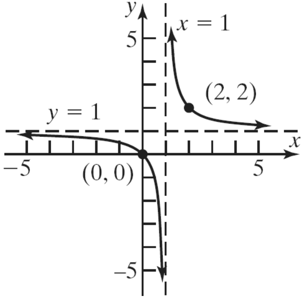
36. $G(x)=\frac{2}{(x+2)^2}=2\left(\frac{1}{(x+2)^2}\right)$; Using the function $y=\frac{1}{x^2}$, shift the graph horizontally 2 units to the left, and stretch vertically by a factor of 2.



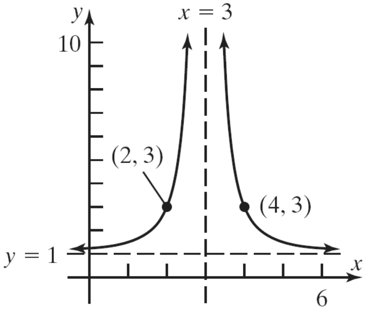
37. $R(x)=\frac{-1}{x^2+4x+4}=-\frac{1}{(x+2)^2}$; Using the function $y=\frac{1}{x^2}$, shift the graph horizontally 2 units to the left, and reflect about the x-axis.



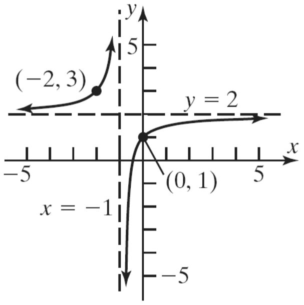
38. $R(x)=\frac{1}{x-1}+1$; Using the function $y=\frac{1}{x}$, shift the graph horizontally 1 unit to the right, and shift vertically 1 unit up.



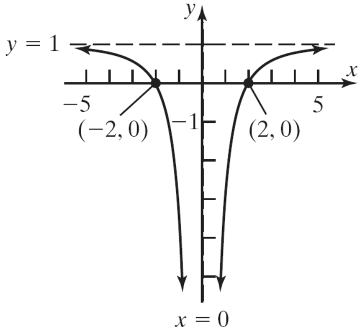
39. $G(x)=1+\dfrac{2}{(x-3)^2}=\dfrac{2}{(x-3)^2}+1=2\left(\dfrac{1}{(x-3)^2}\right)+1$; Using the function $y=\dfrac{1}{x^2}$, shift the graph right 3 units, stretch vertically by a factor of 2, and shift vertically 1 unit up.



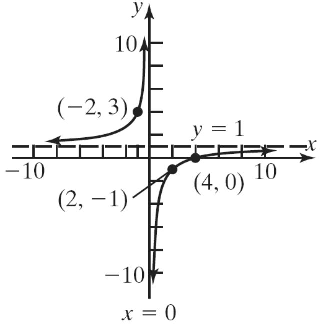
40. $F(x)=2-\dfrac{1}{x+1}=-\left(\dfrac{1}{x+1}\right)+2$; Using the function $y=\dfrac{1}{x}$, shift the graph left 1 unit, reflect about the x-axis, and shift vertically up 2 units.



41. $R(x)=\dfrac{x^2-4}{x^2}=1-\dfrac{4}{x^2}=-4\left(\dfrac{1}{x^2}\right)+1$; Using the function $y=\dfrac{1}{x^2}$, reflect about the x-axis, stretch vertically by a factor of 4, and shift vertically 1 unit up.



42. $R(x)=\dfrac{x-4}{x}=1-\dfrac{4}{x}=-4\left(\dfrac{1}{x}\right)+1$; Using the function $y=\dfrac{1}{x}$, reflect about the x-axis, stretch vertically by a factor of 4, and shift vertically 1 unit up.



43. $R(x)=\dfrac{3x}{x+4}$; The degree of the numerator, $p(x)=3x,$ is $n=1$. The degree of the denominator, $q(x)=x+4,$ is $m=1$. Since $n=m$, the line $y=\dfrac{3}{1}=3$ is a horizontal asymptote. The denominator is zero at $x=-4$, so $x=-4$ is a vertical asymptote.

44. $R(x)=\dfrac{3x+5}{x-6}$; The degree of the numerator, $p(x)=3x+5,$ is $n=1$. The degree of the denominator, $q(x)=x-6,$ is $m=1$. Since $n=m$, the line $y=\dfrac{3}{1}=3$ is a horizontal asymptote. The denominator is zero at $x=6$, so $x=6$ is a vertical asymptote.

45. $H(x)=\dfrac{x^3-8}{x^2-5x+6}=\dfrac{(x-2)\left(x^2+2x+4\right)}{(x-2)(x-3)}$

$=\dfrac{x^2+2x+4}{x-3}$, where $x \neq 2,3$

The degree of the numerator in lowest terms is $n=2$. The degree of the denominator in lowest terms is $m=1$. Since $n=m+1$, there is an oblique asymptote.

Dividing:

$$\begin{array}{r}
x+5 \\
x-3\overline{)x^2+2x+4} \\
-\left(x^2-3x\right) \\
\hline
5x+4 \\
-(5x-15) \\
\hline
19
\end{array}$$

$H(x)=x+5+\dfrac{19}{x-3}$, $x \neq 2,3$

Thus, the oblique asymptote is $y=x+5$.

The denominator in lowest terms is zero at $x=3$ so $x=3$ is a vertical asymptote.

46. $G(x)=\dfrac{x^3+1}{x^2-5x-14}=\dfrac{\left(x^3+1\right)}{(x+2)(x-7)}$

The degree of the numerator, $p(x)=x^3+1,$ is $n=3$. The degree of the denominator, $q(x)=x^2-5x-14,$ is $m=2$. Since $n=m+1$, there is an oblique asymptote.

Dividing:

$$\begin{array}{r}
x+5 \qquad\qquad \\
x^2-5x-14\overline{)x^3 \qquad\qquad +1} \\
-\left(x^3-5x^2-14x\right) \\
\hline
5x^2+14x+1 \\
-\left(5x^2-25x-70\right) \\
\hline
39x+71
\end{array}$$

$G(x)=x+5+\dfrac{39x+71}{x^2-5x-14}$, $x \neq -2,7$

Thus, the oblique asymptote is $y=x+5$.

The denominator is zero at $x=-2$ and $x=7$, so $x=-2$ and $x=7$ are vertical asymptotes.

47. $T(x)=\dfrac{x^3}{x^4-1}$; The degree of the numerator, $p(x)=x^3$, is $n=3$. The degree of the denominator, $q(x)=x^4-1$ is $m=4$. Since $n<m$, the line $y=0$ is a horizontal asymptote. The denominator is zero at $x=-1$ and $x=1$, so $x=-1$ and $x=1$ are vertical asymptotes.

48. $P(x)=\dfrac{4x^2}{x^3-1}$; The degree of the numerator, $p(x)=4x^2$, is $n=2$. The degree of the denominator, $q(x)=x^3-1$ is $m=3$. Since $n<m$, the line $y=0$ is a horizontal asymptote. The denominator is zero at $x=1$, so $x=1$ is a vertical asymptote.

49. $Q(x)=\dfrac{2x^2-5x-12}{3x^2-11x-4}=\dfrac{(2x+3)(x-4)}{(3x+1)(x-4)}$

$=\dfrac{2x+3}{3x+1}$, where $x \neq -\dfrac{1}{3}, 4$

The degree of the numerator in lowest terms is $n=1$. The degree of the denominator in lowest terms is $m=1$. Since $n=m$, the line $y=\dfrac{2}{3}$ is a horizontal asymptote. The denominator in lowest terms is zero at $x=-\dfrac{1}{3}$, so $x=-\dfrac{1}{3}$ is a vertical asymptote.

50. $F(x)=\dfrac{x^2+6x+5}{2x^2+7x+5}=\dfrac{(x+5)(x+1)}{(2x+5)(x+1)}$

$=\dfrac{x+5}{2x+5}$, where $x \neq -\dfrac{5}{2},-1$

The degree of the numerator in lowest terms is $n=1$. The degree of the denominator in lowest terms is $m=1$. Since $n=m$, the line $y=\dfrac{1}{2}$ is a horizontal asymptote. The denominator in lowest terms is zero at $x=-\dfrac{5}{2}$, so $x=-\dfrac{5}{2}$ is a vertical asymptote.

51. $R(x)=\dfrac{6x^2+7x-5}{3x+5}=\dfrac{(3x+5)(2x-1)}{3x+5}$

$=2x-1,$ where $x \neq -\dfrac{5}{3}$

The degree of the numerator in lowest terms is $n=1$. The degree of the denominator in lowest terms is $m=0$. Since $n=m+1$, there is an oblique asymptote. From the simplification shown above, the oblique asymptote is

$y = 2x - 1$. The denominator of $R(x)$ in lowest terms is 1, so there is no vertical asymptote.

52. $R(x) = \dfrac{8x^2 - 26x - 7}{4x - 1} = \dfrac{(4x-1)(2x+7)}{4x-1}$

$= 2x + 7, \text{ where } x \neq -\dfrac{1}{4}$

The degree of the numerator in lowest terms is $n = 1$. The degree of the denominator in lowest terms is $m = 0$. Since $n = m + 1$, there is an oblique asymptote. From the simplification shown above, the oblique asymptote is $y = 2x + 7$. The denominator of $R(x)$ in lowest terms is 1, so there is no vertical asymptote.

53. $G(x) = \dfrac{x^4 - 1}{x^2 - x} = \dfrac{(x^2+1)(x^2-1)}{x(x-1)}$

$= \dfrac{(x^2+1)(x+1)(x-1)}{x(x-1)}$

$= \dfrac{(x^2+1)(x+1)}{x}, \text{ where } x \neq 0, 1$

The degree of the numerator in lowest terms is $n = 3$. The degree of the denominator in lowest terms is $m = 1$. Since $n \geq m + 2$, there is no horizontal asymptote or oblique asymptote. The denominator in lowest terms is zero at $x = 0$, so $x = 0$ is a vertical asymptote.

54. $F(x) = \dfrac{x^4 - 16}{x^2 - 2x} = \dfrac{(x^2+4)(x^2-4)}{x(x-2)}$

$= \dfrac{(x^2+4)(x+2)(x-2)}{x(x-2)}$

$= \dfrac{(x^2+4)(x+2)}{x}, \text{ where } x \neq 0, 2$

The degree of the numerator in lowest terms is $n = 3$. The degree of the denominator in lowest terms is $m = 1$. Since $n \geq m + 2$, there is no horizontal asymptote or oblique asymptote. The denominator in lowest terms is zero at $x = 0$, so $x = 0$ is a vertical asymptote.

55. $g(h) = \dfrac{3.99 \times 10^{14}}{\left(6.374 \times 10^6 + h\right)^2}$

a. $g(0) = \dfrac{3.99 \times 10^{14}}{\left(6.374 \times 10^6 + 0\right)^2} \approx 9.8208 \text{ m/s}^2$

b. $g(443) = \dfrac{3.99 \times 10^{14}}{\left(6.374 \times 10^6 + 443\right)^2}$

$\approx 9.8195 \text{ m/s}^2$

c. $g(8848) = \dfrac{3.99 \times 10^{14}}{\left(6.374 \times 10^6 + 8848\right)^2}$

$\approx 9.7936 \text{ m/s}^2$

d. $g(h) = \dfrac{3.99 \times 10^{14}}{\left(6.374 \times 10^6 + h\right)^2}$

$\approx \dfrac{3.99 \times 10^{14}}{h^2} \to 0 \text{ as } h \to \infty$

Thus, the h-axis is the horizontal asymptote.

e. $g(h) = \dfrac{3.99 \times 10^{14}}{\left(6.374 \times 10^6 + h\right)^2} = 0$, to solve this equation would require that $3.99 \times 10^{14} = 0$, which is impossible. Therefore, there is no height above sea level at which $g = 0$. In other words, there is no point in the entire universe that is unaffected by the Earth's gravity!

56. $P(t) = \dfrac{50(1 + 0.5t)}{2 + 0.01t}$

a. $P(0) = \dfrac{50(1+0)}{2+0} = \dfrac{50}{2} = 25$ insects

b. 5 years = 60 months;

$P(60) = \dfrac{50(1 + 0.5(60))}{2 + 0.01(60)} = \dfrac{1550}{2.6}$

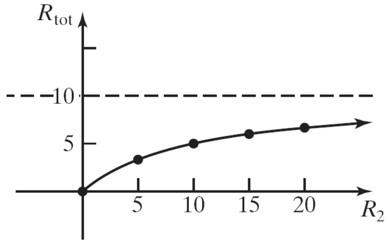
≈ 596 insects

c. $P(t) = \dfrac{50(1+0.5t)}{2+0.01t} \approx \dfrac{50(0.5t)}{0.01t} = 2500$

as $t \to \infty$

Thus, $y = 2500$ is the horizontal asymptote. The area can sustain a maximum population of 2500 insects.

57. a. $R_{tot} = \dfrac{10R_2}{10+R_2}$

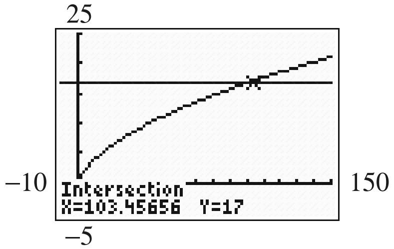


b. Horizontal asymptote: $y = R_{tot} = 10$
As the value of R_2 increases without bound, the total resistance approaches 10 ohms, the resistance of R_1.

c. $R_{tot} = \dfrac{R_1 R_2}{R_1 + R_2}$

$$17 = \frac{R_1 \cdot 2\sqrt{R_1}}{R_1 + 2\sqrt{R_1}}$$

Solving graphically, let $Y_1 = 17$ and $Y_2 = 2x\sqrt{x}\big/\left(x+2\sqrt{x}\right)$.



We would need $R_1 \approx 103.5$ ohms.

58. a. $p(-3) = (-3)^3 - 7(-3) - 40 = -46$
$p(5) = (5)^3 - 7(5) - 40 = 50$

b. Since p is continuous and $p(-3) < 0 < p(5)$, there must be at least one zero in the interval $\left(-3, 5\right)$ [Intermediate Value Theorem].

c. From the problem statement, we find the derivative to be $p'(x) = 3x^2 - 7$.
From part (b), we know there is at least one real zero in the interval $\left(-3, 5\right)$.

$p(-3) = (-3)^3 - 7(-3) - 40 = -34$
Since $p(3) < 0 < p(5)$, we start with $x_0 = 4$.

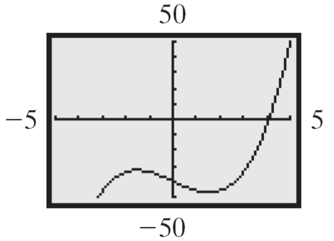
$$x_1 = 4 - \frac{p(4)}{p'(4)} \approx 4.097560976$$

$$x_2 = 4.097560976 - \frac{p(4.097560976)}{p'(4.097560976)}$$
$$\approx 4.094906$$

$$x_3 = 4.094906 - \frac{p(4.094906)}{p'(4.094906)} \approx 4.094904$$

The zero is approximately $x = 4.0949$.

d. $p(x) = x^3 - 7x - 40$



From the graph we can see that there is exactly one real zero in the interval $\left(-3, 5\right)$.

e. $p\left(4.0949\right) = \left(4.0949\right)^3 - 7\left(4.0949\right) - 40$
≈ -0.00017

This result is close to 0. Since 4.0949 is rounded, we expect some error when evaluating the function.

59. Answers will vary. If $x = 4$ is a vertical asymptote, then $x = 4$ is a zero of the denominator. If $x = 4$ is a zero of a polynomial, then $(x-4)$ is a factor of the polynomial. Therefore, if $x = 4$ is a vertical asymptote of a rational function, then $(x-4)$ must be a factor of the denominator.

60. Answers will vary. With rational functions, the only way to get a non-zero horizontal asymptote is if the degree of the numerator equals the degree of the denominator. In such cases, the horizontal asymptote is the ratio of the leading coefficients.

61. A rational function cannot have both a horizontal and oblique asymptote. To have an oblique asymptote, the degree of the numerator must be exactly one larger than the degree of the denominator. However, if the numerator has a higher degree, there is no horizontal asymptote.

62. Answers will vary. We want a rational function such that $r(x) = 2x + 1 + \frac{n(x)}{d(x)}$ where *n* and *d* are polynomial functions and the degree of $n(x)$ is less than the degree of $d(x)$. We could let $n(x) = 1$ and $d(x) = x + 1$. Then our function is $r(x) = 2x + 1 + \frac{1}{x+1}$. Getting a common denominator yields

$$r(x) = \frac{(2x+1)(x+1)}{x+1} + \frac{1}{x+1}$$
$$= \frac{2x^2 + x + 2x + 1 + 1}{x+1}$$
$$= \frac{2x^2 + 3x + 2}{x+1}$$

Therefore, one possibility is $r(x) = \frac{2x^2 + 3x + 2}{x+1}$.

Section 4.5

1. **a.** *y*-intercept:

$$f(0) = \frac{0^2 - 1}{0^2 - 4} = \frac{-1}{-4} = \frac{1}{4}$$

x-intercepts:
Set the numerator equal to 0 and solve for *x*.

$$x^2 - 1 = 0$$
$$(x+1)(x-1) = 0$$
$$x = -1 \text{ or } x = 1$$

The intercepts are $\left(0, \frac{1}{4}\right)$, $(-1, 0)$, and $(1, 0)$.

b. $f(-x) = \frac{(-x)^2 - 1}{(-x)^2 - 4} = \frac{x^2 - 1}{x^2 - 4} = f(x)$.

Thus, *f* is an even function.

2. in lowest terms

3. vertical

4. True

5. True

6. a. range: $\{x \mid x \neq 2\}$ since 2 makes the denominator 0.

b. $0 = \frac{x(x-2)^2}{x-2}$

$0 = x(x-2)^2$

$x = 0$ or $x = 2$

Since R(x) in not defined at x=2, the only x-intercept is x = 0.

In problems 7–44, we will use the terminology: $R(x) = \frac{p(x)}{q(x)}$, *where the degree of* $p(x) = n$ *and the degree of* $q(x) = m$.

7. $R(x) = \frac{x+1}{x(x+4)}$ $p(x) = x + 1;\ q(x) = x(x+4) = x^2 + 4x;\ n = 1;\ m = 2$

Step 1: Domain: $\{x \mid x \neq -4, x \neq 0\}$
Since 0 is not in the domain, there is no y-intercept.

Step 2 & 3:The function is in lowest terms. The *x*-intercept is the zero of $p(x)$: $x = -1$

The *x*-intercept is -1. Near -1, $R(x) \approx -\frac{1}{3}(x+1)$. Plot the point $(-1, 0)$ and show a line with negative slope there.

Step 4: $R(x) = \frac{x+1}{x(x+4)}$ is in lowest terms.

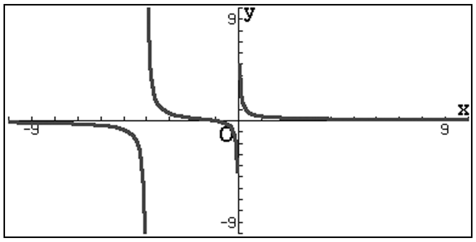
The vertical asymptotes are the zeros of $q(x)$: $x = -4$ and $x = 0$. Plot these lines using dashes.

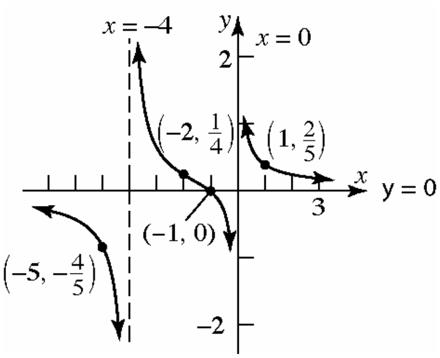
Step 5: Since $n < m$, the line $y = 0$ is the horizontal asymptote. Solve $R(x) = 0$ to find intersection points:

$$\frac{x+1}{x(x+4)} = 0$$
$$x+1 = 0$$
$$x = -1$$

$R(x)$ intersects $y = 0$ at (–1, 0). Plot the point $(-1, 0)$ and the line $y = 0$ using dashes.

Steps 6 & 7: Graphing





8. $R(x) = \dfrac{x}{(x-1)(x+2)}$ $p(x) = x;\; q(x) = (x-1)(x+2) = x^2 + x - 2;\; n = 1;\; m = 2$

Step 1: Domain: $\{x \mid x \neq -2, x \neq 1\}$

The y-intercept; $R(0) = 0$

Step 2 & 3:The function is in lowest terms. The x-intercept is the zero of $p(x)$. 0

Near 0, $R(x) \approx -\frac{1}{2}x$. Plot the point $(0, 0)$ and show a line with negative slope there.

Step 4: $R(x) = \dfrac{x}{(x-1)(x+2)}$ is in lowest terms.

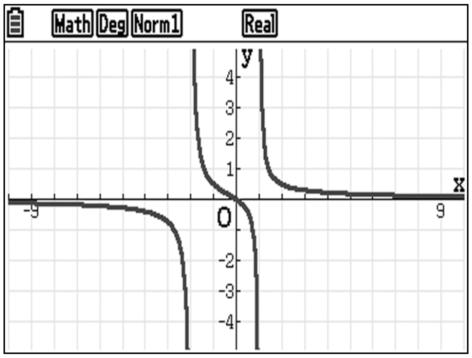
The vertical asymptotes are the zeros of $q(x)$: $x = -2$ and $x = 1$. Graph these asymptotes using dashed lines.

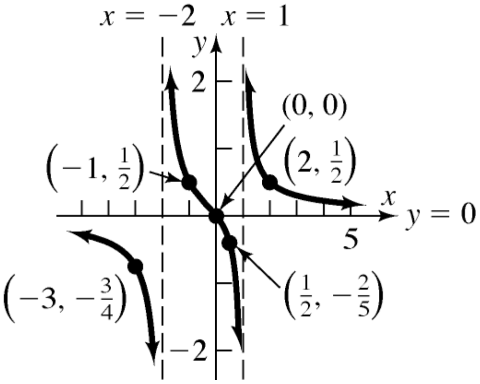
Step 5: Since $n < m$, the line $y = 0$ is the horizontal asymptote. Solve to find intersection points:

$$\frac{x}{(x-1)(x+2)} = 0$$
$$x = 0$$

$R(x)$ intersects $y = 0$ at (0, 0). Plot the point $(0, 0)$ and the line $y = 0$ using dashes.

Steps 6 & 7: Graphing:





9. $R(x)=\dfrac{3x+3}{2x+4}$ $\quad p(x)=3x+3;\ q(x)=2x+4;\ n=1;\ m=1$

Step 1: Domain: $\{x \mid x \neq -2\}$

The y-intercept is $R(0)=\dfrac{3(0)+3}{2(0)+4}=\dfrac{3}{4}$. Plot the point $\left(0, \dfrac{3}{4}\right)$.

Step 2 & 3: $R(x)=\dfrac{3x+3}{2x+4}=\dfrac{3(x+1)}{2(x+2)}$ is in lowest terms. The x-intercept is the zero of $p(x)$, $x=-1$.

Near -1, $R(x)\approx\dfrac{3}{2}(x+1)$. Plot the point $(-1,0)$ and show a line with positive slope there.

Step 4: $R(x)=\dfrac{3x+3}{2x+4}=\dfrac{3(x+1)}{2(x+2)}$ is in lowest terms.

The vertical asymptote is the zero of $q(x)$: $x=-2$. Graph this asymptote using a dashed line.

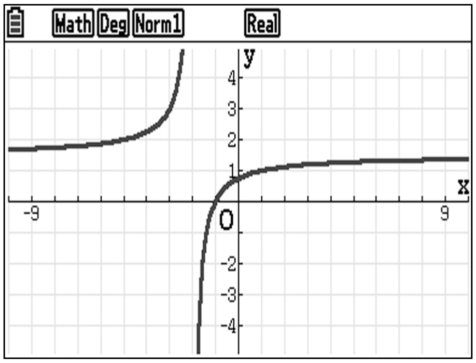
Step 5: Since $n=m$, the line $y=\dfrac{3}{2}$ is the horizontal asymptote.

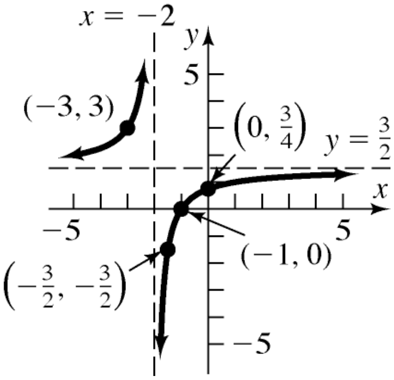
Solve to find intersection points:

$$\begin{aligned} \frac{3x+3}{2x+4}&=\frac{3}{2}\\ 2(3x+3)&=3(2x+4)\\ 6x+6&=6x+4\\ 0&\neq 2 \end{aligned}$$

$R(x)$ does not intersect $y=\dfrac{3}{2}$. Plot the line $y=\dfrac{3}{2}$ with dashes.

Steps 6 & 7: Graphing:





10. $R(x)=\dfrac{2x+4}{x-1}=\dfrac{2(x+2)}{x-1}$ $p(x)=2x+4;\ q(x)=x-1;\ n=1;\ m=1$

Step 1: Domain: $\{x \mid x \neq 1\}$

The *y*-intercept is $R(0)=\dfrac{2(0)+4}{0-1}=\dfrac{4}{-1}=-4$. Plot the point $(0,-4)$.

Step 2 & 3: *R* is in lowest terms. The *x*-intercept is the zero of $p(x)$: $x=-2$

Near -2, $R(x)\approx -\frac{2}{3}(x+2)$. Plot the point $(-2,0)$ and show a line with negative slope there.

Step 4: $R(x)=\dfrac{2x+4}{x-1}=\dfrac{2(x+2)}{x-1}$ is in lowest terms.

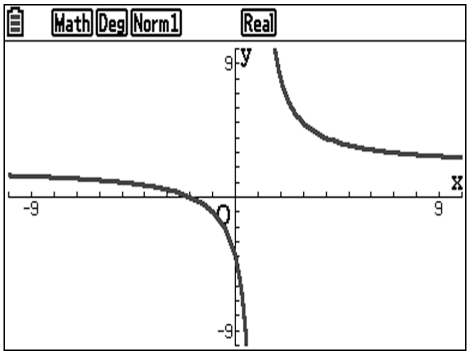
The vertical asymptote is the zero of $q(x)$: $x=1$. Graph this asymptote using a dashed line.

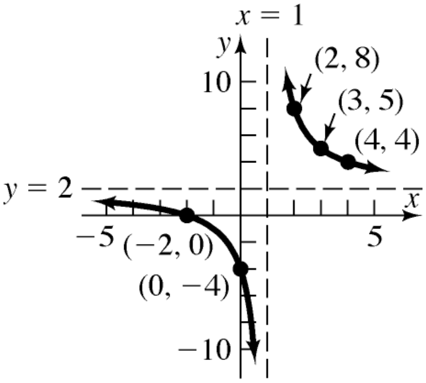
Step 5: Since $n=m$, the line $y=2$ is the horizontal asymptote. Solve to find intersection points:

$$\frac{2x+4}{x-1}=2$$
$$2x+4=2(x-1)$$
$$2x+4=2x-1$$
$$0\neq -5$$

$R(x)$ does not intersect $y=2$. Plot the line $y=2$ with dashes.

Steps 6 & 7: Graphing:





11. $R(x)=\dfrac{3}{x^2-4}=\dfrac{3}{(x-2)(x+2)}$ $p(x)=3;\ q(x)=x^2-4;\ n=0;\ m=2$

Step 1: Domain: $\{x \mid x \neq -2, x \neq 2\}$

The y-intercept is $R(0)=\dfrac{3}{0^2-4}=\dfrac{3}{-4}=-\dfrac{3}{4}$. Plot the point $\left(0,-\dfrac{3}{4}\right)$.

Step 2 & 3: R is in lowest terms. The x-intercepts are the zeros of $p(x)$. Since $p(x)$ is a constant, there are no x-intercepts.

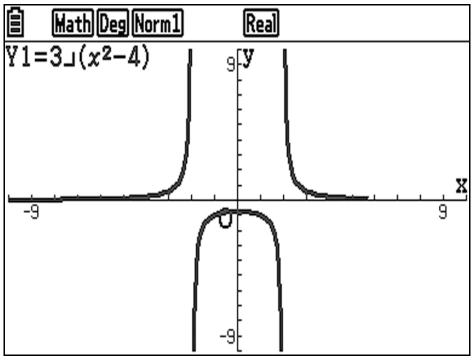
Step 4: $R(x)=\dfrac{3}{x^2-4}$ is in lowest terms. The vertical asymptotes are the zeros of $q(x)$: $x=-2$ and $x=2$. Graph each of these asymptotes using dashed lines.

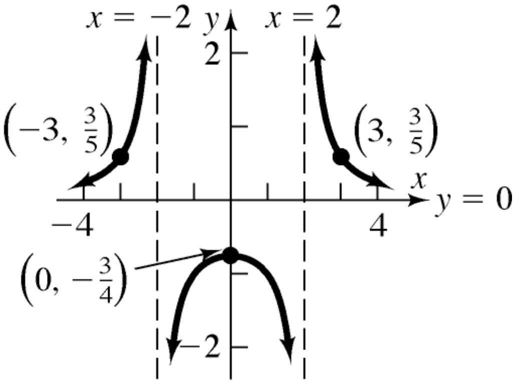
Step 5: Since $n<m$, the line $y=0$ is the horizontal asymptote. Solve to find intersection points:

$$\frac{3}{x^2-4}=0$$
$$3=0\left(x^2-4\right)$$
$$3\neq 0$$

$R(x)$ does not intersect $y=0$. Plot the line $y=0$ with dashes.

Steps 6 & 7: Graphing:





12. $R(x)=\dfrac{6}{x^2-x-6}=\dfrac{6}{(x-3)(x+2)}$ $\quad p(x)=6;\ q(x)=x^2-x-6;\ n=0;\ m=2$

Step 1: Domain: $\{x \mid x \neq -2, x \neq 3\}$

The y-intercept is $R(0)=\dfrac{6}{0^2-0-6}=\dfrac{6}{-6}=-1$. Plot the point $(0,-1)$.

Step 2 & 3: R is in lowest terms. The x-intercepts are the zeros of $p(x)$. Since $p(x)$ is a constant, there are no x-intercepts.

Step 4: $R(x)=\dfrac{6}{x^2-x-6}$ is in lowest terms. The vertical asymptotes are the zeros of $q(x)$: $x=-2$ and $x=3$ Graph each of these asymptotes using dashed lines.

Step 5: Since $n<m$, the line $y=0$ is the horizontal asymptote. Solve to find intersection points:

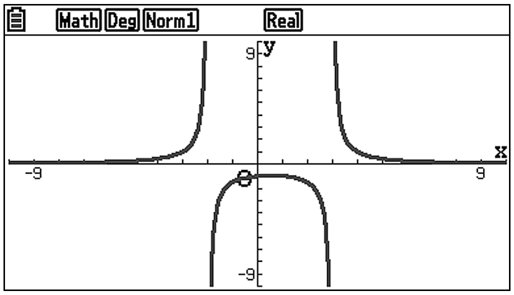
$$\frac{6}{x^2-x-6}=0$$

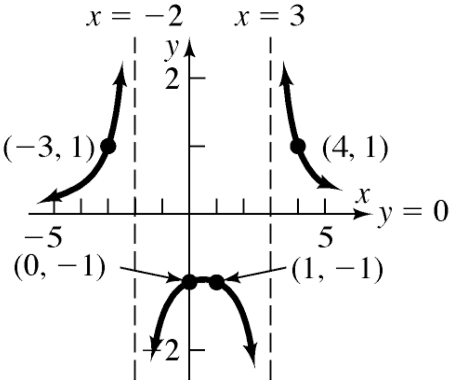
$$6=0\left(x^2-x-6\right)$$

$$6\neq 0$$

$R(x)$ does not intersect $y=0$.

Steps 6 & 7: Graphing:





13. $P(x)=\dfrac{x^4+x^2+1}{x^2-1}$ $p(x)=x^4+x^2+1;\ q(x)=x^2-1;\ n=4;\ m=2$

Step 1: Domain: $\{x \mid x \neq -1, x \neq 1\}$

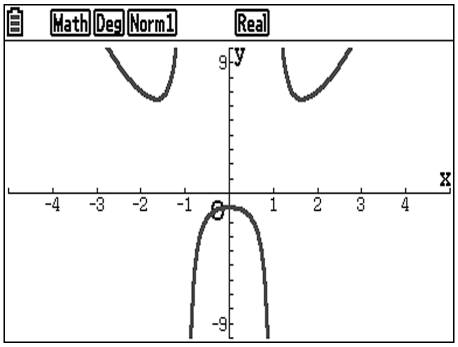
The y-intercept is $P(0)=\dfrac{0^4+0^2+1}{0^2-1}=\dfrac{1}{-1}=-1$. Plot the point $(0,-1)$.

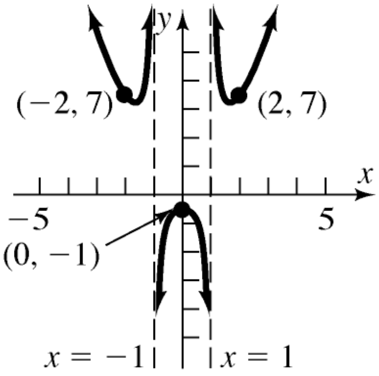
Step 2 & 3: $P(x)=\dfrac{x^4+x^2+1}{x^2-1}$ is in lowest terms. The x-intercept is the zero of $p(x)$. Since $p(x)$ is never 0, there are no x-intercepts.

Step 4: $P(x)=\dfrac{x^4+x^2+1}{x^2-1}$ is in lowest terms. The vertical asymptotes are the zeros of $q(x)$: $x=-1$ and $x=1$. Graph each of these asymptotes using dashed lines.

Step 5: Since $n>m+1$, there is no horizontal or oblique asymptote.

Steps 6 & 7: Graphing:





14. $Q(x)=\dfrac{x^4-1}{x^2-4}=\dfrac{(x^2+1)(x+1)(x-1)}{(x+2)(x-2)}$ $\quad p(x)=x^4-1;\ q(x)=x^2-4;\ n=4;\ m=2$

Step 1: Domain: $\{x\,|\,x\neq -2, x\neq 2\}$

The y-intercept is $Q(0)=\dfrac{0^4-1}{0^2-4}=\dfrac{-1}{-4}=\dfrac{1}{4}$. Plot the point $\left(0,\dfrac{1}{4}\right)$.

Step 2 & 3: $Q(x)=\dfrac{x^4-1}{x^2-4}=\dfrac{(x^2+1)(x+1)(x-1)}{(x+2)(x-2)}$ is in lowest terms. The x-intercepts are the zeros of $p(x)$: -1 and 1.

Near -1, $Q(x)\approx\dfrac{4}{3}(x+1)$; Near 1, $Q(x)\approx-\dfrac{4}{3}(x-1)$.

Plot the point $(-1,0)$ and indicate a line with positive slope there.

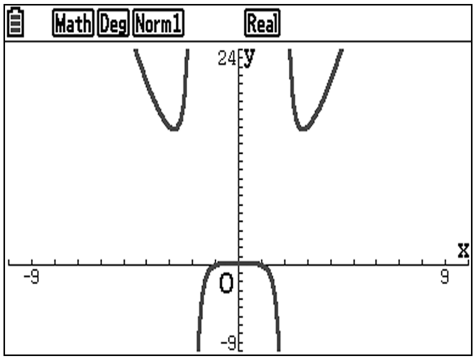
Plot the point $(1,0)$ and indicate a line with negative slope there.

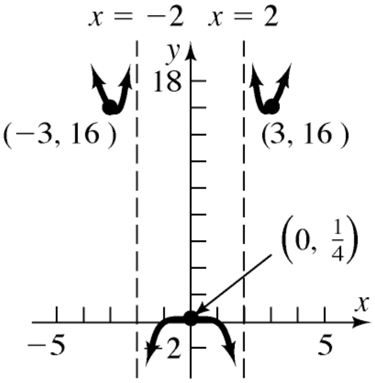
Step 4: $Q(x)=\dfrac{x^4-1}{x^2-4}=\dfrac{(x^2+1)(x+1)(x-1)}{(x+2)(x-2)}$ is in lowest terms.

The vertical asymptotes are the zeros of $q(x)$: $x=-2$ and $x=2$. Graph each of these asymptotes using dashed lines.

Step 5: Since $n>m+1$, there is no horizontal asymptote and no oblique asymptote.

Steps 6 & 7: Graphing:





15. $H(x)=\dfrac{x^3-1}{x^2-9}=\dfrac{(x-1)\left(x^2+x+1\right)}{(x+3)(x-3)}$ $\quad p(x)=x^3-1;\ q(x)=x^2-9;\ n=3;\ m=2$

Step 1: Domain: $\{x\,|\,x\neq -3, x\neq 3\}$

The y-intercept is $H(0)=\dfrac{0^3-1}{0^2-9}=\dfrac{-1}{-9}=\dfrac{1}{9}$. Plot the point $\left(0,\dfrac{1}{9}\right)$.

Step 2 & 3: $H(x)$ is in lowest terms. The x-intercept is the zero of $p(x)$: 1.

Near 1, $H(x)\approx-\dfrac{3}{8}(x-1)$. Plot the point $(1,0)$ and indicate a line with negative slope there.

Step 4: $H(x)$ is in lowest terms. The vertical asymptotes are the zeros of $q(x)$: $x=-3$ and $x=3$. Graph each of these asymptotes using dashed lines.

Step 5: Since $n=m+1$, there is an oblique asymptote. Dividing:

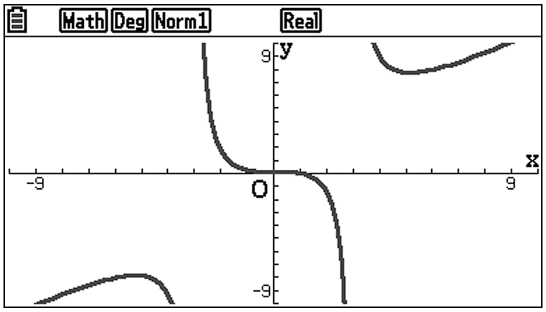
$$\begin{array}{r|l} & x \\ \hline x^2-9 & x^3+0x^2+0x-1 \\ & \underline{x^3 \qquad -9x} \\ & \qquad\qquad 9x-1 \end{array} \qquad\qquad H(x)=x+\frac{9x-1}{x^2-9}$$

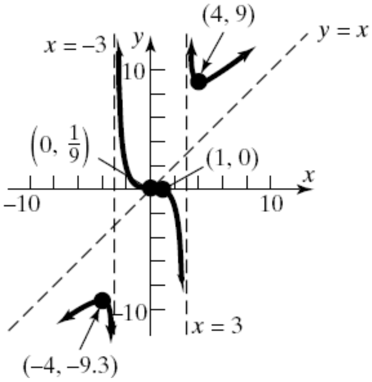
The oblique asymptote is $y=x$. Graph this asymptote with a dashed line. Solve to find intersection points:

$$\frac{x^3-1}{x^2-9}=x$$
$$x^3-1=x^3-9x$$
$$-1=-9x$$
$$x=\frac{1}{9}$$

The oblique asymptote intersects $H(x)$ at $\left(\frac{1}{9},\frac{1}{9}\right)$.

Steps 6 & 7: Graphing:





16. $G(x)=\dfrac{x^3+1}{x^2+2x}=\dfrac{(x+1)(x^2-x+1)}{x(x+2)}$ $\quad p(x)=x^3+1;\ q(x)=x^2+2x;\ n=3;\ m=2$

Step 1: Domain: $\{x \mid x\neq -2, x\neq 0\}$

There is no y-intercept since $G(0)=\dfrac{0^3+1}{0^2+2(0)}=\dfrac{1}{0}$.

Step 2 & 3: $G(x)=\dfrac{x^3+1}{x^2+2x}$ is in lowest terms. The x-intercept is the zero of $p(x)$: -1.

Near -1, $G(x)\approx -3(x+1)$. Plot the point $(-1,0)$ and indicate a line with negative slope there.

Step 4: $G(x)=\dfrac{x^3+1}{x^2+2x}$ is in lowest terms. The vertical asymptotes are the zeros of $q(x)$: $x=-2$ and $x=0$. Graph each of these asymptotes using dashed lines.

Step 5: Since $n=m+1$, there is an oblique asymptote. Dividing:

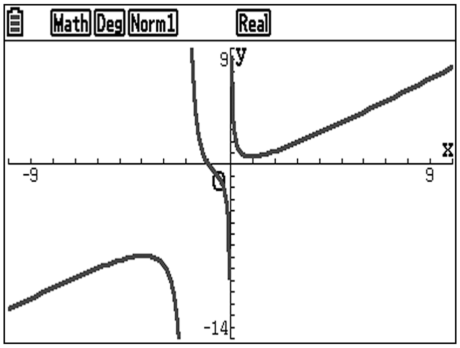
$$\begin{array}{r|l} & x-2 \\ \hline x^2+2x & x^3+0x^2+0x+1 \\ & \underline{x^3+2x^2} \\ & \quad -2x^2 \qquad +1 \\ & \quad \underline{-2x^2-4x} \\ & \qquad\quad 4x+1 \end{array} \qquad G(x)=x-2+\frac{4x+1}{x^2+2x}$$

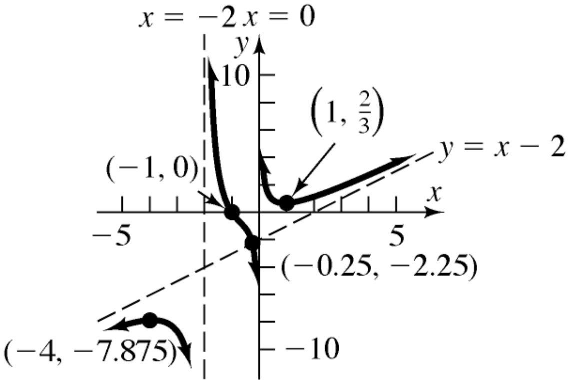
The oblique asymptote is $y=x-2$. Graph this asymptote with a dashed line. Solve to find intersection points:

$$\begin{aligned} \frac{x^3+1}{x^2+2x}&=x-2 \\ x^3+1&=x^3-4x \\ 1&=-4x \\ x&=-\frac{1}{4} \end{aligned}$$

The oblique asymptote intersects $G(x)$ at $\left(-\frac{1}{4},-\frac{9}{4}\right)$.

Steps 6 & 7: Graphing:





17. $R(x)=\dfrac{x^2}{x^2+x-6}=\dfrac{x^2}{(x+3)(x-2)}$ $\quad p(x)=x^2;\ q(x)=x^2+x-6;\ n=2;\ m=2$

Step 1: Domain: $\{x \mid x\neq -3,\ x\neq 2\}$

The y-intercept is $R(0)=\dfrac{0^2}{0^2+0-6}=\dfrac{0}{-6}=0$. Plot the point $(0,0)$.

Step 2 & 3: $R(x)=\dfrac{x^2}{x^2+x-6}$ is in lowest terms. The x-intercept is the zero of $p(x)$: 0

Near 0, $R(x)\approx -\dfrac{1}{6}x^2$. Plot the point $(0,0)$ and indicate a parabola opening down there.

Step 4: $R(x)=\dfrac{x^2}{x^2+x-6}$ is in lowest terms. The vertical asymptotes are the zeros of $q(x)$:

$x=-3$ and $x=2$. Graph each of these asymptotes using dashed lines.

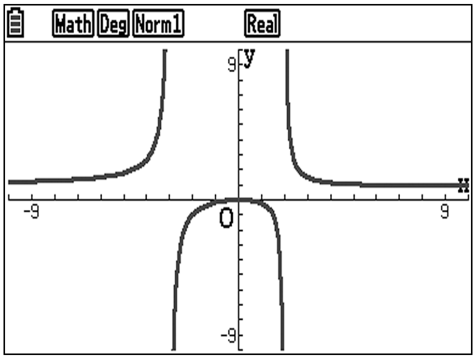
Step 5: Since $n=m$, the line $y=1$ is the horizontal asymptote. Graph this asymptote with a dashed line.

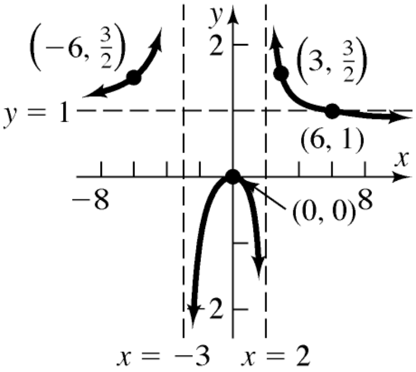
Solve to find intersection points:

$$\begin{aligned}\frac{x^2}{x^2+x-6}&=1\\ x^2&=x^2+x-6\\ 0&=x-6\\ x&=6\end{aligned}$$

$R(x)$ intersects $y=1$ at (6, 1).

Steps 6 & 7: Graphing:





18. $R(x)=\dfrac{x^2+x-12}{x^2-4}=\dfrac{(x+4)(x-3)}{(x+2)(x-2)}$ $p(x)=x^2+x-12;\ q(x)=x^2-4;\ n=2;\ m=2$

Step 1: Domain: $\{x \mid x\neq -2, x\neq 2\}$

The y-intercept is $R(0)=\dfrac{0^2+0-12}{0^2-4}=\dfrac{-12}{-4}=3$. Plot the point $(0,3)$.

Step 2 & 3: $R(x)=\dfrac{x^2+x-12}{x^2-4}=\dfrac{(x+4)(x-3)}{(x+2)(x-2)}$ is in lowest terms.

The x-intercepts are the zeros of $p(x)$: –4 and 3

Near –4, $R(x)\approx -\dfrac{7}{12}(x+4)$; Near 3, $R(x)\approx \dfrac{7}{5}(x-3)$.

Plot the point $\left(-4,0\right)$ and indicate a line with negative slope there.

Plot the point $\left(3,0\right)$ and indicate a line with positive slope there.

Step 4: $R(x)=\dfrac{x^2+x-12}{x^2-4}$ is in lowest terms.

The vertical asymptotes are the zeros of $q(x)$: $x=-2$ and $x=2$. Graph each of these asymptotes using a dashed line.

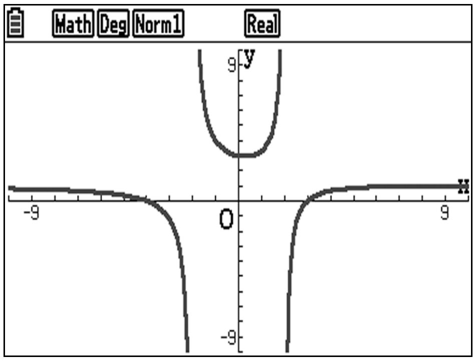
Step 5: Since $n=m$, the line $y=1$ is the horizontal asymptote. Graph this asymptote using a dashed line.

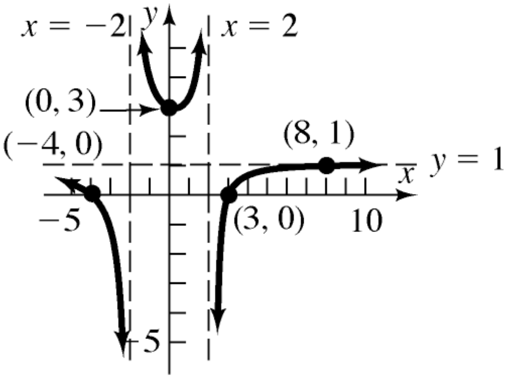
Solve to find intersection points:

$$\frac{x^2+x-12}{x^2-4}=1$$
$$x^2+x-12=x^2-4$$
$$x=8$$

$R(x)$ intersects $y=1$ at (8, 1).

Steps 6 & 7: Graphing:





19. $G(x)=\dfrac{x}{x^2-4}=\dfrac{x}{(x+2)(x-2)}$ $\quad p(x)=x;\ q(x)=x^2-4;\ n=1;\ m=2$

Step 1: Domain: $\left\{x\,\middle|\, x\neq -2, x\neq 2\right\}$

The y-intercept is $G(0)=\dfrac{0}{0^2-4}=\dfrac{0}{-4}=0$. Plot the point $\left(0,0\right)$.

Step 2 & 3: $G(x)=\dfrac{x}{x^2-4}$ is in lowest terms. The x-intercept is the zero of $p(x)$: 0

Near 0, $G\left(x\right)\approx -\dfrac{1}{4}x$. Plot the point $\left(0,0\right)$ and indicate a line with negative slope there.

Step 4: $G(x)=\dfrac{x}{x^2-4}$ is in lowest terms. The vertical asymptotes are the zeros of $q(x)$: $x=-2$ and $x=2$.

Graph each of these asymptotes using a dashed line.

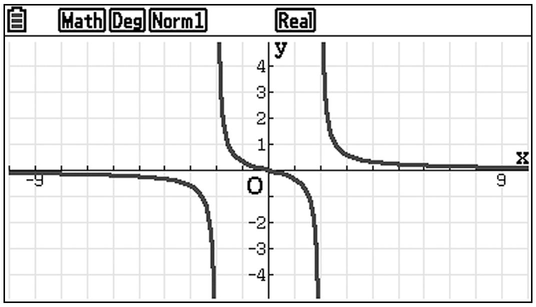
Step 5: Since $n < m$, the line $y = 0$ is the horizontal asymptote. Graph this asymptote using a dashed line. Solve to find intersection points:

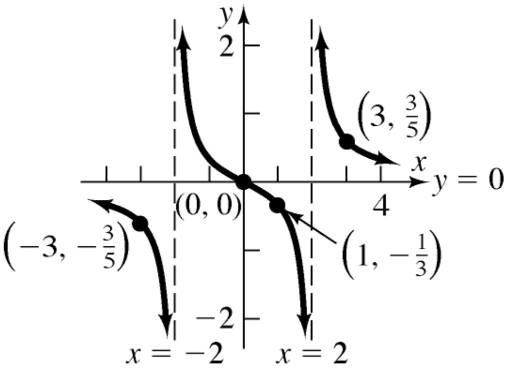
$$\frac{x}{x^2 - 4} = 0$$

$$x = 0$$

$G(x)$ intersects $y = 0$ at (0, 0).

Steps 6 & 7: Graphing:





20. $G(x) = \dfrac{3x}{x^2 - 1} = \dfrac{3x}{(x+1)(x-1)}$ $p(x) = 3x;\ q(x) = x^2 - 1;\ n = 1;\ m = 2$

Step 1: Domain: $\{x \mid x \neq -1, x \neq 1\}$

The y-intercept is $G(0) = \dfrac{3(0)}{0^2 - 1} = \dfrac{0}{-1} = 0$. Plot the point $(0,0)$.

Step 2 & 3: $G(x) = \dfrac{3x}{x^2 - 1}$ is in lowest terms. The x-intercept is the zero of $p(x)$: 0

Near 0, $G(x) \approx -3x$. Plot the point $(0,0)$ and indicate a line with negative slope there.

Step 4: $G(x) = \dfrac{3x}{x^2 - 1}$ is in lowest terms. The vertical asymptotes are the zeros of $q(x)$: $x = -1$ and $x = 1$

Graph each of these asymptotes using a dashed line.

Step 5: Since $n < m$, the line $y = 0$ is the horizontal asymptote. Graph this asymptote using a dashed line. Solve to find intersection points:

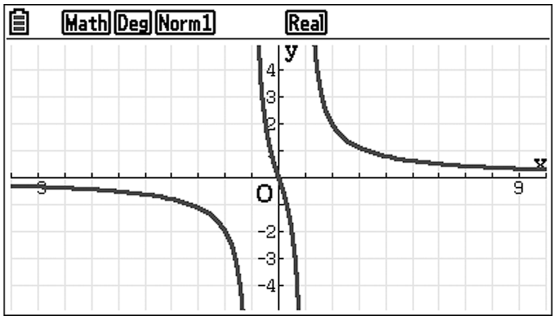
$$\frac{3x}{x^2 - 1} = 0$$

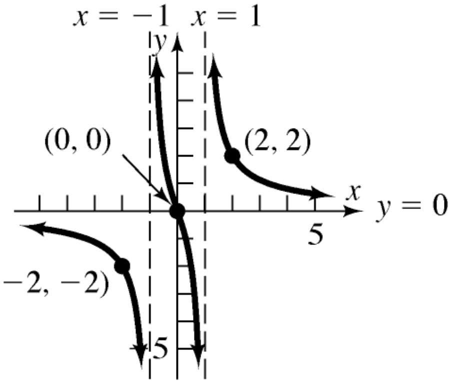
$$3x = 0$$

$$x = 0$$

$G(x)$ intersects $y = 0$ at (0, 0).

Steps 6 & 7: Graphing:





21. $R(x)=\dfrac{3}{(x-1)(x^2-4)}=\dfrac{3}{(x-1)(x+2)(x-2)}$ $p(x)=3;$ $q(x)=(x-1)(x^2-4);$ $n=0;$ $m=3$

Step 1: Domain: $\{x \mid x \neq -2, x \neq 1, x \neq 2\}$

The y-intercept is $R(0)=\dfrac{3}{(0-1)(0^2-4)}=\dfrac{3}{4}$. Plot the point $\left(0,\dfrac{3}{4}\right)$.

Step 2 & 3: $R(x)=\dfrac{3}{(x-1)(x^2-4)}$ is in lowest terms. There is no x-intercept.

Step 4: $R(x)=\dfrac{3}{(x-1)(x^2-4)}$ is in lowest terms.

The vertical asymptotes are the zeros of $q(x)$: $x=-2, x=1,$ and $x=2$.

Graph each of these asymptotes using a dashed line.

Step 5: Since $n<m$, the line $y=0$ is the horizontal asymptote. Graph this asymptote with a dashed line.

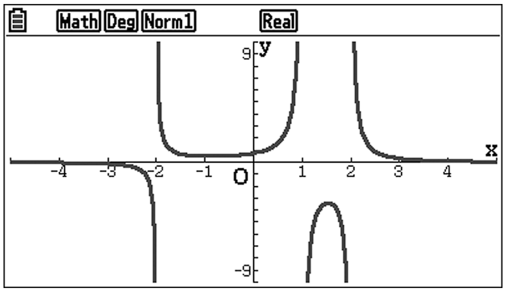
Solve to find intersection points:

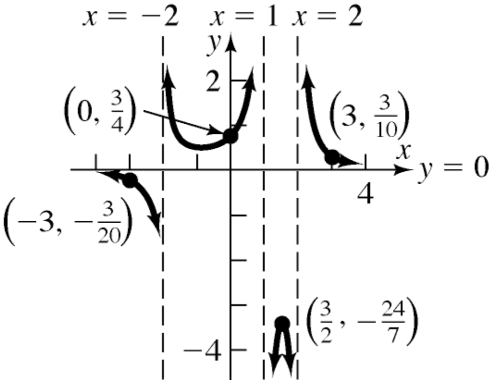
$$\frac{3}{(x-1)(x^2-4)}=0$$

$$3 \neq 0$$

$R(x)$ does not intersect $y=0$.

Steps 6 & 7: Graphing:





22. $R(x) = \dfrac{-4}{(x+1)(x^2-9)} = \dfrac{-4}{(x+1)(x+3)(x-3)}$ $\quad p(x) = -4;\ q(x) = (x+1)(x^2-9); \quad n = 0;\ m = 3$

Step 1: Domain: $\{x \mid x \neq -3, x \neq -1, x \neq 3\}$

The y-intercept is $R(0) = \dfrac{-4}{(0+1)(0^2-9)} = \dfrac{-4}{-9} = \dfrac{4}{9}$. Plot the point $\left(0, \dfrac{4}{9}\right)$.

Step 2 & 3: $R(x) = \dfrac{-4}{(x+1)(x^2-9)}$ is in lowest terms. There is no x-intercept.

Step 4: $R(x) = \dfrac{-4}{(x+1)(x^2-9)}$ is in lowest terms.

The vertical asymptotes are the zeros of $q(x)$: $x = -3, x = -1,$ and $x = 3$

Graph each of these asymptotes using a dashed line.

Step 5: Since $n < m$, the line $y = 0$ is the horizontal asymptote. Graph this asymptote with a dashed line.

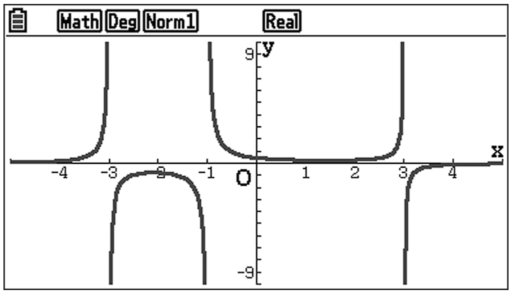
Solve to find intersection points:

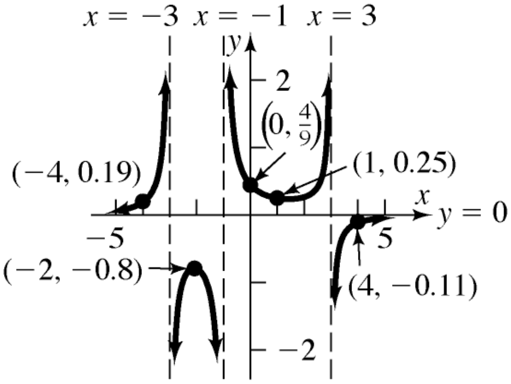
$$\frac{-4}{(x+1)(x^2-9)} = 0$$

$$-4 \neq 0$$

$R(x)$ does not intersect $y = 0$.

Steps 6 & 7: Graphing:





23. $H(x)=\dfrac{x^2-1}{x^4-16}=\dfrac{(x-1)(x+1)}{(x^2+4)(x+2)(x-2)}$ $\quad p(x)=x^2-1;\ q(x)=x^4-16;\quad n=2;\ m=4$

Step 1: Domain: $\{x \mid x\neq -2, x\neq 2\}$

The y-intercept is $H(0)=\dfrac{0^2-1}{0^4-16}=\dfrac{-1}{-16}=\dfrac{1}{16}$. Plot the point $\left(0,\dfrac{1}{16}\right)$.

Step 2 & 3: $H(x)=\dfrac{x^2-1}{x^4-16}$ is in lowest terms. The x-intercepts are the zeros of $p(x)$: –1 and 1

Near -1, $H(x)\approx \dfrac{2}{15}(x+1)$; Near 1, $H(x)\approx -\dfrac{2}{15}(x-1)$

Plot $(-1,0)$ and indicate a line with positive slope there.

Plot $(1,0)$ and indicate a line with negative slope there.

Step 4: $H(x)=\dfrac{x^2-1}{x^4-16}$ is in lowest terms. The vertical asymptotes are the zeros of $q(x)$: $x=-2$ and $x=2$

Graph each of these asymptotes using a dashed line.

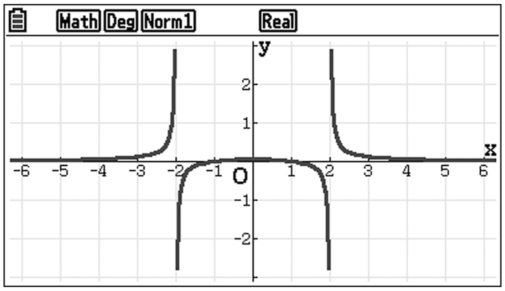
Step 5: Since $n<m$, the line $y=0$ is the horizontal asymptote. Graph this asymptote using a dashed line.

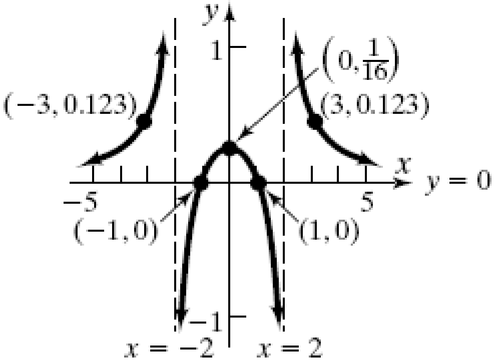
Solve to find intersection points:

$$\frac{x^2-1}{x^4-16}=0$$
$$x^2-1=0$$
$$x=\pm 1$$

$H(x)$ intersects $y=0$ at (–1, 0) and (1, 0).

Steps 6 & 7: Graphing:





24. $H(x)=\dfrac{x^2+4}{x^4-1}=\dfrac{x^2+4}{(x^2+1)(x+1)(x-1)}$ $p(x)=x^2+4;\ q(x)=x^4-1;\quad n=2;\ m=4$

Step 1: Domain: $\{x \mid x \neq -1,\ x \neq 1\}$

The y-intercept is $H(0)=\dfrac{0^2+4}{0^4-1}=\dfrac{4}{-1}=-4$. Plot the point $(0,-4)$.

Step 2 & 3: $H(x)=\dfrac{x^2+4}{x^4-1}$ is in lowest terms. There are no x-intercepts.

Step 4: $H(x)=\dfrac{x^2+4}{x^4-1}$ is in lowest terms. The vertical asymptotes are the zeros of $q(x)$: $x=-1$ and $x=1$

Graph each of these asymptotes using a dashed line.

Step 5: Since $n<m$, the line $y=0$ is the horizontal asymptote. Graph this asymptote using a dashed line.

Solve to find intersection points:

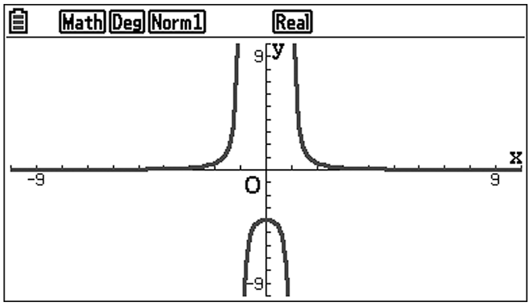
$$\frac{x^2+4}{x^4-1}=0$$

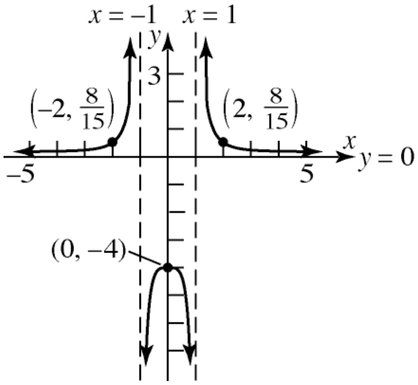
$$x^2+4=0$$

no real solution

$H(x)$ does not intersect $y=0$.

Steps 6 & 7: Graphing:





25. $F(x)=\dfrac{x^2-3x-4}{x+2}=\dfrac{(x+1)(x-4)}{x+2}$ $\quad p(x)=x^2-3x-4;\ q(x)=x+2;\ n=2;\ m=1$

Step 1: Domain: $\{x \mid x \neq -2\}$

The y-intercept is $F(0)=\dfrac{0^2-3(0)-4}{0+2}=\dfrac{-4}{2}=-2$. Plot the point $(0,-2)$.

Step 2 & 3: $F(x)=\dfrac{x^2-3x-4}{x+2}$ is in lowest terms. The x-intercepts are the zeros of $p(x)$: -1 and 4.

Near -1, $F(x)\approx -5(x+1)$; Near 4, $F(x)\approx \dfrac{5}{6}(x-4)$.

Plot $(-1,0)$ and indicate a line with negative slope there.

Plot $(4,0)$ and indicate a line with positive slope there.

Step 4: $F(x)=\dfrac{x^2-3x-4}{x+2}$ is in lowest terms. The vertical asymptote is the zero of $q(x)$: $x=-2$

Graph this asymptote using a dashed line.

Step 5: Since $n=m+1$, there is an oblique asymptote. Dividing:

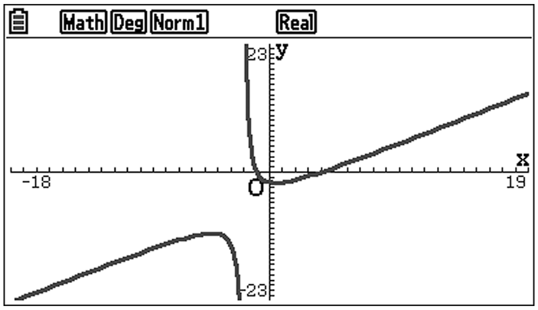
$$\begin{array}{r@{}l}
 & x-5 \\ \cline{2-2}
x+2\big) & x^2-3x-4 \\
 & \underline{x^2+2x} \\
 & \quad -5x-\ \ 4 \\
 & \quad \underline{-5x-10} \\
 & \qquad\qquad 6
\end{array}
\qquad\qquad F(x)=x-5+\frac{6}{x+2}$$

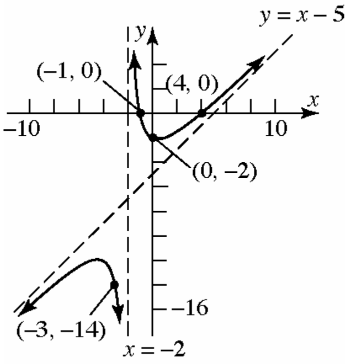
The oblique asymptote is $y=x-5$. Graph this asymptote using a dashed line. Solve to find intersection points:

$$\frac{x^2-3x-4}{x+2}=x-5$$
$$x^2-3x-4=x^2-3x-10$$
$$-4\neq-10$$

The oblique asymptote does not intersect $F(x)$.

Steps 6 & 7: Graphing:





26. $F(x)=\dfrac{x^2+3x+2}{x-1}=\dfrac{(x+2)(x+1)}{x-1}\quad p(x)=x^2+3x+2;\ q(x)=x-1;\ n=2;\ m=1$

Step 1: Domain: $\{x\,|\,x\neq1\}$

The y-intercept is $F(0)=\dfrac{0^2+3(0)+2}{0-1}=\dfrac{2}{-1}=-2$. Plot the point $(0,-2)$.

Step 2 & 3: $F(x)=\dfrac{x^2+3x+2}{x-1}$ is in lowest terms. The x-intercepts are the zeros of $p(x)$: -2 and -1.

Near -2, $F(x)\approx\frac{1}{3}(x+2)$; Near -1, $F(x)\approx-\frac{1}{2}(x+1)$.

Plot $(-2,0)$ and indicate a line with positive slope there.

Plot $(-1,0)$ and indicate a line with negative slope there.

Step 4: $F(x)=\dfrac{x^2+3x+2}{x-1}$ is in lowest terms. The vertical asymptote is the zero of $q(x)$: $x=1$

Graph this asymptote using a dashed line.

Step 5: Since $n = m+1$, there is an oblique asymptote. Dividing:

$$\begin{array}{r|l} & x+4 \\ \hline x-1 & x^2+3x+2 \\ & \underline{x^2-\ \ x} \\ & \quad 4x+2 \\ & \quad \underline{4x-4} \\ & \qquad 6 \end{array} \qquad F(x) = x+4+\frac{6}{x-1}$$

The oblique asymptote is $y = x+4$. Graph this asymptote using a dashed line. Solve to find intersection points:

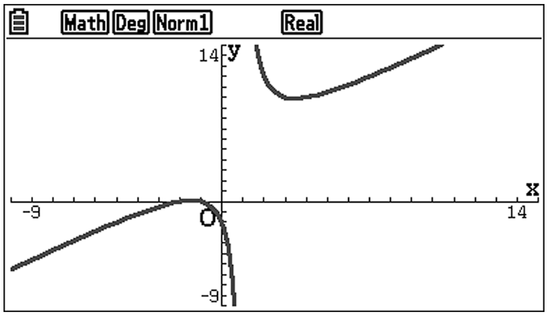
$$\frac{x^2+3x+2}{x-1} = x+4$$

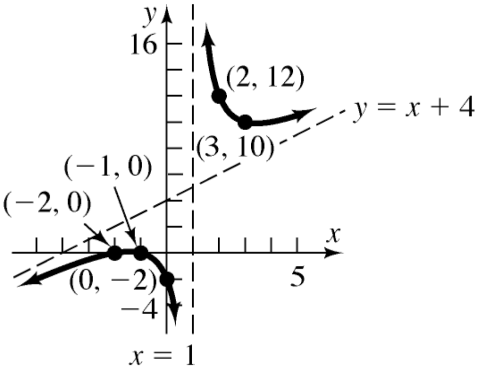
$$x^2+3x+2 = x^2+3x-4$$

$$2 \neq -4$$

The oblique asymptote does not intersect $F(x)$.

Steps 6 & 7: Graphing:





27. $R(x) = \dfrac{x^2+x-12}{x-4} = \dfrac{(x+4)(x-3)}{x-4}$ $\quad p(x) = x^2+x-12;\ q(x) = x-4;\ n = 2;\ m = 1$

Step 1: Domain: $\{x \mid x \neq 4\}$

The y-intercept is $R(0) = \dfrac{0^2+0-12}{0-4} = \dfrac{-12}{-4} = 3$. Plot the point $(0,3)$.

Step 2 & 3: $R(x) = \dfrac{x^2+x-12}{x-4}$ is in lowest terms. The x-intercepts are the zeros of $p(x)$: –4 and 3.

Near -4, $R(x) \approx \frac{7}{8}(x+4)$; Near 3, $R(x) \approx -7(x-3)$.

Plot $(-4,0)$ and indicate a line with positive slope there.

Plot $(3,0)$ and indicate a line with negative slope there.

Step 4: $R(x)=\dfrac{x^2+x-12}{x-4}$ is in lowest terms. The vertical asymptote is the zero of $q(x)$: $x=4$

Graph this asymptote using a dashed line.

Step 5: Since $n=m+1$, there is an oblique asymptote. Dividing:

$$\begin{array}{r|l} & \;\; x+5 \\ \hline x-4 & x^2+\;\;x-12 \\ & \underline{x^2-4x} \\ & \qquad 5x-\;12 \\ & \qquad \underline{5x-20} \\ & \qquad\qquad\;\; 8 \end{array} \qquad\qquad R(x)=x+5+\frac{8}{x-4}$$

The oblique asymptote is $y=x+5$. Graph this asymptote using a dashed line. Solve to find intersection points:

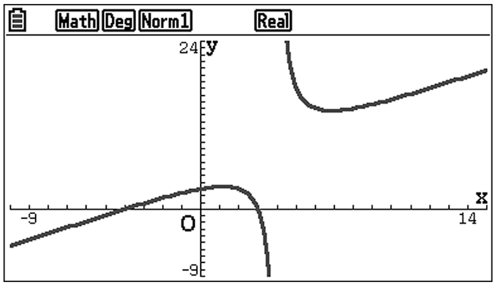
$$\frac{x^2+x-12}{x-4}=x+5$$

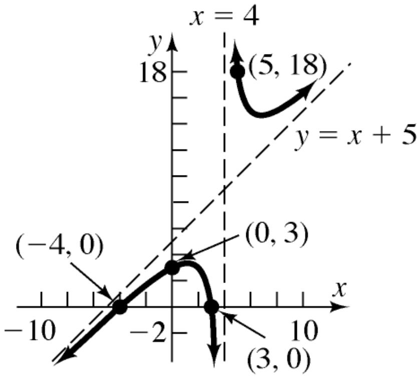
$$x^2+x-12=x^2+x-20$$

$$-12\neq-20$$

The oblique asymptote does not intersect $R(x)$.

Steps 6 & 7: Graphing:





28. $R(x)=\dfrac{x^2-x-12}{x+5}=\dfrac{(x-4)(x+3)}{x+5}$ $\quad p(x)=x^2-x-12;\; q(x)=x+5;\; n=2;\; m=1$

Step 1: Domain: $\{x\,|\,x\neq-5\}$

The y-intercept is $R(0)=\dfrac{0^2-0-12}{0+5}=-\dfrac{12}{5}$. Plot the point $\left(0,-\dfrac{12}{5}\right)$.

Step 2 & 3: $R(x)=\dfrac{x^2-x-12}{x+5}$ is in lowest terms. The x-intercepts are the zeros of $p(x)$: –3 and 4.

Near -3, $R(x)\approx-\dfrac{7}{2}(x+3)$; Near 4, $R(x)\approx\dfrac{7}{9}(x-4)$.

Plot $(-3,0)$ and indicate a line with negative slope there.

Plot $(4,0)$ and indicate a line with positive slope there.

Step 4: $R(x)=\dfrac{x^2-x-12}{x+5}$ is in lowest terms. The vertical asymptote is the zero of $q(x)$: $x=-5$

Graph this asymptote using a dashed line.

Step 5: Since $n=m+1$, there is an oblique asymptote. Dividing:

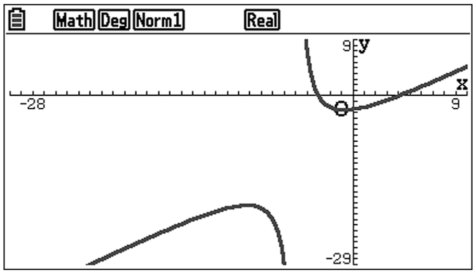
$$\begin{array}{r|l} & x-6 \\ \hline x+5 & x^2-\ \ x-12 \\ & \underline{x^2+5x} \\ & \quad -6x-12 \\ & \quad \underline{-6x-30} \\ & \qquad\quad 18 \end{array} \qquad R(x)=x-6+\frac{18}{x+5}$$

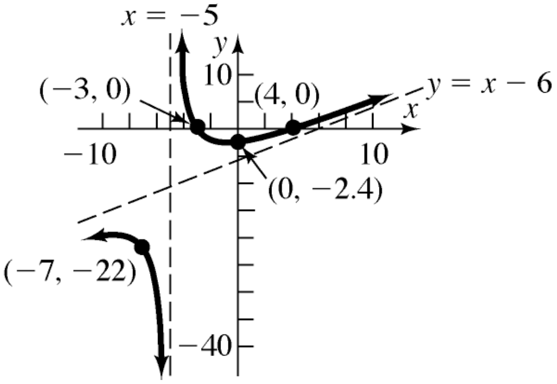
The oblique asymptote is $y=x-6$. Graph this asymptote using a dashed line. Solve to find intersection points:

$$\frac{x^2-x-12}{x+5}=x-6$$
$$x^2-x-12=x^2-x-30$$
$$-12\neq-30$$

The oblique asymptote does not intersect $R(x)$.

Steps 6 & 7: Graphing:





29. $F(x)=\dfrac{x^2+x-12}{x+2}=\dfrac{(x+4)(x-3)}{x+2}$ $\quad p(x)=x^2+x-12;\ q(x)=x+2;\ n=2;\ m=1$

Step 1: Domain: $\{x \mid x\neq-2\}$

The y-intercept is $F(0)=\dfrac{0^2+0-12}{0+2}=\dfrac{-12}{2}=-6$. Plot the point $(0,-6)$.

Step 2 & 3: $F(x)=\dfrac{x^2+x-12}{x+2}$ is in lowest terms. The x-intercepts are the zeros of $p(x)$: –4 and 3.

Near -4, $F(x)\approx\dfrac{7}{2}(x+4)$; Near 3, $F(x)\approx\dfrac{7}{5}(x-3)$.

Plot $(-4,0)$ and indicate a line with positive slope there.

Plot $(3,0)$ and indicate a line with positive slope there.

Step 4: $F(x)=\dfrac{x^2+x-12}{x+2}$ is in lowest terms. The vertical asymptote is the zero of $q(x)$: $x=-2$

Graph this asymptote using a dashed line.

Step 5: Since $n=m+1$, there is an oblique asymptote. Dividing:

$$\begin{array}{r|l} & x-1 \\ \hline x+2 & x^2+\ \ x-12 \\ & \underline{x^2+2x} \\ & \quad -x-12 \\ & \quad \underline{-x-\ 2} \\ & \qquad -10 \end{array} \qquad F(x)=x-1+\frac{-10}{x+2}$$

The oblique asymptote is $y=x-1$. Graph this asymptote using a dashed line. Solve to find intersection points:

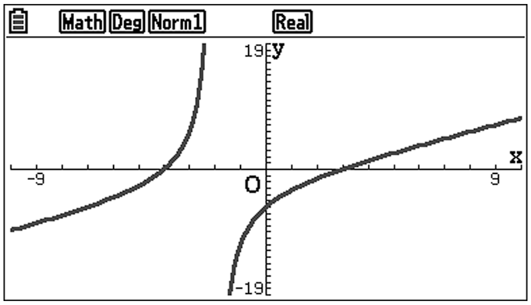
$$\frac{x^2+x-12}{x+2}=x-1$$

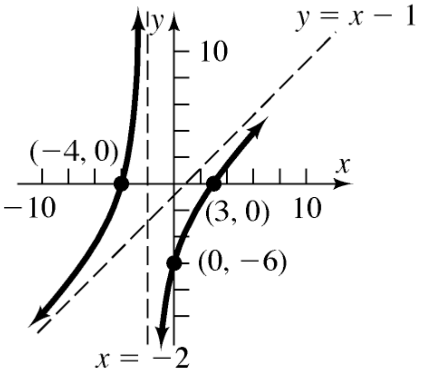
$$x^2+x-12=x^2+x-2$$

$$-12\neq-2$$

The oblique asymptote does not intersect $F(x)$.

Steps 6 & 7: Graphing:





30. $G(x)=\dfrac{x^2-x-12}{x+1}=\dfrac{(x+3)(x-4)}{x+1}$ $\quad p(x)=x^2-x-12;\ q(x)=x+1;\ n=2;\ m=1$

Step 1: Domain: $\{x \mid x\neq -1\}$

The y-intercept is $F(0)=\dfrac{0^2-0-12}{0+1}=\dfrac{-12}{1}=-12$. Plot the point $(0,-12)$.

Step 2 & 3: $G(x)=\dfrac{x^2-x-12}{x+1}$ is in lowest terms. The x-intercepts are the zeros of $p(x)$: –3 and 4.

Near -3, $G(x)\approx\dfrac{7}{2}(x+3)$; Near 4, $G(x)\approx\dfrac{7}{5}(x-4)$.

Plot $(-3,0)$ and indicate a line with positive slope there.

Plot $(4,0)$ and indicate a line with positive slope there.

Step 4: $G(x)=\dfrac{x^2-x-12}{x+1}$ is in lowest terms. The vertical asymptote is the zero of $q(x)$: $x=-1$

Graph this asymptote using a dashed line.

Step 5: Since $n=m+1$, there is an oblique asymptote. Dividing:

$$\begin{array}{r|l} & x-2 \\ \hline x+1 & x^2-x-12 \\ & \underline{x^2+x} \\ & \quad -2x-12 \\ & \quad \underline{-2x-\ 2} \\ & \qquad\ -10 \end{array}$$

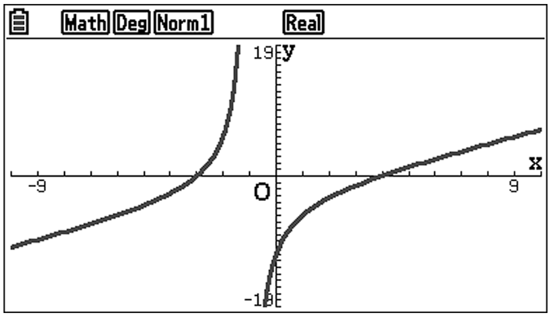
$$G(x)=x-2+\frac{-10}{x+1}$$

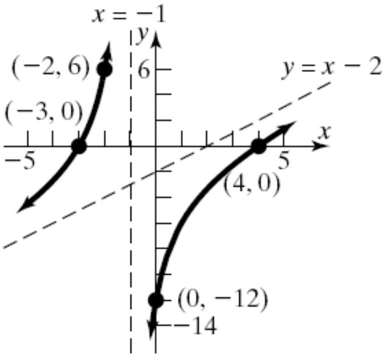
The oblique asymptote is $y=x-2$. Graph this asymptote using a dashed line. Solve to find intersection points:

$$\frac{x^2-x-12}{x+1}=x-2$$
$$x^2-x-12=x^2-x-2$$
$$-12\neq -2$$

The oblique asymptote does not intersect $G(x)$.

Steps 6 & 7: Graphing:





31. $R(x)=\dfrac{x(x-1)^2}{(x+3)^3}$ $\quad p(x)=x(x-1)^2;\ q(x)=(x+3)^3;\ n=3;\ m=3$

Step 1: Domain: $\{x \mid x\neq -3\}$

The y-intercept is $R(0)=\dfrac{0(0-1)^2}{(0+3)^3}=\dfrac{0}{27}=0$. Plot the point $(0,0)$.

Step 2 & 3: $R(x)=\dfrac{x(x-1)^2}{(x+3)^3}$ is in lowest terms. The x-intercepts are the zeros of $p(x)$: 0 and 1

Near 0, $R(x)\approx\dfrac{1}{27}x$; Near 1, $R(x)\approx\dfrac{1}{64}(x-1)^2$.

Plot $(0,0)$ and indicate a line with positive slope there.

Plot $(1,0)$ and indicate a parabola that opens up there.

Step 4: $R(x)=\dfrac{x(x-1)^2}{(x+3)^3}$ is in lowest terms. The vertical asymptote is the zero of $q(x)$: $x=-3$

Graph this asymptote with a dashed line.

Step 5: Since $n=m$, the line $y=1$ is the horizontal asymptote. Graph this asymptote with a dashed line. Solve to find intersection points:

$$\frac{x(x-1)^2}{(x+3)^3}=1$$

$$x^3-2x^2+x=x^3+9x^2+27x+27$$

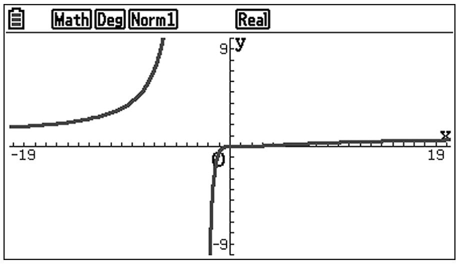
$$0=11x^2+26x+27$$

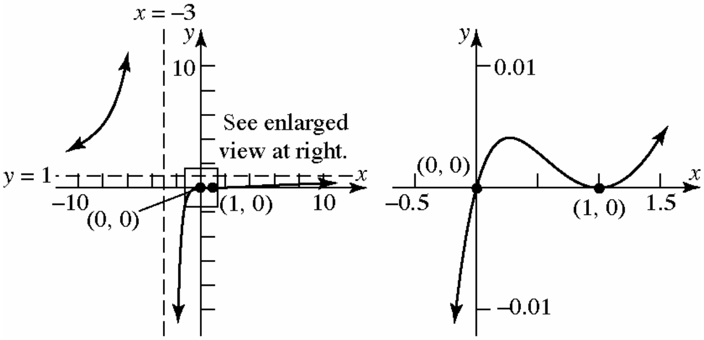
$$b^2-4ac=26^2-4(11)(27)=-512$$

no real solution

$R(x)$ does not intersect $y=1$.

Steps 6 & 7: Graphing:





Enlarged view

32. $R(x)=\dfrac{(x-1)(x+2)(x-3)}{x(x-4)^2}$ $\quad p(x)=(x-1)(x+2)(x-3);\; q(x)=x(x-4)^2; \quad n=3;\; m=3$

Step 1: Domain: $\{x \mid x\neq 0,\, x\neq 4\}$

There is no y-intercept since $R(0)=\dfrac{(0-1)(0+2)(0-3)}{0(0-4)^2}=\dfrac{6}{0}$.

Step 2 & 3: $R(x)=\dfrac{(x-1)(x+2)(x-3)}{x(x-4)^2}$ is in lowest terms. The x-intercepts are the zeros of $p(x)$: –2, 1, and 3

Near -2, $R(x)\approx -\dfrac{5}{24}(x+2)$; Near 1, $R(x)\approx -\dfrac{2}{3}(x-1)$; Near 3, $R(x)\approx \dfrac{10}{3}(x-3)$.

Plot $(-2,0)$ and indicate a line with negative slope there. Plot $(1,0)$ and indicate a line with negative slope there. Plot $(3,0)$ and indicate a line with positive slope there.

Step 4: $R(x)=\dfrac{(x-1)(x+2)(x-3)}{x(x-4)^2}$ is in lowest terms.

The vertical asymptotes are the zeros of $q(x)$: $x=0$ and $x=4$

Graph each of these asymptotes with a dashed line.

Step 5: Since $n=m$, the line $y=1$ is the horizontal asymptote. Graph this asymptote with a dashed line.

Solve to find intersection points:

$$\frac{(x-1)(x+2)(x-3)}{x(x-4)^2}=1$$

$$(x^2+x-2)(x-3)=x(x^2-8x+16)$$

$$x^3-2x^2-5x+6=x^3-8x^2+16x$$

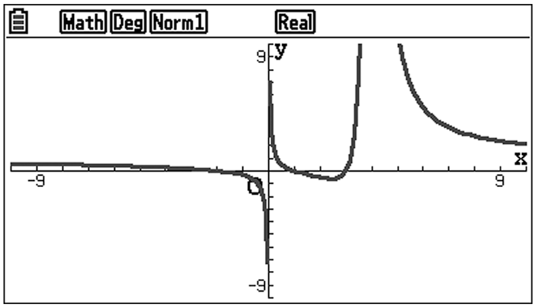
$$6x^2-21x+6=0$$

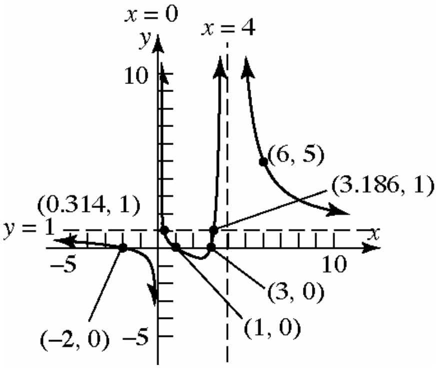
$$2x^2-7x+2=0$$

$$x=\frac{7\pm\sqrt{49-4(2)(2)}}{2(2)}=\frac{7\pm\sqrt{33}}{4}$$

$R(x)$ intersects $y=1$ at $\left(\frac{7-\sqrt{33}}{4},1\right)$ and $\left(\frac{7+\sqrt{33}}{4},1\right)$.

Steps 6 & 7: Graphing:





33. $R(x)=\frac{x^2+x-12}{x^2-x-6}=\frac{(x+4)(x-3)}{(x-3)(x+2)}=\frac{x+4}{x+2}$ $\quad p(x)=x^2+x-12;\ q(x)=x^2-x-6;\quad n=2;\ m=2$

Step 1: Domain: $\{x \mid x\neq -2,\ x\neq 3\}$

The y-intercept is $R(0)=\frac{0^2+0-12}{0^2-0-6}=\frac{-12}{-6}=2$. Plot the point $(0,2)$.

Step 2 & 3:In lowest terms, $R(x)=\frac{x+4}{x+2},\ x\neq 3$. Note: $R(x)$ is still undefined at both 3 and -2.

The x-intercept is the zero of $y=x+4$: –4

Near -4, $R(x)\approx -\frac{1}{2}(x+4)$. Plot $(-4,0)$ and indicate a line with negative slope there.

Step 4: In lowest terms, $R(x)=\frac{x+4}{x+2},\ x\neq 3$. The vertical asymptote is the zero of $f(x)=x+2$: $x=-2$;

Graph this asymptote using a dashed line. Note: $x=3$ is not a vertical asymptote because the reduced form must be used to find the asymptotes. The graph has a hole at $\left(3,\frac{7}{5}\right)$.

Step 5: Since $n = m$, the line $y = 1$ is the horizontal asymptote. Graph this asymptote using a dashed line. Solve to find intersection points:

$$\frac{x^2 + x - 12}{x^2 - x - 6} = 1$$

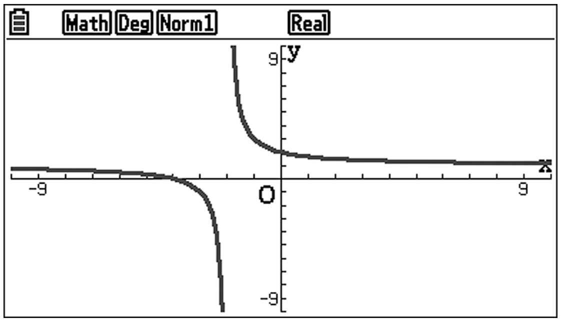
$$x^2 + x - 12 = x^2 - x - 6$$

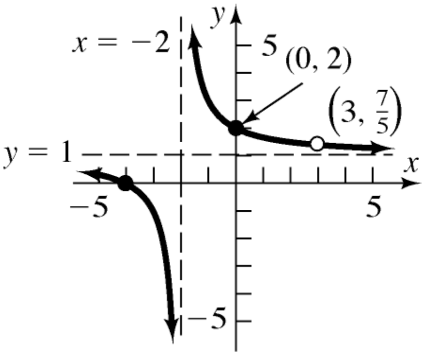
$$2x = 6$$

$$x = 3$$

$R(x)$ does not intersect $y = 1$ because $R(x)$ is not defined at $x = 3$.

Steps 6 & 7: Graphing:





34. $R(x) = \dfrac{x^2 + 3x - 10}{x^2 + 8x + 15} = \dfrac{(x+5)(x-2)}{(x+5)(x+3)} = \dfrac{x-2}{x+3}$ $p(x) = x^2 + 3x - 10;$ $q(x) = x^2 + 8x + 15;$ $n = 2;$ $m = 2$

Step 1: Domain: $\{x \mid x \neq -5, x \neq -3\}$

The y-intercept is $R(0) = \dfrac{0^2 + 3(0) - 10}{0^2 + 8(0) + 15} = \dfrac{-10}{15} = -\dfrac{2}{3}$. Plot the point $\left(0, -\dfrac{2}{3}\right)$.

Step 2 & 3: In lowest terms, $R(x) = \dfrac{x-2}{x+3}$, $x \neq -5$. The x-intercept is the zero of $y = x - 2$: 2;

Note: –5 is not a zero because reduced form must be used to find the zeros.

Near 2, $R(x) \approx \frac{1}{5}(x - 2)$. Plot the point $(2, 0)$ and indicate a line with positive slope there.

Step 4: In lowest terms, $R(x) = \dfrac{x-2}{x+3}$, $x \neq -5$. The vertical asymptote is the zero of $f(x) = x + 3$: $x = -3$; Graph this asymptote using a dashed line.

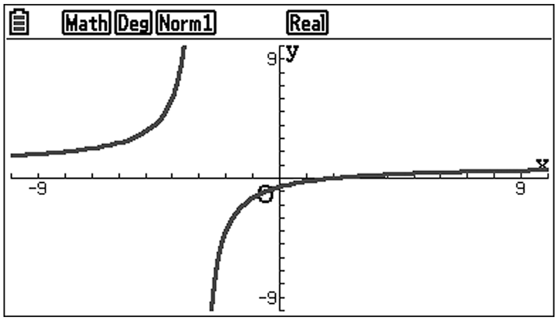
Note: $x = -5$ is not a vertical asymptote because reduced form must be used to find the asymptotes. The graph has a hole at $(-5, 3.5)$.

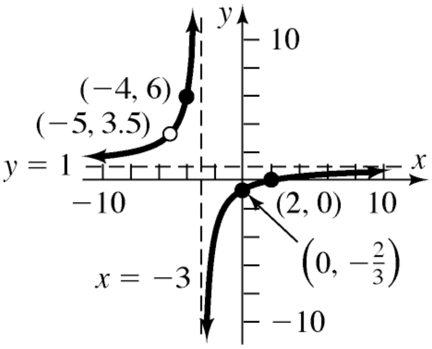
Step 5: Since $n = m$, the line $y = 1$ is the horizontal asymptote. Graph this asymptote using a dashed line. Solve to find intersection points:

$$\frac{x^2+3x-10}{x^2+8x+15}=1$$
$$x^2+3x-10=x^2+8x+15$$
$$-5x=25$$
$$x=-5$$

$R(x)$ does not intersect $y=1$ because $R(x)$ is not defined at $x=-5$.

Steps 6 & 7: Graphing:





35. $R(x)=\dfrac{6x^2-7x-3}{2x^2-7x+6}=\dfrac{(3x+1)(2x-3)}{(2x-3)(x-2)}=\dfrac{3x+1}{x-2}$ $\quad p(x)=6x^2-7x-3;\quad q(x)=2x^2-7x+6;\quad n=2;\ m=2$

Step 1: Domain: $\left\{x \,\middle|\, x\neq\frac{3}{2},\, x\neq 2\right\}$

The y-intercept is $R(0)=\dfrac{6(0)^2-7(0)-3}{2(0)^2-7(0)+6}=\dfrac{-3}{6}=-\dfrac{1}{2}$. Plot the point $\left(0,-\frac{1}{2}\right)$.

Step 2 & 3: In lowest terms, $R(x)=\dfrac{3x+1}{x-2},\ x\neq\dfrac{3}{2}$. The x-intercept is the zero of $y=3x+1$: $-\dfrac{1}{3}$;

Note: $x=\dfrac{3}{2}$ is not a zero because reduced form must be used to find the zeros.

Near $-\dfrac{1}{3}$, $R(x)\approx-\dfrac{3}{7}(3x+1)$. Plot the point $\left(-\frac{1}{3},0\right)$ and indicate a line with negative slope there.

Step 4: In lowest terms, $R(x)=\dfrac{3x+1}{x-2},\ x\neq\dfrac{3}{2}$. The vertical asymptote is the zero of $f(x)=x-2$: $x=2$;

Graph this asymptote using a dashed line.

Note: $x=\dfrac{3}{2}$ is not a vertical asymptote because reduced form must be used to find the asymptotes.

The graph has a hole at $\left(\frac{3}{2},-11\right)$.

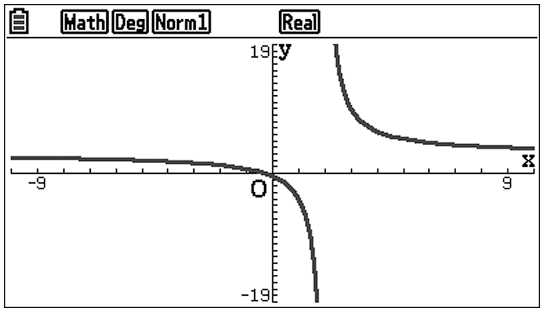
Step 5: Since $n=m$, the line $y=3$ is the horizontal asymptote. Graph this asymptote using a dashed line.

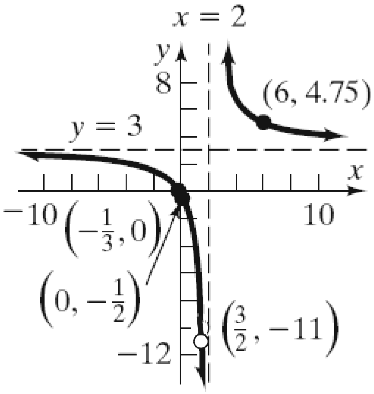
Solve to find intersection points:

$$\begin{aligned}\frac{6x^2-7x-3}{2x^2-7x+6}&=3\\ 6x^2-7x-3&=6x^2-21x+18\\ 14x&=21\\ x&=\frac{3}{2}\end{aligned}$$

$R(x)$ does not intersect $y=3$ because $R(x)$ is not defined at $x=\frac{3}{2}$.

Steps 6 & 7: Graphing:





36. $R(x)=\dfrac{8x^2+26x+15}{2x^2-x-15}=\dfrac{(4x+3)(2x+5)}{(2x+5)(x-3)}=\dfrac{4x+3}{x-3}$ $p(x)=8x^2+26x+15$; $q(x)=2x^2-x-15$; $n=2$; $m=2$

Step 1: Domain: $\left\{x \,\middle|\, x\neq -\frac{5}{2},\ x\neq 3\right\}$

The y-intercept is $R(0)=\dfrac{8(0)^2+26(0)+15}{2(0)^2-0-15}=\dfrac{15}{-15}=-1$. Plot the point $(0,-1)$.

Step 2 & 3:In lowest terms, $R(x)=\dfrac{4x+3}{x-3}$, $x\neq-\dfrac{5}{2}$. The x-intercept is the zero of $y=4x+3$: $-\dfrac{3}{4}$;

Note: $-\dfrac{5}{2}$ is not a zero because reduced form must be used to find the zeros.

Near $-\dfrac{3}{4}$, $R(x)\approx-\dfrac{4}{15}(4x+3)$. Plot the point $\left(-\dfrac{3}{4},0\right)$ and indicate a line with negative slope there.

Step 4: In lowest terms, $R(x)=\dfrac{4x+3}{x-3}$, $x\neq-\dfrac{5}{2}$. The vertical asymptote is the zero of $f(x)=x-3$: $x=3$; Graph this asymptote using a dashed line.

Note: $x=-\frac{5}{2}$ is not a vertical asymptote because reduced form must be used to find the asymptotes.

The graph has a hole at $\left(-\frac{5}{2},\frac{14}{11}\right)$.

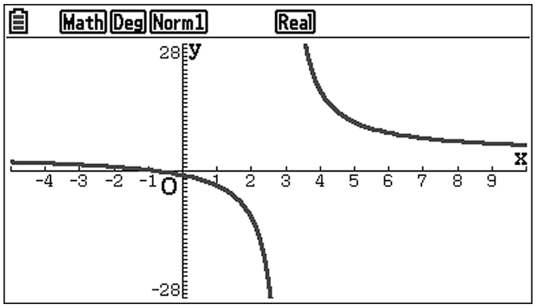
Step 5: Since $n=m$, the line $y=4$ is the horizontal asymptote. Graph this asymptote using a dashed line.

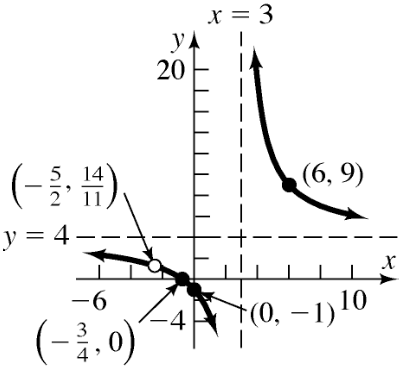
Solve to find intersection points:

$$\frac{8x^2+26x+15}{2x^2-x-15}=4$$
$$8x^2+26x+15=8x^2-4x-60$$
$$30x=-75$$
$$x=-\frac{5}{2}$$

$R(x)$ does not intersect $y=4$ because $R(x)$ is not defined at $x=-\frac{5}{2}$.

Steps 6 & 7: Graphing:





37. $R(x)=\frac{x^2+5x+6}{x+3}=\frac{(x+2)(x+3)}{x+3}=x+2 \qquad p(x)=x^2+5x+6; \quad q(x)=x+3; \quad n=2; \; m=1$

Step 1: Domain: $\{x \mid x\neq -3\}$

The y-intercept is $R(0)=\frac{0^2+5(0)+6}{0+3}=\frac{6}{3}=2$. Plot the point $(0,2)$.

Step 2 & 3:In lowest terms, $R(x)=x+2,\ x\neq -3$. The x-intercept is the zero of $y=x+2$: –2;

Note: –3 is not a zero because reduced form must be used to find the zeros.

Near -2, $R(x)=x+2$. Plot the point $(0,-2)$ and indicate the line $y=x+2$ there.

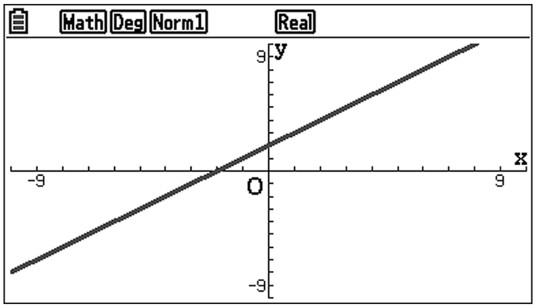
Step 4: In lowest terms, $R(x)=x+2,\ x\neq -3$. There are no vertical asymptotes. Note: $x=-3$ is not a vertical asymptote because reduced form must be used to find the asymptotes. The graph has a hole at $(-3,-1)$.

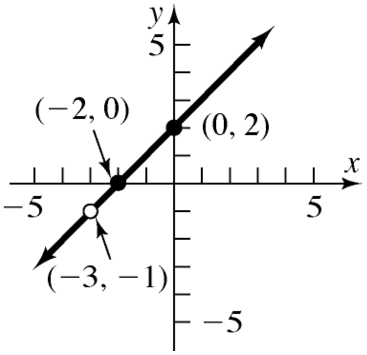
Step 5: Since $n = m + 1$ there is an oblique asymptote. The line $y = x + 2$ is the oblique asymptote. Solve to find intersection points:

$$\frac{x^2+5x+6}{x+3} = x+2$$
$$x^2+5x+6 = (x+2)(x+3)$$
$$x^2+5x+6 = x^2+5x+6$$
$$0 = 0$$

The oblique asymptote intersects $R(x)$ at every point of the form $(x, x+2)$ except (–3, –1).

Steps 6 & 7: Graphing:





38. $R(x) = \dfrac{x^2+x-30}{x+6} = \dfrac{(x+6)(x-5)}{x+6} = x-5 \qquad p(x) = x^2+x-30; \quad q(x) = x+6; \quad n = 2; \; m = 1$

Step 1: Domain: $\{x \mid x \neq -6\}$

The y-intercept is $R(0) = \dfrac{0^2+(0)-30}{0+6} = \dfrac{-30}{6} = -5$. Plot the point $(0,-5)$.

Step 2 & 3: In lowest terms, $R(x) = x-5,\ x \neq -6$. The x-intercept is the zero of $y = x-5$: 5;
Note: –6 is not a zero because reduced form must be used to find the zeros.
Near 5, $R(x) = x-5$. Plot the point $(5,0)$ and indicate the line $y = x-5$ there.

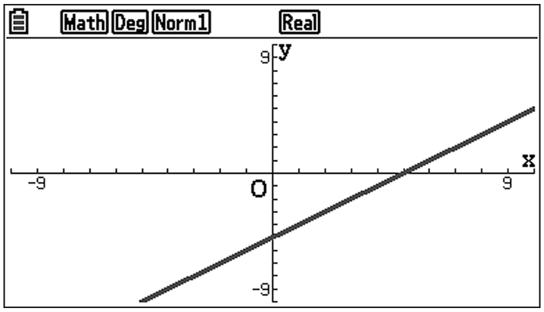
Step 4: In lowest terms, $R(x) = x-5,\ x \neq -6$. There are no vertical asymptotes. Note: $x = -6$ is not a vertical asymptote because reduced form must be used to find the asymptotes. The graph has a hole at $(-6,-11)$.

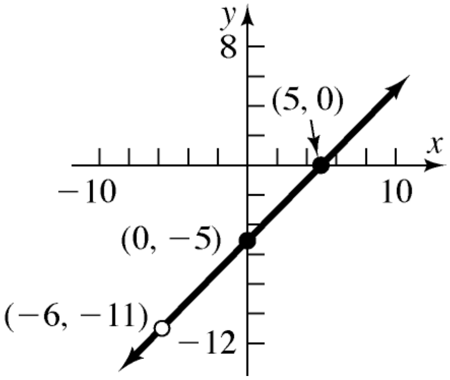
Step 5: Since $n = m+1$ there is an oblique asymptote. The line $y = x-5$ is the oblique asymptote. Solve to find intersection points:

$$\frac{x^2+x-30}{x+6}=x-5$$
$$x^2+x-30=(x+6)(x-5)$$
$$x^2+x-30=x^2+x-30$$
$$0=0$$

The oblique asymptote intersects $R(x)$ at every point of the form $(x, x-5)$ except (–6, –11).

Steps 6 & 7: Graphing:





39. $f(x)=x+\frac{1}{x}=\frac{x^2+1}{x}$ $\quad p(x)=x^2+1;\ q(x)=x;\ n=2;\ m=1$

Step 1: Domain: $\{x \mid x \neq 0\}$

There is no y-intercept because 0 is not in the domain.

Step 2 & 3: $f(x)=\frac{x^2+1}{x}$ is in lowest terms. There are no x-intercepts since $x^2+1=0$ has no real solutions.

Step 4: $f(x)=\frac{x^2+1}{x}$ is in lowest terms. The vertical asymptote is the zero of $q(x)$: $x=0$ Graph this asymptote using a dashed line.

Step 5: Since $n=m+1$, there is an oblique asymptote.

Dividing:

$$\begin{array}{r} x \\ x\overline{)\,x^2+1} \\ \underline{x^2} \\ 1 \end{array}$$

$f(x)=x+\frac{1}{x}$

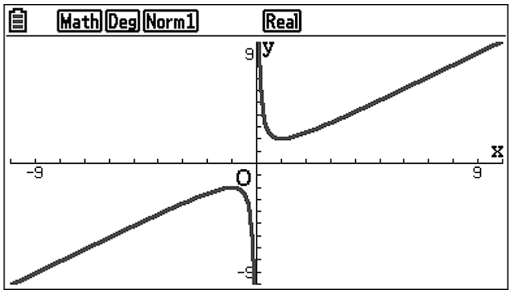
The oblique asymptote is y=x.

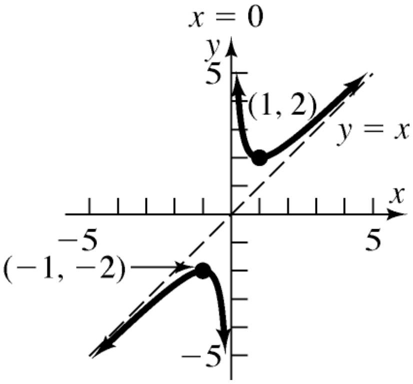
Graph this asymptote using a dashed line. Solve to find intersection points:

$$\frac{x^2+1}{x}=x$$
$$x^2+1=x^2$$
$$1\neq 0$$

The oblique asymptote does not intersect $f(x)$.

Steps 6 & 7: Graphing :





40. $f(x)=2x+\frac{9}{x}=\frac{2x^2+9}{x}$ $p(x)=2x^2+9$; $q(x)=x$; $n=2$; $m=1$

Step 1: Domain: $\{x \mid x \neq 0\}$

There is no y-intercept because 0 is not in the domain.

Step 2 & 3: $f(x)=\frac{2x^2+9}{x}$ is in lowest terms. There are no x-intercepts since $2x^2+9=0$ has no real solutions.

Step 4: $f(x)=\frac{2x^2+9}{x}$ is in lowest terms. The vertical asymptote is the zero of $q(x)$: $x=0$

Graph this asymptote using a dashed line.

Step 5: Since $n=m+1$, there is an oblique asymptote. Dividing:

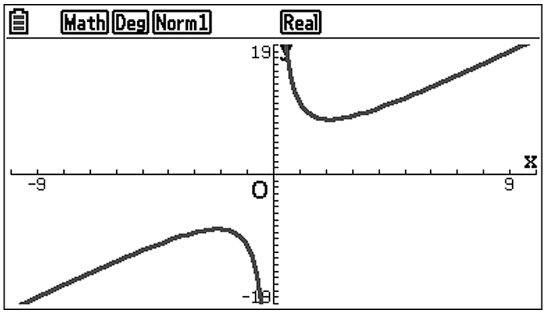
$$\begin{array}{r} 2x \\ x\overline{)2x^2+9} \\ \underline{2x^2} \\ 9 \end{array} \qquad f(x)=2x+\frac{9}{x}$$

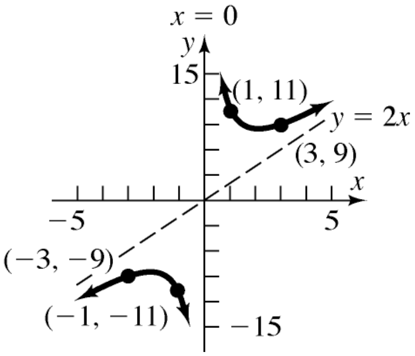
The oblique asymptote is $y=2x$. Graph this asymptote using a dashed line. Solve to find intersection points:

$$\frac{2x^2+9}{x}=2x$$
$$2x^2+9=2x^2$$
$$9\neq 0$$

The oblique asymptote does not intersect $f(x)$.

Steps 6 & 7: Graphing:





41. $f(x)=x^2+\frac{1}{x}=\frac{x^3+1}{x}=\frac{(x+1)\left(x^2-x+1\right)}{x}$ $\quad p(x)=x^3+1;\ q(x)=x;\ n=3;\ m=1.$

Step 1: Domain: $\{x \mid x\neq 0\}$

There is no y-intercept because 0 is not in the domain.

Step 2 & 3: $f(x)=\frac{x^3+1}{x}$ is in lowest terms. The x-intercept is the zero of $p(x)$: -1

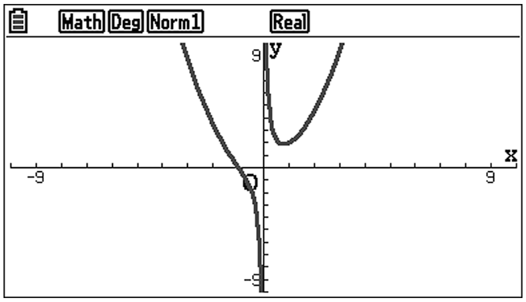
Near -1, $f(x)\approx -3(x+1)$. Plot the point $(-1,0)$ and indicate a line with negative slope there.

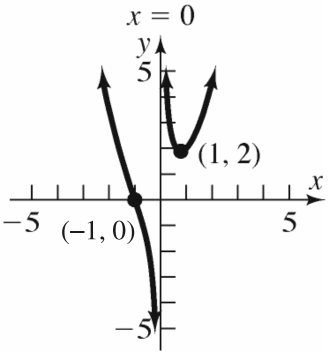
Step 4: $f(x)=\frac{x^3+1}{x}$ is in lowest terms. The vertical asymptote is the zero of $q(x)$: $x=0$

Graph this asymptote using a dashed line.

Step 5: Since $n>m+1$, there is no horizontal or oblique asymptote.

Steps 6 & 7: Graphing:





42. $f(x)=2x^2+\frac{16}{x}=\frac{2x^3+16}{x}=\frac{2\left(x^3+8\right)}{x}=\frac{2(x+2)\left(x^2-2x+4\right)}{x}$ $\quad p(x)=2x^3+16;\ q(x)=x;\ n=3;\ m=1$

Step 1: Domain: $\{x \mid x\neq 0\}$

There is no y-intercept because 0 is not in the domain.

Step 2 & 3: $f(x)=\dfrac{2x^3+16}{x}$ is in lowest terms. The *x*-intercept is the zero of $p(x)$: -2

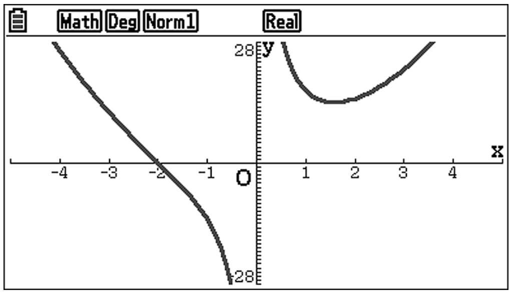
Near -2, $f(x)\approx -12(x+2)$. Plot $(-2,0)$ and indicate a line with negative slope there.

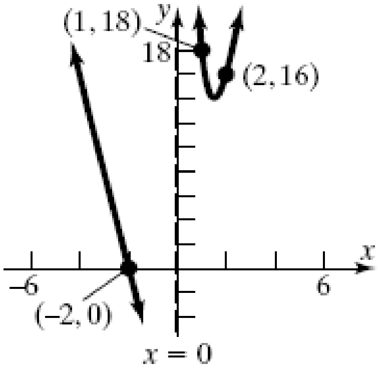
Step 4: $f(x)=\dfrac{2x^3+16}{x}$ is in lowest terms. The vertical asymptote is the zero of $q(x)$: $x=0$

Graph this asymptote using a dashed line.

Step 5: Since $n>m+1$, there is no horizontal or oblique asymptote.

Steps 6 & 7: Graphing:





43. $f(x)=x+\dfrac{1}{x^3}=\dfrac{x^4+1}{x^3}$ $\quad p(x)=x^4+1;\ q(x)=x^3;\ n=4;\ m=3$

Step 1: Domain: $\{x \mid x\neq 0\}$

There is no *y*-intercept because 0 is not in the domain.

Step 2 & 3: $f(x)=\dfrac{x^4+1}{x^3}$ is in lowest terms. There are no *x*-intercepts since $x^4+1=0$ has no real solutions.

Step 4: $f(x)=\dfrac{x^4+1}{x^3}$ is in lowest terms. The vertical asymptote is the zero of $q(x)$: $x=0$

Graph this asymptote using a dashed line.

Step 5: Since $n=m+1$, there is an oblique asymptote. Dividing:

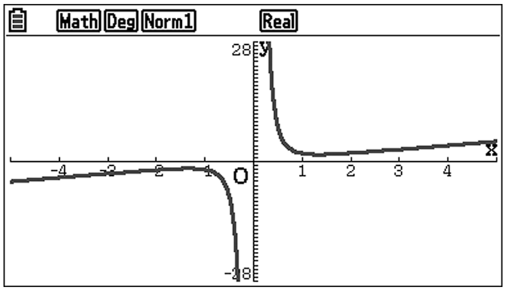
$$\begin{array}{r} x \\ x^3 \overline{\big) x^4+1} \\ \underline{x^4} \\ 1 \end{array} \qquad\qquad f(x)=x+\frac{1}{x^3}$$

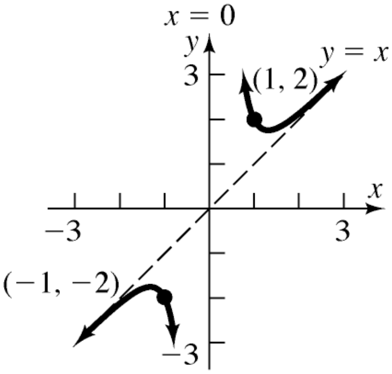
The oblique asymptote is $y=x$. Graph this asymptote using a dashed line. Solve to find intersection points:

$$\frac{x^4+1}{x^3}=x$$
$$x^4+1=x^4$$
$$1\neq 0$$

The oblique asymptote does not intersect $f(x)$.

Steps 6 & 7: Graphing:





44. $f(x)=2x+\dfrac{9}{x^3}=\dfrac{2x^4+9}{x^3}$ $p(x)=2x^4+9$; $q(x)=x^3$; $n=4$; $m=3$

Step 1: Domain: $\{x\mid x\neq 0\}$

There is no y-intercept because 0 is not in the domain.

Step 2 & 3: $f(x)=\dfrac{2x^4+9}{x^3}$ is in lowest terms. There are no x-intercepts since $2x^4+9=0$ has no real solutions.

Step 4: $f(x)=\dfrac{2x^4+9}{x^3}$ is in lowest terms. The vertical asymptote is the zero of $q(x)$: $x=0$

Graph this asymptote using a dashed line.

Step 5: Since $n=m+1$, there is an oblique asymptote. Dividing:

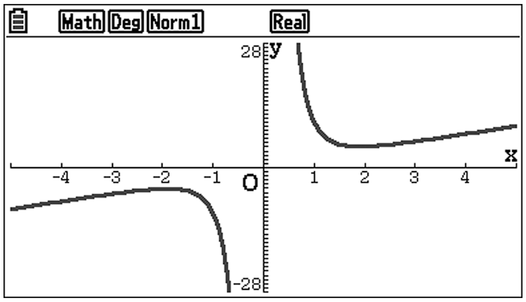
$$\begin{array}{r} 2x \\ x^3\overline{\big)\,2x^4+9} \\ \underline{2x^4} \\ 9 \end{array} \qquad f(x)=2x+\frac{9}{x^3}$$

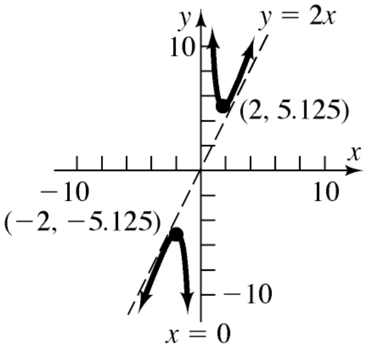
The oblique asymptote is $y=2x$. Graph this asymptote using a dashed line. Solve to find intersection points:

$$\frac{2x^4+9}{x^3}=2x$$
$$2x^4+9=2x^4$$
$$9\neq 0$$

The oblique asymptote does not intersect $f(x)$.

Steps 6 & 7: Graphing:





45. One possibility: $R(x)=\dfrac{x^2}{x^2-4}$

46. One possibility: $R(x)=-\dfrac{x}{x^2-1}$

47. One possibility: $R(x)=\dfrac{(x-1)(x-3)\left(x^2+a\right)}{(x+1)^2(x-2)^2}$

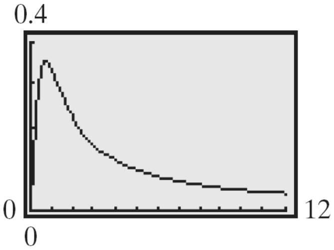
(Using the point $(0,1)$ leads to $a=4/3$.) Thus,

$$R(x)=\frac{(x-1)(x-3)\left(x^2+\frac{4}{3}\right)}{(x+1)^2(x-2)^2}.$$

48. One possibility: $R(x)=\dfrac{3(x+2)(x-1)^2}{(x+3)(x-4)^2}$

49. **a.** The degree of the numerator is 1 and the degree of the denominator is 2. Thus, the horizontal asymptote is $y=0$. The concentration of the drug decreases to 0 as time increases.

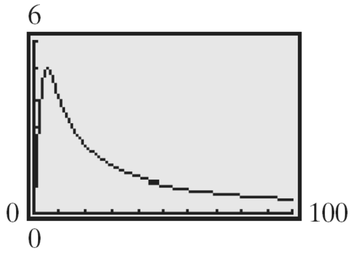
b. Graphing:



c. Using MAXIMUM, the concentration is highest after $t\approx 0.71$ hours.

50. **a.** The degree of the numerator is 1 and the degree of the denominator is 2. Thus, the horizontal asymptote is $y=0$. The concentration of the drug decreases to 0 as time increases.

b. Graphing:



c. Using MAXIMUM, the concentration is highest after $t=5$ minutes.

51. **a.** The cost of the project is the sum of the cost for the parallel side, the two other sides, and the posts.

$A=xy$

$1000=xy$

$y=\dfrac{1000}{x}$

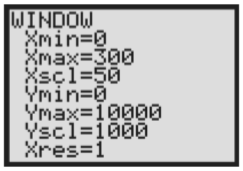
If the length of a perpendicular side is x feet, the length of the parallel side is

$y = \frac{1000}{x}$ feet. Thus,

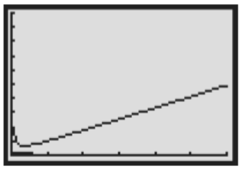
$$\begin{aligned} C(x) &= 2 \cdot 8 \cdot x + 5 \cdot \frac{1000}{x} + 4(25) \\ &= 16x + \frac{5000}{x} + 100 \end{aligned}$$

b. The domain is $x > 0$. Note that x is a length so it cannot be negative. In addition, if $x = 0$, there is no rectangle (that is, the area is 0 square feet).

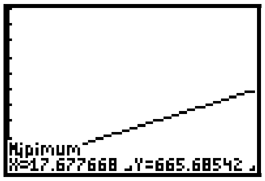
c. $C(x) = 16x + \frac{5000}{x} + 100$



```
WINDOW
 Xmin=0
 Xmax=300
 Xscl=50
 Ymin=0
 Ymax=10000
 Yscl=1000
 Xres=1
```



d. Using MINIMUM, the dimensions of cheapest cost are about 17.7 feet by 56.6 feet (longer side parallel to river).



```
Minimum
X=17.677668  Y=665.68542
```

Note: $x = 17\frac{2}{3} = \frac{53}{3}$ feet and

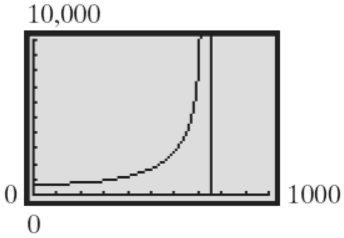
$y = \frac{1000}{53/3} = \frac{3000}{53}$ feet .

52. a. $$\begin{aligned} f'(v_s) &= 600\left(\frac{772.4 - 45}{772.4 - v_s}\right) \\ &= 600\left(\frac{727.4}{772.4 - v_s}\right) \end{aligned}$$

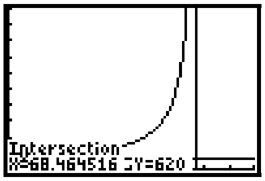
b. $$\begin{aligned} 620 &= 600\left(\frac{727.4}{772.4 - v_s}\right) \\ 620 &= \frac{436,440}{772.4 - v_s} \\ 620(772.4 - v_s) &= 436,440 \\ 772.4 - v_s &= \frac{436,440}{620} \\ v_s &= 772.4 - \frac{436,440}{620} \approx 68.5 \end{aligned}$$

If $f' = 620$ Hz, the speed of the ambulance is roughly 68.5 miles per hour.

c. $y = \frac{436,440}{772.4 - x}$



d. Let $Y_1 = \frac{436,440}{772.4 - x}$ and $Y_2 = 620$, then find the intersection point.



```
Intersection
X=68.464516  Y=620
```

The graph agrees with our direct calculation.

53. a. The surface area is the sum of the areas of the six sides.

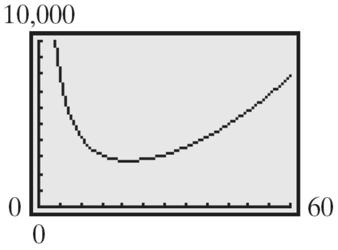
$S = xy + xy + xy + xy + x^2 + x^2 = 4xy + 2x^2$

The volume is $x \cdot x \cdot y = x^2 y = 10,000$.

Thus, $y = \dfrac{10,000}{x^2}$, so

$$S(x) = 4x\left(\frac{10,000}{x^2}\right) + 2x^2$$
$$= 2x^2 + \frac{40,000}{x}$$
$$= \frac{2x^3 + 40,000}{x}$$

b. Graphing:



c. Using MINIMUM, the minimum surface area (amount of cardboard) is about 2784.95 square inches.

d. The surface area is a minimum when $x \approx 21.54$ inches.

$$y = \frac{10,000}{(21.544)^2} \approx 21.54 \text{ inches}$$

The dimensions of the box are: 21.54 in. by 21.54 in. by 21.54 in.

e. Answers will vary. One possibility is to save costs or reduce weight by minimizing the material needed to construct the box.

54. a. The surface area is the sum of the areas of the five sides.

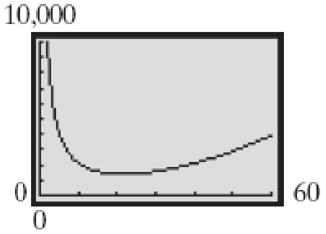
$S = xy + xy + xy + xy + x^2 = 4xy + x^2$

The volume is $x \cdot x \cdot y = x^2 y = 5000.$

Thus, $y = \dfrac{5000}{x^2}$, so

$$S(x) = 4x\left(\frac{5000}{x^2}\right) + x^2$$
$$= x^2 + \frac{20,000}{x}$$
$$= \frac{x^3 + 20,000}{x}$$

b. Graphing:



c. Using MINIMUM, the minimum surface area (amount of cardboard) is about 1392.48 square inches.

d. The surface area is a minimum when $x = 21.54$.

$$y = \frac{5000}{(21.54)^2} \approx 10.78$$

The dimensions of the box are: 21.54 in. by 21.54 in. by 10.78 in.

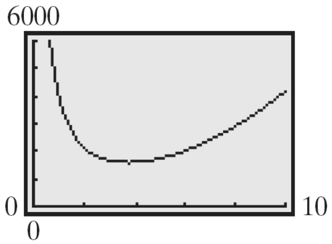
e. Answers will vary. One possibility is to save costs or reduce weight by minimizing the material needed to construct the box.

55. a. $500 = \pi r^2 h$

$$h = \frac{500}{\pi r^2}$$

$$C(r) = 6(2\pi r^2) + 4(2\pi rh)$$
$$= 12\pi r^2 + 8\pi r\left(\frac{500}{\pi r^2}\right)$$
$$= 12\pi r^2 + \frac{4000}{r}$$

b. Graphing:



Using MINIMUM, the cost is least for $r \approx 3.76$ cm.

56. **a.** $100 = \pi r^2 h$

$$h = \frac{100}{\pi r^2}$$

$$A(r) = 2\pi r^2 + 2\pi rh$$
$$= 2\pi r^2 + 2\pi r\left(\frac{100}{\pi r^2}\right)$$
$$= 2\pi r^2 + \frac{200}{r}$$

b. $A(3) = 2\pi \cdot 3^2 + \frac{200}{3}$

$= 18\pi + \frac{200}{3} \approx 123.22$ square feet

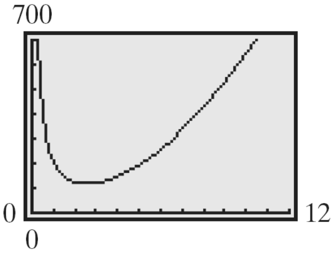
c. $A(4) = 2\pi \cdot 4^2 + \frac{200}{4}$

$= 32\pi + 50 \approx 150.53$ square feet

d. $A(5) = 2\pi \cdot 5^2 + \frac{200}{5}$

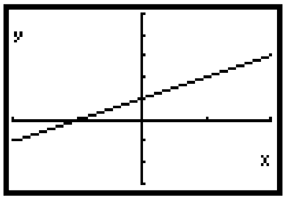
$= 50\pi + 40 \approx 197.08$ square feet

e. Graphing:

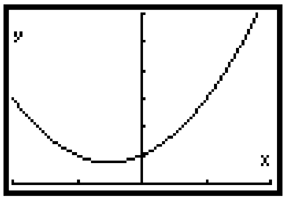


Using MINIMUM, the area is smallest when $r \approx 2.52$ feet.

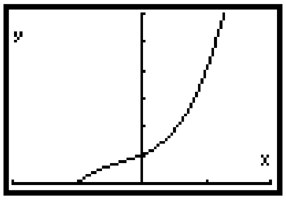
57. $y = \frac{x^2 - 1}{x - 1}$



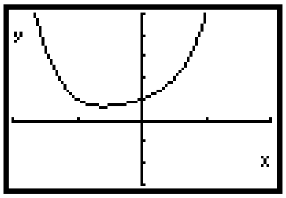
$y = \frac{x^3 - 1}{x - 1}$



$y = \frac{x^4 - 1}{x - 1}$



$y = \frac{x^5 - 1}{x - 1}$



$x = 1$ is not a vertical asymptote because of the following behavior:

When $x \neq 1$:

$$y = \frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$$

$$y = \frac{x^3 - 1}{x - 1} = \frac{(x-1)\left(x^2 + x + 1\right)}{x-1} = x^2 + x + 1$$

$$y = \frac{x^4 - 1}{x - 1} = \frac{\left(x^2+1\right)\left(x^2-1\right)}{x-1}$$
$$= \frac{\left(x^2+1\right)(x-1)(x+1)}{x-1}$$
$$= x^3 + x^2 + x + 1$$

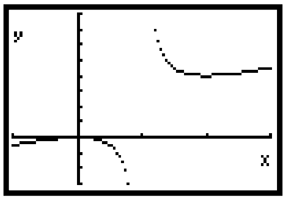
$$y = \frac{x^5 - 1}{x - 1} = \frac{(x^4 + x^3 + x^2 + x + 1)(x-1)}{x-1}$$
$$= x^4 + x^3 + x^2 + x + 1$$

In general, the graph of

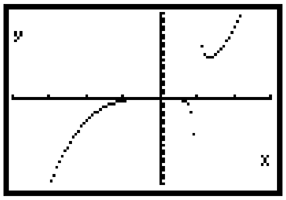
$y = \dfrac{x^n - 1}{x-1},\ n \ge 1,$ an integer,

will have a "hole" with coordinates $(1,\ n)$.

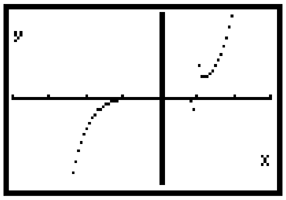
58. $y = \dfrac{x^2}{x-1}$



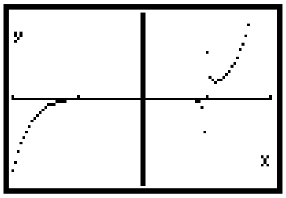
$y = \dfrac{x^4}{x-1}$



$y = \dfrac{x^6}{x-1}$



$y = \dfrac{x^8}{x-1}$



All four graphs have a vertical asymptote at $x = 1$. $y = \dfrac{x^2}{x-1}$ has an oblique asymptote at $y = x + 1$.

59. Answers will vary.

60. Answers will vary. One example is

$$R(x) = \frac{3(x-2)(x+1)^2}{(x+5)(x-6)^2}.$$

61. Answers will vary. One example is

$$R(x) = \frac{2(x-3)(x+2)^2}{(x-1)^3}.$$

62. Answers will vary.

63. Answers will vary.

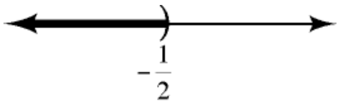
Section 4.6

1. $3 - 4x > 5$

$-4x > 2$

$x < -\dfrac{1}{2}$

The solution set is $\left\{x \middle| x < -\dfrac{1}{2}\right\}$ or, using interval notation, $\left(-\infty, -\dfrac{1}{2}\right)$.



2. $x^2 - 5x \le 24$

$x^2 - 5x - 24 \le 0$

$(x+3)(x-8) \le 0$

$f(x) = x^2 - 5x - 24 = (x+3)(x-8)$

$x = -3, x = 8$ are the zeros of *f*.

Interval	$(-\infty, -3)$	$(-3, 8)$	$(8, \infty)$
Number Chosen	–4	0	9
Value of *f*	12	–24	12
Conclusion	Positive	Negative	Positive

The solution set is $\{x \mid -3 \le x \le 8\}$ or, using interval notation, $[-3, 8]$.

3. True

4. False. The value 3 is not in the domain of *f*, so it must be restricted from the solution. The solution set would be $\{x \mid x \le 0 \text{ or } x > 3\}$.

5. The *x*-intercepts of the graph of *f* are 0, 1, and 2.

a. The graph of *f* is above the *x*-axis (so *f* is positive) for $0 < x < 1$ or $x > 2$. Therefore, the solution set is $\left\{ x \middle| 0 < x < 1 \text{ or } x > 2 \right\}$ or, using interval notation, $(0, 1) \cup (2, \infty)$.

b. The graph of *f* is below the *x*-axis (so *f* is negative) for $x < 0$ or $1 < x < 2$. Since the inequality is not strict, we include 0, 1, and 2 in the solution set. Therefore, the solution set is $\left\{ x \middle| x \le 0 \text{ or } 1 \le x \le 2 \right\}$ or, using interval notation $(-\infty, 0] \cup [1, 2]$.

6. The *x*-intercepts of the graph of *f* are -1, 1, and 2.

a. The graph of *f* is below the *x*-axis (so *f* is negative) for $-1 < x < 1$ or $x > 2$. Therefore, the solution set is $\left\{ x \middle| -1 < x < 1 \text{ or } x > 2 \right\}$ or, using interval notation, $(-1, 1) \cup (2, \infty)$.

b. The graph of *f* is above the *x*-axis (so *f* is positive) for $x < -1$ or $1 < x < 2$. Since the inequality is not strict, we include 0, 1, and 2 in the solution set. Therefore, the solution set is $\left\{ x \middle| x \le -1 \text{ or } 1 \le x \le 2 \right\}$ or, using interval notation $(-\infty, -1] \cup [1, 2]$.

7. The *x*-intercept of the graph of *f* is 0.

a. The graph of *f* is below the *x*-axis (so *f* is negative) for $-1 < x < 0$ or $x > 1$. Therefore, the solution set is $\left\{ x \middle| -1 < x < 0 \text{ or } x > 1 \right\}$ or, using interval notation, $(-1, 0) \cup (1, \infty)$.

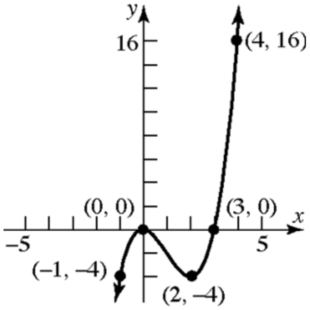
b. The graph of *f* is above the *x*-axis (so *f* is positive) for $x < -1$ or $0 < x < 1$. Since the inequality is not strict, we include 0 in the solution set. Therefore, the solution set is $\left\{ x \middle| x < -1 \text{ or } 0 \le x < 1 \right\}$ or, using interval notation $(-\infty, -1) \cup [0, 1)$.

8. The *x*-intercepts of the graph of *f* are 1 and 3.

a. The graph of *f* is above the *x*-axis (so *f* is positive) for $x < -1$ or $-1 < x < 1$ or $x > 3$. Therefore, the solution set is $\left\{ x \middle| x < -1 \text{ or } -1 < x < 1 \text{ or } x > 3 \right\}$ or, using interval notation, $(-\infty, -1) \cup (-1, 1) \cup (3, \infty)$.

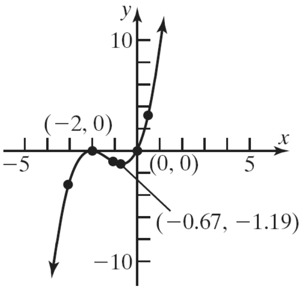
b. The graph of *f* is below the *x*-axis (so *f* is negative) for $1 < x < 2$ or $2 < x < 3$. Since the inequality is not strict, we include 1 and 3 in the solution set. Therefore, the solution set is $\left\{ x \middle| 1 \le x < 2 \text{ or } 2 < x \le 3 \right\}$ or, using interval notation $[1, 2) \cup (2, 3]$.

9. We graphed $f(x) = x^2(x-3)$ in Problem 69 of Section 4.1. The graph is reproduced below.



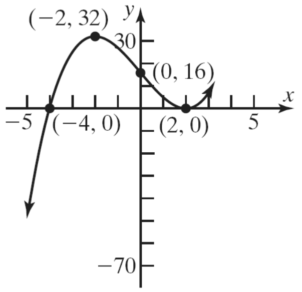
From the graph, we see that *f* is below the *x*-axis (so *f* is negative) for $x < 0$ or $0 < x < 3$. Thus, the solution set is $\left\{ x \middle| x < 0 \text{ or } 0 < x < 3 \right\}$ or, using interval notation $(-\infty, 0) \cup (0, 3)$.

10. We graphed $f(x) = x(x+2)^2$ in Problem 70 of Section 4.1. The graph is reproduced below.



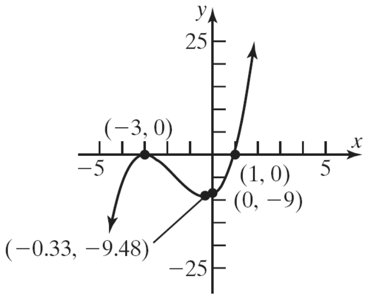
From the graph, we see that *f* is below the *x*-axis (so *f* is negative) for $x < -2$ or $-2 < x < 0$. Since the inequality is not strict, we include -2 and 0 in the solution set. Therefore, the solution set is $\left\{ x \middle| x \le 0 \right\}$ or, using interval notation $(-\infty, 0]$.

11. We graphed $f(x) = (x+4)(x-2)^2$ in Problem 71 of Section 4.1. The graph is reproduced below.



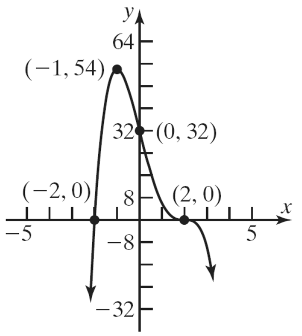
From the graph, we that *f* is above the *x*-axis (so *f* is positive) for $-4 < x < 2$ or $x > 2$. Since the inequality is not strict, we include -4 and 2 in the solution set. Therefore, the solution set is $\left\{ x \middle| x \ge -4 \right\}$ or, using interval notation $[-4, \infty)$.

12. We graphed $f(x) = (x-1)(x+3)^2$ in Problem 72 of Section 4.1. The graph is reproduced below.



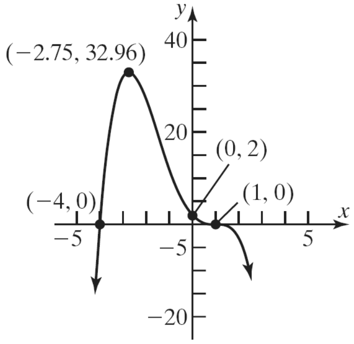
From the graph, we that *f* is above the *x*-axis (so *f* is positive) for $x > 1$. Therefore, the solution set is $\left\{ x \middle| x > 1 \right\}$ or, using interval notation $(1, \infty)$.

13. We graphed $f(x) = -2(x+2)(x-2)^3$ in Problem 73 of Section 4.1. The graph is reproduced below.



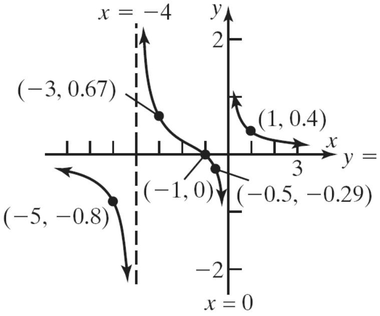
From the graph, we see that *f* is below the *x*-axis (so *f* is negative) for $x < -2$ or $x > 2$. Since the inequality is not strict, we include -2 and 2 in the solution set. Therefore, the solution set is $\left\{ x \middle| x \le -2 \text{ or } x \ge 2 \right\}$ or, using interval notation $(-\infty, -2] \cup [2, \infty)$.

14. We graphed $f(x) = -\frac{1}{2}(x+4)(x-1)^3$ in Problem 74 of Section 4.1. The graph is reproduced below.



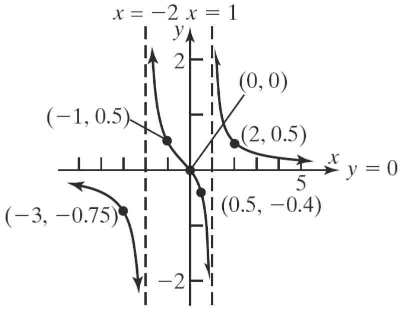
From the graph, we see that *f* is below the *x*-axis (so *f* is negative) for $x < -4$ or $x > 1$. Therefore, the solution set is $\left\{ x \middle| x < -4 \text{ or } x > 1 \right\}$ or, using interval notation $(-\infty, -4) \cup (1, \infty)$.

15. We graphed $R(x)=\dfrac{x+1}{x(x+4)}$ in Problem 7 of Section 4.5. The graph is reproduced below.



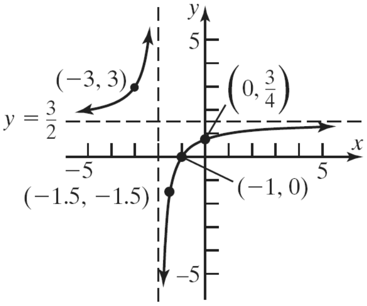
From the graph, we that *R* is above the *x*-axis (so *R* is positive) for $-4<x<-1$ or $x>0$. Therefore, the solution set is $\left\{x\,\middle|\,-4<x<-1 \text{ or } x>0\right\}$ or, using interval notation $(-4,-1)\cup(0,\infty)$.

16. We graphed $R(x)=\dfrac{x}{(x-1)(x+2)}$ in Problem 8 of Section 4.5. The graph is reproduced below.



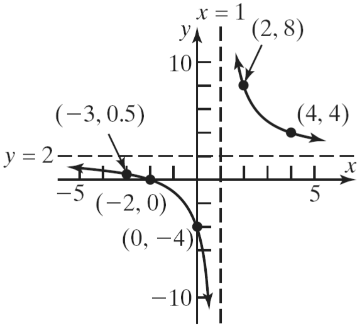
From the graph, we that *R* is below the *x*-axis (so *R* is negative) for $x<-2$ or $0<x<1$. Thus, the solution set is $\left\{x\,\middle|\,x<-2 \text{ or } 0<x<1\right\}$ or, using interval notation $(-\infty,-2)\cup(0,1)$.

17. We graphed $R(x)=\dfrac{3x+3}{2x+4}$ in Problem 9 of Section 4.5. The graph is reproduced below.



From the graph, we that *R* is below the *x*-axis (so *R* is negative) for $-2<x<-1$. Since the inequality is not strict, we include -1 in the solution set. Therefore, the solution set is $\left\{x\,\middle|\,-2<x\le -1\right\}$ or, using interval notation $(-2,-1]$.

18. We graphed $R(x)=\dfrac{2x+4}{x-1}$ in Problem 10 of Section 4.5. The graph is reproduced below.



From the graph, we that *R* is above the *x*-axis (so *R* is positive) for $x<-2$ or $x>1$. Since the inequality is not strict, we include -2 in the solution set. Therefore, the solution set is $\left\{x\,\middle|\,x\le -2 \text{ or } x>1\right\}$ or, using interval notation $(-\infty,-2]\cup(1,\infty)$.

19. $(x-5)^2(x+2)<0$

$f(x)=(x-5)^2(x+2)$

$x=5,\ x=-2$ are the zeros of f.

Interval	$(-\infty,-2)$	$(-2,5)$	$(5,\infty)$
Number Chosen	–3	0	6
Value of *f*	–64	50	8
Conclusion	Negative	Positive	Positive

The solution set is $\left\{x\,\middle|\,x<-2\right\}$ or, using interval notation, $(-\infty,-2)$.

20. $(x-5)(x+2)^2>0$

$f(x)=(x-5)(x+2)^2$

$x=5,\ x=-2$ are the zeros of f.

Interval	$(-\infty,-2)$	$(-2,5)$	$(5,\infty)$
Number Chosen	–3	0	6
Value of *f*	–8	–20	64
Conclusion	Negative	Negative	Positive

The solution set is $\left\{ x \middle| x > 5 \right\}$ or, using interval notation, $(5, \infty)$.

21. $x^3 - 4x^2 > 0$

$x^2(x-4) > 0$

$f(x) = x^3 - 4x^2 = x^2(x-4)$

$x = 0,\ x = 4$ are the zeros of f.

Interval	$(-\infty, 0)$	$(0, 4)$	$(4, \infty)$
Number Chosen	-1	1	5
Value of *f*	-5	-3	25
Conclusion	Negative	Negative	Positive

The solution set is $\left\{ x \middle| x > 4 \right\}$ or, using interval notation, $(4, \infty)$.

22. $x^3 + 8x^2 < 0$

$x^2(x+8) < 0$

$f(x) = x^3 + 8x^2 = x^2(x+8)$

$x = -8,\ x = 0$ are the zeros of f.

Interval	$(-\infty, -8)$	$(-8, 0)$	$(0, \infty)$
Number Chosen	-9	-1	1
Value of *f*	-81	7	9
Conclusion	Negative	Positive	Positive

The solution set is $\left\{ x \middle| x < -8 \right\}$ or, using interval notation, $(-\infty, -8)$.

23. $2x^3 > -8x^2$

$2x^3 + 8x^2 > 0$

$2x^2(x+4) > 0$

$f(x) = 2x^3 + 8x^2$

$x = 0, x = -4$ are the zeros of *f*.

Interval	$(-\infty, -4)$	$(-4, 0)$	$(0, \infty)$
Number Chosen	-5	-1	1
Value of *f*	-50	6	10
Conclusion	Negative	Positive	Positive

The solution set is $\{x \mid -4 < x < 0 \text{ or } x > 0\}$ or, using interval notation, $(-4, 0) \cup (0, \infty)$.

24. $3x^3 < -15x^2$

$3x^3 + 15x^2 < 0$

$3x^2(x+5) < 0$

$f(x) = 3x^3 + 15x^2$

$x = 0, x = -5$ are the zeros of *f*.

Interval	$(-\infty, -5)$	$(-5, 0)$	$(0, \infty)$
Number Chosen	-6	-1	1
Value of *f*	-108	12	18
Conclusion	Negative	Positive	Positive

The solution set is $\{x \mid x < -5\}$ or, using interval notation, $(-\infty, -5)$.

25. $(x-1)(x-2)(x-3) \le 0$

$f(x) = (x-1)(x-2)(x-3)$

$x = 1,\ x = 2,\ x = 3$ are the zeros of f.

Interval	$(-\infty, 1)$	$(1, 2)$	$(2, 3)$	$(3, \infty)$
Number Chosen	0	1.5	2.5	4
Value of *f*	-6	0.375	-0.375	6
Conclusion	Negative	Positive	Negative	Positive

The solution set is $\left\{ x \middle| x \le 1 \text{ or } 2 \le x \le 3 \right\}$ or, using interval notation, $(-\infty, 1] \cup [2, 3]$.

26. $(x+1)(x+2)(x+3) \le 0$

$f(x) = (x+1)(x+2)(x+3)$

$x = -1,\ x = -2,\ x = -3$ are the zeros of f.

Interval	$(-\infty, -3)$	$(-3, -2)$	$(-2, -1)$	$(-1, \infty)$
Number Chosen	-4	-2.5	-1.5	0
Value of *f*	-6	0.375	-0.375	6
Conclusion	Negative	Positive	Negative	Positive

The solution set is $\left\{ x \middle| x \le -3 \text{ or } -2 \le x \le -1 \right\}$ or, using interval notation, $(-\infty, -3] \cup [-2, -1]$.

27. $x^3 - 2x^2 - 3x > 0$

$x\left(x^2 - 2x - 3\right) > 0$

$x(x+1)(x-3) > 0$

$f(x) = x^3 - 2x^2 - 3x$

$x=-1, x=0, x=3$ are the zeros of f .

Interval	$(-\infty,-1)$	$(-1,0)$	$(0,3)$	$(3,\infty)$
Number Chosen	–2	–0.5	1	4
Value of f	–10	0.875	–4	20
Conclusion	Negative	Positive	Negative	Positive

The solution set is $\{x \mid -1 < x < 0 \text{ or } x > 3\}$ or, using interval notation, $(-1,0)\cup(3,\infty)$.

28. $x^3+2x^2-3x>0$

$x(x^2+2x-3)>0$

$x(x+3)(x-1)>0$

$f(x)=x(x+3)(x-1)$

$x=0, x=-3, x=1$ are the zeros of f

Interval	$(-\infty,-3)$	$(-3,0)$	$(0,1)$	$(1,\infty)$
Number Chosen	–4	–1	0.5	2
Value of f	–20	4	–0.875	10
Conclusion	Negative	Positive	Negative	Positive

The solution set is $\{x \mid -3 < x < 0 \text{ or } x > 1\}$ or, using interval notation, $(-3,0)\cup(1,\infty)$.

29. $x^4>x^2$

$x^4-x^2>0$

$x^2(x^2-1)>0$

$x^2(x-1)(x+1)>0$

$f(x)=x^2(x-1)(x+1)$

$x=-1,\ x=0, x=1$ are the zeros of f

Interval	$(-\infty,-1)$	$(-1,0)$	$(0,1)$	$(1,\infty)$
Number Chosen	–2	–0.5	0.5	2
Value of f	12	–0.1875	–0.1875	12
Conclusion	Positive	Negative	Negative	Positive

The solution set is $\{x \mid x < -1 \text{ or } x > 1\}$ or, using interval notation, $(-\infty,-1)\cup(1,\infty)$.

30. $x^4<9x^2$

$x^4-9x^2<0$

$x^2(x^2-9)<0$

$x^2(x-3)(x+3)<0$

$f(x)=x^2(x-3)(x+3)$

$x=0, x=3, x=-3$ are the zeros of f

Interval	$(-\infty,-3)$	$(-3,0)$	$(0,3)$	$(3,\infty)$
Number Chosen	–4	–1	1	4
Value of f	112	–8	–8	112
Conclusion	Positive	Negative	Negative	Positive

The solution set is $\{x \mid -3 < x < 0 \text{ or } 0 < x < 3\}$ or, using interval notation, $(-3,0)\cup(0,3)$.

31. $x^4>1$

$x^4-1>0$

$(x^2-1)(x^2+1)>0$

$(x-1)(x+1)(x^2+1)>0$

$f(x)=(x-1)(x+1)(x^2+1)$

$x=1, x=-1$ are the zeros of f ; x^2+1 has no real solution

Interval	$(-\infty,-1)$	$(-1,1)$	$(1,\infty)$
Number Chosen	–2	0	2
Value of f	15	–1	15
Conclusion	Positive	Negative	Positive

The solution set is $\{x \mid x < -1 \text{ or } x > 1\}$ or, using interval notation, $(-\infty,-1)\cup(1,\infty)$.

32. $x^3>1$

$x^3-1>0$

$(x-1)(x^2+x+1)>0$

$f(x)=(x-1)(x^2+x+1)$

$x=1$ is a zero of f; x^2+x+1 has no real solution.

Interval	$(-\infty,1)$	$(1,\infty)$
Number Chosen	0	2
Value of f	–1	7
Conclusion	Negative	Positive

The solution set is $\{x \mid x > 1\}$ or, using interval notation, $(1,\infty)$.

33. $\frac{x+1}{x-1} > 0$

$f(x) = \frac{x+1}{x-1}$

The zeros and values where f is undefined are $x = -1$ and $x = 1$.

Interval	$(-\infty, -1)$	$(-1, 1)$	$(1, \infty)$
Number Chosen	-2	0	2
Value of f	$\frac{1}{3}$	-1	3
Conclusion	Positive	Negative	Positive

The solution set is $\{x \mid x < -1 \text{ or } x > 1\}$ or, using interval notation, $(-\infty, -1) \cup (1, \infty)$.

34. $\frac{x-3}{x+1} > 0$

$f(x) = \frac{x-3}{x+1}$

The zeros and values where f is undefined are $x = -1$ and $x = 3$.

Interval	$(-\infty, -1)$	$(-1, 3)$	$(3, \infty)$
Number Chosen	-2	0	4
Value of f	5	-3	0.2
Conclusion	Positive	Negative	Positive

The solution set is $\{x \mid x < -1 \text{ or } x > 3\}$ or, using interval notation, $(-\infty, -1) \cup (3, \infty)$.

35. $\frac{(x-1)(x+1)}{x} \le 0$

$f(x) = \frac{(x-1)(x+1)}{x}$

The zeros and values where f is undefined are $x = -1$, $x = 0$ and $x = 1$.

Interval	$(-\infty, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \infty)$
Number Chosen	-2	-0.5	0.5	2
Value of f	-1.5	1.5	-1.5	1.5
Conclusion	Negative	Positive	Negative	Positive

The solution set is $\{x \mid x \le -1 \text{ or } 0 < x \le 1\}$ or, using interval notation, $(-\infty, -1] \cup (0, 1]$.

36. $\frac{(x-3)(x+2)}{x-1} \le 0$

$f(x) = \frac{(x-3)(x+2)}{x-1}$

The zeros and values where f is undefined are $x = -2$, $x = 1$ and $x = 3$.

Interval	$(-\infty, -2)$	$(-2, 1)$	$(1, 3)$	$(3, \infty)$
Number Chosen	-3	0	2	4
Value of f	-1.5	6	-4	2
Conclusion	Negative	Positive	Negative	Positive

The solution set is $\{x \mid x \le -2 \text{ or } 1 < x \le 3\}$ or, using interval notation, $(-\infty, -2] \cup (1, 3]$.

37. $\frac{(x-2)^2}{x^2-1} \ge 0$

$\frac{(x-2)^2}{(x+1)(x-1)} \ge 0$

$f(x) = \frac{(x-2)^2}{x^2-1}$

The zeros and values where f is undefined are $x = -1$, $x = 1$ and $x = 2$.

Interval	$(-\infty, -1)$	$(-1, 1)$	$(1, 2)$	$(2, \infty)$
Number Chosen	-2	0	1.5	3
Value of f	$\frac{16}{3}$	-4	0.2	0.125
Conclusion	Positive	Negative	Positive	Positive

The solution set is $\{x \mid x < -1 \text{ or } x > 1\}$ or, using interval notation, $(-\infty, -1) \cup (1, \infty)$.

38. $\frac{(x+5)^2}{x^2-4} \ge 0$

$\frac{(x+5)^2}{(x+2)(x-2)} \ge 0$

$f(x) = \frac{(x+5)^2}{x^2-4}$

The zeros and values where f is undefined are $x = -5$, $x = -2$ and $x = 2$.

Interval	$(-\infty, -5)$	$(-5, -2)$	$(-2, 2)$	$(2, \infty)$
Number Chosen	–6	–3	0	3
Value of f	0.03125	0.8	–6.25	12.8
Conclusion	Positive	Positive	Negative	Positive

The solution set is $\{x \mid x < -2 \text{ or } x > 2\}$ or, using interval notation, $(-\infty, -2) \cup (2, \infty)$.

39.
$$\frac{x+4}{x-2} \le 1$$
$$\frac{x+4}{x-2} - 1 \le 0$$
$$\frac{x+4-(x-2)}{x-2} \le 0$$
$$\frac{6}{x-2} \le 0$$
$$f(x) = \frac{6}{x-2}$$
The value where f is undefined is $x = 2$.

Interval	$(-\infty, 2)$	$(2, \infty)$
Number Chosen	0	3
Value of f	–3	6
Conclusion	Negative	Positive

The solution set is $\{x \mid x < 2\}$ or, using interval notation, $(-\infty, 2)$.

40.
$$\frac{x+2}{x-4} \ge 1$$
$$\frac{x+2}{x-4} - 1 \ge 0$$
$$\frac{x+2-(x-4)}{x-4} \ge 0$$
$$\frac{6}{x-4} \ge 0$$
$$f(x) = \frac{6}{x-4}$$
The value where f is undefined is $x = 4$.

Interval	$(-\infty, 4)$	$(4, \infty)$
Number Chosen	0	5
Value of f	–1.5	6
Conclusion	Negative	Positive

The solution set is $\{x \mid x > 4\}$ or, using interval notation, $(4, \infty)$.

41.
$$\frac{3x-5}{x+2} \le 2$$
$$\frac{3x-5}{x+2} - 2 \le 0$$
$$\frac{3x-5-2(x+2)}{x+2} \le 0$$
$$\frac{x-9}{x+2} \le 0$$
$$f(x) = \frac{x-9}{x+2}$$
The zeros and values where f is undefined are $x = -2$ and $x = 9$.

Interval	$(-\infty, -2)$	$(-2, 9)$	$(9, \infty)$
Number Chosen	–3	0	10
Value of f	12	–4.5	$\frac{1}{12}$
Conclusion	Positive	Negative	Positive

The solution set is $\{x \mid -2 < x \le 9\}$ or, using interval notation, $(-2, 9]$.

42.
$$\frac{x-4}{2x+4} \ge 1$$
$$\frac{x-4}{2x+4} - 1 \ge 0$$
$$\frac{x-4-2x-4}{2x+4} \ge 0$$
$$\frac{x+8}{2(x+2)} \le 0$$
$$f(x) = \frac{x+8}{2(x+2)}$$
The zeros and values where f is undefined are $x = -8$ and $x = -2$.

Interval	$(-\infty, -8)$	$(-8, -2)$	$(-2, \infty)$
Number Chosen	–9	–3	0
Value of f	$\frac{1}{14}$	–2.5	2
Conclusion	Positive	Negative	Positive

The solution set is $\{x \mid -8 \le x < -2\}$ or, using interval notation, $[-8, -2)$.

43. $$\frac{1}{x-2}<\frac{2}{3x-9}$$
$$\frac{1}{x-2}-\frac{2}{3x-9}<0$$
$$\frac{3x-9-2(x-2)}{(x-2)(3x-9)}<0$$
$$\frac{x-5}{(x-2)(3x-9)}<0$$
$$f(x)=\frac{x-5}{(x-2)(3x-9)}$$

The zeros and values where f is undefined are $x=2,\quad x=3,$ and $x=5$.

Interval	$(-\infty, 2)$	$(2, 3)$	$(3, 5)$	$(5, \infty)$
Number Chosen	0	2.5	4	6
Value of f	$-\frac{5}{18}$	$\frac{10}{3}$	$-\frac{1}{6}$	$\frac{1}{36}$
Conclusion	Negative	Positive	Negative	Positive

The solution set is $\{x \mid x<2 \text{ or } 3<x<5\}$ or, using interval notation, $(-\infty,2)\cup(3,5)$.

44. $$\frac{5}{x-3}>\frac{3}{x+1}$$
$$\frac{5}{x-3}-\frac{3}{x+1}>0$$
$$\frac{5x+5-3x+9}{(x-3)(x+1)}>0$$
$$\frac{2(x+7)}{(x-3)(x+1)}>0$$
$$f(x)=\frac{2(x+7)}{(x-3)(x+1)}$$

The zeros and values where f is undefined are $x=-7,\ x=-1,$ and $x=3$.

Interval	$(-\infty, -7)$	$(-7,-1)$	$(-1, 3)$	$(3, \infty)$
Number Chosen	-8	-2	0	4
Value of f	$-\frac{2}{77}$	2	$-\frac{14}{3}$	$\frac{22}{5}$
Conclusion	Negative	Positive	Negative	Positive

The solution set is $\{x \mid -7<x<-1 \text{ or } x>3\}$ or, using interval notation, $(-7,-1)\cup(3,\infty)$.

45. $$\frac{x^2(3+x)(x+4)}{(x+5)(x-1)}\ge 0$$
$$f(x)=\frac{x^2(3+x)(x+4)}{(x+5)(x-1)}$$

The zeros and values where f is undefined are $x=-5,\ x=-4,\ x=-3,\ x=0$ and $x=1$.

Interval	Number Chosen	Value of f	Conclusion
$(-\infty, -5)$	-6	$\frac{216}{7}$	Positive
$(-5,-4)$	-4.5	$-\frac{243}{44}$	Negative
$(-4, -3)$	-3.5	$\frac{49}{108}$	Positive
$(-3, 0)$	-1	-0.75	Negative
$(0, 1)$	0.5	$-\frac{63}{44}$	Negative
$(1, \infty)$	2	$\frac{120}{7}$	Positive

The solution set is $\{x \mid x<-5 \text{ or } -4\le x\le -3 \text{ or } x=0 \text{ or } x>1\}$ or, using interval notation, $(-\infty,-5)\cup[-4,-3]\cup\{0\}\cup(1,\infty)$.

46. $$\frac{x(x^2+1)(x-2)}{(x-1)(x+1)}\ge 0$$
$$f(x)=\frac{x(x^2+1)(x-2)}{(x-1)(x+1)}$$

The zeros and values where f is undefined are $x=-1,\ x=0,\ x=1$ and $x=2$.

Interval	$(-\infty, -1)$	$(-1, 0)$	$(0, 1)$	$(1, 2)$	$(2, \infty)$
Number Chosen	-2	-0.5	0.5	1.5	3
Value of f	$\frac{40}{3}$	$-\frac{25}{12}$	1.25	-1.95	3.75
Conclusion	Positive	Negative	Positive	Negative	Positive

The solution set is $\{x \mid x<-1 \text{ or } 0\le x<1 \text{ or } x\ge 2\}$ or, using interval notation, $(-\infty,-1)\cup[0,1)\cup[2,\infty)$.

47. $\dfrac{(3-x)^3(2x+1)}{x^3-1}<0$

$\dfrac{(3-x)^3(2x+1)}{(x-1)(x^2+x+1)}<0$

$f(x)=\dfrac{(3-x)^3(2x+1)}{(x-1)(x^2+x+1)}$

The zeros and values where *f* is undefined are $x=3, x=-\frac{1}{2},$ and $x=1$.

Interval	Number Chosen	Value of *f*	Conclusion
$\left(-\infty,-\frac{1}{2}\right)$	-1	32	Positive
$\left(-\frac{1}{2},1\right)$	0	-27	Negative
$(1, 3)$	2	$5/7$	Positive
$(3,\infty)$	4	$-1/7$	Negative

The solution set is $\left\{x \middle| -\frac{1}{2}<x<1 \text{ or } x>3\right\}$ or, using interval notation, $\left(-\frac{1}{2},1\right)\cup(3,\infty)$.

48. $\dfrac{(2-x)^3(3x-2)}{x^3+1}<0$

$\dfrac{(2-x)^3(3x-2)}{(x+1)(x^2-x+1)}<0$

$f(x)=\dfrac{(2-x)^3(3x-2)}{(x+1)(x^2-x+1)}$

The zeros and values where *f* is undefined are $x=2, x=\frac{2}{3},$ and $x=-1$.

Interval	Number Chosen	Value of *f*	Conclusion
$(-\infty,-1)$	-2	$512/7$	Positive
$\left(-1,\frac{2}{3}\right)$	0	-16	Negative
$\left(\frac{2}{3}, 2\right)$	1	$1/2$	Positive
$(2,\infty)$	3	$-1/4$	Negative

The solution set is $\left\{x \middle| -1<x<\frac{2}{3} \text{ or } x>2\right\}$ or, using interval notation, $\left(-1,\frac{2}{3}\right)\cup(2,\infty)$.

49. $(x+1)(x-3)(x-5)>0$

$f(x)=(x+1)(x-3)(x-5)$

$x=-1, x=3, x=5$ are the zeros of *f*.

Interval	$(-\infty,-1)$	$(-1,3)$	$(3, 5)$	$(5,\infty)$
Number Chosen	-2	0	4	6
Value of *f*	-35	15	-5	21
Conclusion	Negative	Positive	Negative	Positive

We want to know where $f(x)>0$, so the solution set is $\{x \mid -1<x<3 \text{ or } x>5\}$ or, using interval notation, $(-1, 3)\cup(5,\infty)$.

50. $(2x-1)(x+2)(x+5)<0$

$f(x)=(2x-1)(x+2)(x+5)$

$x=\frac{1}{2}, x=-2, x=-5$ are the zeros of *f*.

Interval	$(-\infty,-5)$	$(-5,-2)$	$\left(-2,\frac{1}{2}\right)$	$\left(\frac{1}{2},\infty\right)$
Number Chosen	-6	-4	0	1
Value of *f*	-52	18	-10	18
Conclusion	Negative	Positive	Negative	Positive

We want to know where $f(x)<0$, so the solution set is $\left\{x \middle| x<-5 \text{ or } -2<x<\frac{1}{2}\right\}$ or, using interval notation, $(-\infty,-5)\cup\left(-2,\frac{1}{2}\right)$.

51. $7x-4\ge -2x^2$

$2x^2+7x-4\ge 0$

$(2x-1)(x+4)\ge 0$

$f(x)=2x^2+7x-4$

$x=\frac{1}{2}, x=-4$ are the zeros of *f*.

Interval	$(-\infty,-4)$	$\left(-4,\frac{1}{2}\right)$	$\left(\frac{1}{2},\infty\right)$
Number Chosen	-5	0	1
Value of *f*	11	-4	5
Conclusion	Positive	Negative	Positive

We want to know where $f(x)\ge 0$, so the solution set is $\left\{x \middle| x\le -4 \text{ or } x\ge \frac{1}{2}\right\}$ or, using interval notation, $(-\infty,-4]\cup\left[\frac{1}{2},\infty\right)$.

52. $x^2+3x \ge 10$

$x^2+3x-10 \ge 0$

$(x+5)(x-2) \ge 0$

$f(x)=x^2+3x-10$

$x=-5, x=2$ are the zeros of f.

Interval	$(-\infty,-5)$	$(-5,2)$	$(2,\infty)$
Number Chosen	-6	0	3
Value of f	8	-10	8
Conclusion	Positive	Negative	Positive

We want to know where $f(x) \ge 0$, so the solution set is $\{x \mid x \le -5 \text{ or } x \ge 2\}$ or, using interval notation, $(-\infty,-5] \cup [2,\infty)$.

53. $\dfrac{x+1}{x-3} \le 2$

$\dfrac{x+1}{x-3}-2 \le 0$

$\dfrac{x+1-2(x-3)}{x-3} \le 0$

$\dfrac{x+1-2x+6)}{x-3} \le 0$

$\dfrac{-x+7}{x-3} \le 0$

$f(x)=\dfrac{-x+7}{x-3}$

The zeros and values where f is undefined are $x=3$ and $x=7$.

Interval	$(-\infty,3)$	$(3,7)$	$(7,\infty)$
Number Chosen	1	5	8
Value of f	-3	1	$-\frac{1}{5}$
Conclusion	Negative	Positive	Negative

We want to know where $f(x) \le 0$, so the solution set is $\{\, x \mid x<3 \text{ or } x \ge 7 \,\}$ or, using interval notation, $(-\infty,3) \cup [7,\infty)$. Note that 3 is not in the solution set because 3 is not in the domain of f.

54. $\dfrac{x-1}{x+2} \ge -2$

$\dfrac{x-1}{x+2}+2 \ge 0$

$\dfrac{x-1+2(x+2)}{x+2} \ge 0$

$\dfrac{x-1+2x+4}{x+2} \ge 0$

$\dfrac{3x+3}{x+2} \ge 0$

$\dfrac{3(x+1)}{x+2} \ge 0$

$f(x)=\dfrac{3(x+1)}{x+2}$

The zeros and values where f is undefined are $x=-2$ and $x=-1$.

Interval	$(-\infty,-2)$	$(-2,-1)$	$(-1,\infty)$
Number Chosen	-3	$-\frac{3}{2}$	1
Value of f	6	-3	2
Conclusion	Positive	Negative	Positive

We want to know where $f(x) \ge 0$, so the solution set is $\{\, x \mid x<-2 \text{ or } x \ge -1 \,\}$ or, using interval notation, $(-\infty,-2) \cup [-1,\infty)$. Note that -2 is not in the solution set because -2 is not in the domain of f.

55. $3(x^2-2) < 2(x-1)^2+x^2$

$3x^2-6 < 2(x^2-2x+1)+x^2$

$3x^2-6 < 2x^2-4x+2+x^2$

$3x^2-6 < 3x^2-4x+2$

$-6 < -4x+2$

$4x-6 < 2$

$4x < 8$

$x < 2$

The solution set is $\{x \mid x<2\}$ or, using interval notation, $(-\infty,2)$.

56. $(x-3)(x+2) < x^2+3x+5$

$$x^2+2x-3x-6 < x^2+3x+5$$
$$x^2-x-6 < x^2+3x+5$$
$$-x-6 < 3x+5$$
$$-4x-6 < 5$$
$$-4x < 11$$
$$x > -\frac{11}{4}$$

The solution set is $\left\{x \middle| x > -\frac{11}{4}\right\}$ or, using interval notation, $\left(-\frac{11}{4}, \infty\right)$.

57. $6x-5 < \frac{6}{x}$

$$6x-5-\frac{6}{x} < 0$$
$$\frac{6x^2-5x-6}{x} < 0$$
$$\frac{(2x-3)(3x+2)}{x} < 0$$
$$f(x) = \frac{(2x-3)(3x+2)}{x}$$

The zeros and values where f is undefined are $x = -\frac{2}{3}$, $x = 0$ and $x = \frac{3}{2}$.

Interval	$\left(-\infty, -\frac{2}{3}\right)$	$\left(-\frac{2}{3}, 0\right)$	$\left(0, \frac{3}{2}\right)$	$\left(\frac{3}{2}, \infty\right)$
Number Chosen	-1	-0.5	1	2
Value of f	-5	4	-5	4
Conclusion	Negative	Positive	Negative	Positive

We want to know where $f(x) < 0$, so the solution set is $\left\{ x \middle| x < -\frac{2}{3} \text{ or } 0 < x < \frac{3}{2} \right\}$ or, using interval notation, $\left(-\infty, -\frac{2}{3}\right) \cup \left(0, \frac{3}{2}\right)$.

58. $x + \frac{12}{x} < 7$

$$x+\frac{12}{x}-7 < 0$$
$$\frac{x^2-7x+12}{x} < 0$$
$$\frac{(x-3)(x-4)}{x} < 0$$

$f(x) = \frac{(x-3)(x-4)}{x}$; The zeros and values where f is undefined are $x = 0$, $x = 3$ and $x = 4$.

Interval	$(-\infty, 0)$	$(0, 3)$	$(3, 4)$	$(4, \infty)$
Number Chosen	-1	1	3.5	5
Value of f	-20	6	$-\frac{1}{14}$	0.4
Conclusion	Negative	Positive	Negative	Positive

We want to know where $f(x) < 0$, so the solution set is $\{ x \mid x < 0 \text{ or } 3 < x < 4 \}$ or, using interval notation, $(-\infty, 0) \cup (3, 4)$.

59. $x^3 - 9x \le 0$

$$x(x-3)(x+3) \le 0$$
$$f(x) = x^3 - 9x$$

$x = -3$, $x = 0$, $x = 3$ are the zeros of f.

Interval	$(-\infty, -3)$	$(-3, 0)$	$(0, 3)$	$(3, \infty)$
Number Chosen	-4	-1	1	4
Value of f	-28	8	-8	28
Conclusion	Negative	Positive	Negative	Positive

We want to know where $f(x) \le 0$, so the solution set is $\{x \mid x \le -3 \text{ or } 0 \le x \le 3\}$ or, using interval notation, $(-\infty, -3] \cup [0, 3]$.

60. $x^3 - x \ge 0$

$$x(x-1)(x+1) \ge 0$$
$$f(x) = x^3 - x$$

$x = -1$, $x = 0$, $x = 1$ are the zeros of f.

Interval	$(-\infty, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \infty)$
Number Chosen	-2	$-\frac{1}{2}$	$\frac{1}{2}$	2
Value of f	-6	0.375	-0.375	6
Conclusion	Negative	Positive	Negative	Positive

We want to know where $f(x) \ge 0$, so the solution set is $\{x \mid -1 \le x \le 0 \text{ or } x \ge 1\}$ or, using interval notation, $[-1, 0] \cup [1, \infty)$.

61. $f(x)=2x^4+11x^3-11x^2-104x-48$

a. Step 1: $f(x)$ has at most 4 real zeros.

Step 2:Possible rational zeros:

$p:\pm1,\pm2,\pm3,\pm4,\pm6,\pm8,$
$\pm12,\pm24,\pm48$

$q:\pm1,\pm2$

$\frac{p}{q}=\pm1,\pm2,\pm3,\pm4,\pm6,\pm8,\pm12,$
$\pm24,\pm\frac{1}{2},\pm\frac{3}{2},\pm48$

Step 3:From the graph it appears that there are *x*-intercepts at –4, –0.5, and 3. Using synthetic division:

$$\begin{array}{r|rrrrr} -4 & 2 & 11 & -11 & -104 & -48 \\ & & -8 & -12 & 92 & 48 \\ \hline & 2 & 3 & -23 & -12 & 0 \end{array}$$

Since the remainder is 0, $x+4$ is a factor. Using synthetic division again:

$$\begin{array}{r|rrrr} -4 & 2 & 3 & -23 & -12 \\ & & -8 & 20 & 12 \\ \hline & 2 & -5 & -3 & 0 \end{array}$$

Since the remainder is 0, $x+4$ is also a factor again. Thus,

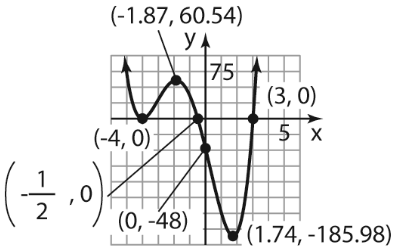
$$f(x)=(x+4)^2(2x^2-5x-3)$$
$$=(x-4)^2(2x+1)(x-3)$$

The zeros are –4, $-\frac{1}{2}$, and 3.

b. The factoring is:

$$f(x)=(x-4)^2(2x+1)(x-3)$$

c.



d. Looking at the graph we have $f(x)<0$ at $\left(-\frac{1}{2},3\right)$

62. $f(x)=4x^5-19x^4+32x^3-31x^2+28x-12$

a. Step 1: $f(x)$ has at most 5 real zeros.

Step 2:Possible rational zeros:

$p:\pm1,\pm2,\pm3,\pm4,\pm6,\pm12$

$q:\pm1,\pm2,\pm4$

$\frac{p}{q}=\pm1,\pm2,\pm3,\pm4,\pm6,\pm12,$
$\pm\frac{1}{2},\pm\frac{3}{2},\pm\frac{1}{4},\pm\frac{3}{4}$

Step 3:From the graph it appears that there are *x*-intercepts at 3/4 and 2. Using synthetic division:

$$\begin{array}{r|rrrrrr} 2 & 4 & -19 & 32 & -31 & 28 & -12 \\ & & 8 & -22 & 20 & -22 & 12 \\ \hline & 4 & -11 & 10 & -11 & 6 & 0 \end{array}$$

Since the remainder is 0, $x-2$ is a factor. Using synthetic division again:

$$\begin{array}{r|rrrrr} 2 & 4 & -11 & 10 & -11 & 6 \\ & & 8 & -6 & 8 & -6 \\ \hline & 4 & -3 & 4 & -3 & 0 \end{array}$$

Since the remainder is 0, $x-2$ is also a factor again. Using synthetic division again:

$$\begin{array}{r|rrrr} \frac{3}{4} & 4 & -3 & 4 & -3 \\ & & 3 & 0 & 3 \\ \hline & 4 & 0 & 4 & 0 \end{array}$$

Since the remainder is 0, $4x-3$ is a factor. Using synthetic division again: Thus,

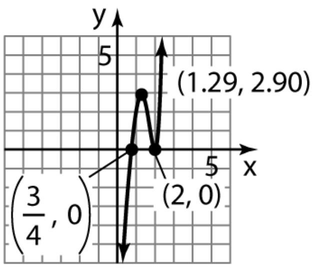
$$f(x)=(x-2)^2(4x^3-3x^2+4x-3)$$
$$=(x-2)^2(4x-3)(x^2+1)$$

The zeros are 2 and $\frac{3}{4}$.

b. The factoring is:

$$f(x)=(x-2)^2(4x-3)(x^2+1)$$

c.



d. Looking at the graph we have $f(x) < 0$ at $\left(-\infty, \frac{3}{4}\right)$

63. a. $R(x) = \dfrac{x^2+5x-6}{x^2-4} = \dfrac{(x+6)(x-1)}{(x-2)(x-2)}$ $\quad p(x) = x^2+5x-6; \quad q(x) = x^2-4; \quad n = 2; \; m = 2$

Step 1: Domain: $\{x \mid x \neq 2\}$

The y-intercept is $R(0) = \dfrac{(0)^2+5(0)-6}{(0)^2-4} = \dfrac{-6}{4} = -\dfrac{3}{2}$. Plot the point $\left(0, -\dfrac{3}{2}\right)$.

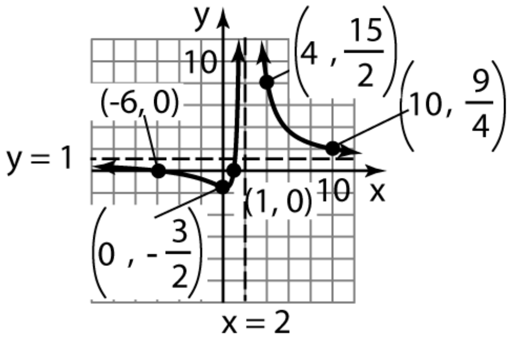
Step 2 & 3:In lowest terms, $R(x) = \dfrac{(x+6)(x-1)}{(x-2)(x-1)}$, $x \neq 2$. The x-intercepts are the zeros of $y = x+6$ and $y = x-1$: $-6, 1$;

Step 4: In lowest terms, $R(x) = \dfrac{(x+6)(x-1)}{(x-2)(x-2)}$, $x \neq 2$. The vertical asymptote is the zero of $f(x) = x-2$: $x = 2$;
Graph this asymptote using a dashed line.

Step 5: Since $n = m$, the line $y = 1$ is the horizontal asymptote. Graph this asymptote using a dashed line.
Solve to find intersection points:

$$\frac{x^2+5x-6}{x^2-4} = 1$$
$$x^2+5x-6 = x^2-4$$
$$5x = 2$$
$$x = \frac{2}{5}$$

Steps 6 & 7: Graphing:



b. $\dfrac{(x+6)(x-1)}{(x-2)(x-2)} \geq 0$

The zeros and values where f is undefined are $x = -6$, $x = 1$, and $x = 2$.

Interval	Number Chosen	Value of f	Conclusion
$(-\infty, -6)$	-6	$\frac{216}{7}$	Positive
$(-6, -1)$	-4.5	$-\frac{243}{44}$	Negative
$(-1, 2)$	-3.5	$\frac{49}{108}$	Positive
$(2, \infty)$	2	$\frac{120}{7}$	Positive

The solution set is $\{ x \mid x < -6 \text{ or } 1 \le x \le 2 \text{ or } x > 2 \}$ or, using interval notation, $(-\infty, -6] \cup [1, 2) \cup (2, \infty)$

64. **a.** $R(x) = \dfrac{2x^2 + 9x + 9}{x^2 - 4} = \dfrac{(2x+3)(x+3)}{(x+2)(x-2)}$ $p(x) = 2x^2 + 9x + 9;$ $q(x) = x^2 - 4;$ $n = 2;$ $m = 2$

Step 1: Domain: $\{x \mid x \ne 2, -2\}$

The y-intercept is $R(0) = \dfrac{2(0)^2 + 9(0) + 9}{(0)^2 - 4} = \dfrac{9}{-4} = -\dfrac{9}{4}$. Plot the point $\left(0, -\dfrac{9}{4}\right)$.

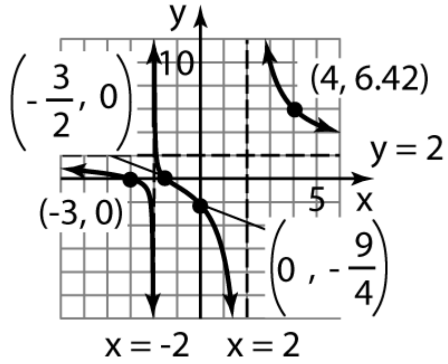
Step 2 & 3: In lowest terms, $R(x) = \dfrac{(2x+3)(x+3)}{(x+2)(x-2)}$, $x \ne 2, -2$. The x-intercepts are the zeros of $y = 2x + 3$ and $y = x + 3$: $-\dfrac{3}{2}, -3$;

Step 4: In lowest terms, $R(x) = \dfrac{(2x+3)(x+3)}{(x+2)(x-2)}$, $x \ne 2, -2$. The vertical asymptotes are the zeros of $f(x) = x - 2$ and $f(x) = x + 2$: $x = 2$ and $x = -2$;
Graph these asymptotes using dashed lines.

Step 5: Since $n = m$, the line $y = 1$ is the horizontal asymptote. Graph this asymptote using a dashed line.
Solve to find intersection points:

$$\frac{2x^2 + 9x + 9}{x^2 - 4} = 1$$
$$x^2 + 9x + 9 = x^2 - 4$$
$$9x = -13$$
$$x = -\frac{13}{9}$$

Steps 6 & 7: Graphing:



b. $\frac{(2x+3)(x+3)}{(x+2)(x-2)} \geq 0$

The zeros and values where f is undefined are $x = -\frac{3}{2}, x = -3, x = -2$ and $x = 2$.

Interval	Number Chosen	Value of *f*	Conclusion
$(-\infty, -3)$	–4	$\frac{5}{12}$	Positive
$(-3, -2)$	–2.5	$-\frac{4}{9}$	Negative
$\left(-2, -\frac{3}{2}\right)$	–1.75	$\frac{2}{3}$	Positive
$\left(-\frac{3}{2}, 2\right)$	0	$-\frac{9}{4}$	Negative
$(2, \infty)$	3	$\frac{54}{5}$	Positive

The solution set is $\left\{ x \,\middle|\, x < -3 \text{ or } -2 < x \leq -\frac{3}{2} \text{ or } x > 2 \right\}$ or, using interval notation,

$(-\infty, -3) \cup \left(-2, -\frac{3}{2}\right] \cup (2, \infty)$

65. a. $R(x) = \frac{x^3 + 2x^2 - 11x - 12}{x^2 - x - 6} = \frac{(x-3)(x+4)(x+1)}{(x-3)(x+2)} = \frac{(x+4)(x+1)}{(x+2)}$ $\quad p(x) = x^3 + 2x^2 - 11x - 12;$

$q(x) = x^2 - x - 6; \quad n = 3; \quad m = 2$

Step 1: Domain: $\{x \mid x \neq -2, 3\}$

The y-intercept is $R(0) = \frac{(0)^3 + 2(0)^2 - 11(0) - 12}{(0)^2 - (0) - 6} = \frac{-12}{-6} = 2$. Plot the point $(0, 2)$.

Step 2 & 3:In lowest terms, $R(x)=\dfrac{(x+4)(x+1)}{(x+2)},\ x\neq -2$. The x-intercepts are the zeros of $y=x+4$ and $y=x+1$: $-4,-1$;

Note: $x=3$ is not a zero because reduced form must be used to find the zeros.

Step 4: In lowest terms, $R(x)=\dfrac{(x+4)(x+1)}{(x+2)},\ x\neq -2$. The vertical asymptote is the zero of $f(x)=x+2$: $x=-2$;

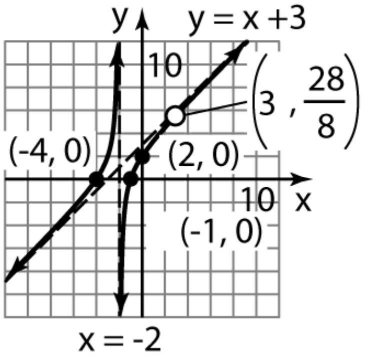
Graph this asymptote using a dashed line.

Step 5: Since $n=m+1$, there is an oblique asymptote. Dividing:

$$\begin{array}{r}
x+3 \\
x^2-x-6\overline{)\,x^3+2x^2-11x-12} \\
\underline{x^3-x^2\ \ -6x} \\
3x^2\ -5x\ -12 \\
\underline{3x^2\ -3x-18} \\
-2x+6
\end{array}
\qquad G(x)=x+3+\frac{-2x+6}{x^2-x-6}$$

The oblique asymptote is $y=x+3$. Graph this asymptote with a dashed line.

Steps 6 & 7: Graphing:



b. $\dfrac{(x-3)(x+4)(x+1)}{(x-3)(x+2)}\geq 0$

The zeros and values where f is undefined are $x=-4,\ x=-2,$ and $x=-1$.

Interval	Number Chosen	Value of f	Conclusion
$(-\infty,-4)$	-5	$-\frac{4}{3}$	Negative
$(-4,-2)$	-3	2	Positive
$(-2,-1)$	-1.5	$-\frac{5}{2}$	Negative
$(-1,\infty)$	0	2	Positive

The solution set is $\{x\mid -4\leq x<-2$ or $-1\leq x<3$ or $x>3\}$ or, using interval notation, $[-4,-2)\cup[-1,3)\cup(3,\infty)$

66. **a.** $R(x)=\dfrac{x^3-6x^2+9x-4}{x^2+x-20}=\dfrac{(x-1)(x-1)(x-4)}{(x+5)(x-4)}=\dfrac{(x-1)(x-1)}{(x+5)}$ $p(x)=x^3-6x^2+9x-4;$

$q(x)=x^2+x-20;$ $n=3;$ $m=2$

Step 1: Domain: $\{x \mid x\neq -5, 4\}$

The y-intercept is $R(0)=\dfrac{(0)^3-6(0)^2+9(0)-4}{(0)^2+(0)-20}=\dfrac{-4}{-20}=\dfrac{1}{5}$. Plot the point $\left(0,\dfrac{1}{5}\right)$.

Step 2 & 3:In lowest terms, $R(x)=\dfrac{(x-1)(x-1)}{(x+5)}$, $x\neq -5$. The x-intercept is the zero of $y=x-1$: 1;

Note: $x=4$ is not a zero because reduced form must be used to find the zeros.

Step 4: In lowest terms, $R(x)=\dfrac{(x-1)(x-1)}{(x+5)}$, $x\neq -5$. The vertical asymptote is the zero of $f(x)=x+5$:

$x=-5$;
Graph this asymptote using a dashed line.

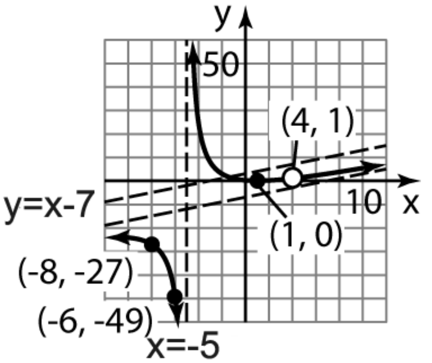
Step 5: Since $n=m+1$, there is an oblique asymptote. Dividing:

$$\begin{array}{r} x-7 \\ x^2+x-20\overline{\big)x^3-6x^2+9x-4} \\ \underline{x^3+x^2\quad -20x} \\ -7x^2+29x-4 \\ -7\underline{x^2\quad -7x+140} \\ 36x-144 \end{array}$$

$$G(x)=x+3+\frac{-2x+6}{x^2-x-6}$$

The oblique asymptote is $y=x-7$. Graph this asymptote with a dashed line.

Steps 6 & 7: Graphing:



b. $\dfrac{(x-1)(x-1)(x-4)}{(x+5)(x-4)}\geq 0$

The zeros and values where f is undefined are $x=-5$, $x=1$, and $x=4$.

Interval	Number Chosen	Value of f	Conclusion
$(-\infty, -5)$	-6	-49	Negative
$(-5, 1)$	0	$\frac{1}{5}$	Positive
$(1, 4)$	3	$\frac{1}{2}$	Positive
$(4, \infty)$	5	$\frac{8}{5}$	Positive

The solution set is $\{x \mid -5 < x < 4 \text{ or } x > 4\}$ or, using interval notation, $(-5, 4) \cup (4, \infty)$

67. Let x be the positive number. Then

$x^3 > 4x^2$

$x^3 - 4x^2 > 0$

$x^2(x-4) > 0$

$f(x) = x^2(x-4)$

$x = 0$ and $x = 4$ are the zeros of f.

Interval	$(-\infty, 0)$	$(0, 4)$	$(4, \infty)$
Number Chosen	-1	1	5
Value of f	-5	-3	25
Conclusion	Negative	Negative	Positive

Since x must be positive, all real numbers greater than 4 satisfy the condition. The solution set is $\{x \mid x > 4\}$ or, using interval notation, $(4, \infty)$.

68. Let x be the positive number. Then

$x^3 < x$

$x^3 - x < 0$

$x(x-1)(x+1) < 0$

$f(x) = x(x-1)(x+1)$

$x = -1, x = 0,$ and $x = 1$ are the zeros of f.

Interval	$(-\infty, -1)$	$(-1, 0)$	$(0, 1)$	$(1, \infty)$
Number Chosen	-2	$-1/2$	$1/2$	2
Value of f	-6	0.375	-0.375	6
Conclusion	Negative	Positive	Negative	Positive

Since x must be positive, all real numbers between (but not including) 0 and 1 satisfy the condition. The solution set is $\{x \mid 0 < x < 1\}$ or, using interval notation, $(0, 1)$.

69. The domain of $f(x) = \sqrt{x^4 - 16}$ consists of all real numbers x for which

$x^4 - 16 \ge 0$

$(x^2+4)(x^2-4) \ge 0$

$(x^2+4)(x-2)(x+2) \ge 0$

$p(x) = (x^2+4)(x-2)(x+2)$

$x = -2$ and $x = 2$ are the zeros of p.

Interval	$(-\infty, -2)$	$(-2, 2)$	$(2, \infty)$
Number Chosen	-3	0	3
Value of p	65	-16	65
Conclusion	Positive	Negative	Positive

The domain of f will be where $p(x) \ge 0$. Thus, the domain of f is $\{x \mid x \le -2 \text{ or } x \ge 2\}$ or, using interval notation, $(-\infty, -2] \cup [2, \infty)$.

70. The domain of $f(x) = \sqrt{x^3 - 3x^2}$ consists of all real numbers x for which

$x^3 - 3x^2 \ge 0$

$x^2(x-3) \ge 0$

$p(x) = x^2(x-3)$

$x = 0$ and $x = 3$ are the zeros of p.

Interval	$(-\infty, 0)$	$(0,3)$	$(3, \infty)$
Number Chosen	-1	1	4
Value of p	-4	-2	16
Conclusion	Negative	Negative	Positive

The domain of *f* will be where $p(x) \ge 0$. Thus, the domain of *f* is $\{x | x = 0 \text{ or } x \ge 3\}$ or, using interval notation, $\{0\} \cup [3,\infty)$.

71. The domain of $f(x) = \sqrt{\frac{x-2}{x+4}}$ includes all values for which $\frac{x-2}{x+4} \ge 0$.

$R(x) = \frac{x-2}{x+4}$

The zeros and values where *R* is undefined are $x = -4$ and $x = 2$.

Interval	$(-\infty, -4)$	$(-4, 2)$	$(2, \infty)$
Number Chosen	-5	0	3
Value of R	7	$-\frac{1}{2}$	$\frac{1}{7}$
Conclusion	Positive	Negative	Positive

The domain of *f* will be where $R(x) \ge 0$. Thus, the domain of *f* is $\{x | x < -4 \text{ or } x \ge 2\}$ or, using interval notation, $(-\infty,-4) \cup [2,\infty)$.

72. The domain of $f(x) = \sqrt{\frac{x-1}{x+4}}$ includes all values for which

$\frac{x-1}{x+4} \ge 0$

$R(x) = \frac{x-1}{x+4}$

The zeros and values where the expression is undefined are $x = -4$ and $x = 1$.

Interval	$(-\infty, -4)$	$(-4,1)$	$(1, \infty)$
Number Chosen	-5	0	2
Value of R	6	$-\frac{1}{4}$	$\frac{1}{6}$
Conclusion	Positive	Negative	Positive

The domain of *f* will be where $R(x) \ge 0$. Thus, the domain of *f* is $\{x | x < -4 \text{ or } x \ge 1\}$ or, using interval notation, $(-\infty,-4) \cup [1,\infty)$.

73.

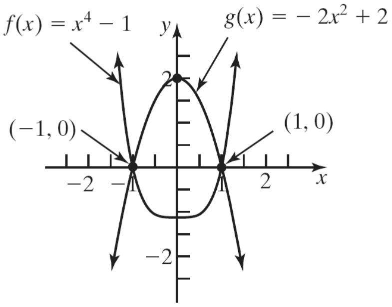
$$\begin{aligned} f(x) &\le g(x) \\ x^4 - 1 &\le -2x^2 + 2 \\ x^4 + 2x^2 - 3 &\le 0 \\ (x^2+3)(x^2-1) &\le 0 \\ (x^2+3)(x-1)(x+1) &\le 0 \end{aligned}$$

$h(x) = (x^2+3)(x-1)(x+1)$

$x = -1$ and $x = 1$ are the zeros of *h*.

Interval	$(-\infty, -1)$	$(-1,1)$	$(1, \infty)$
Number Chosen	-2	0	2
Value of h	21	-3	21
Conclusion	Positive	Negative	Positive

$f(x) \le g(x)$ if $-1 \le x \le 1$. That is, on the interval $[-1,1]$.



74.

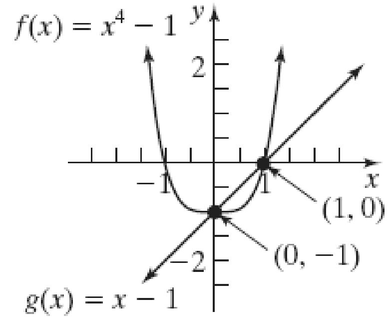
$$\begin{aligned} f(x) &\le g(x) \\ x^4 - 1 &\le x - 1 \\ x^4 - x &\le 0 \\ x(x^3-1) &\le 0 \\ x(x-1)(x^2+x+1) &\le 0 \end{aligned}$$

$h(x) = x(x-1)(x^2+x+1)$

$x = 0$ and $x = 1$ are the zeros of *h*.

Interval	$(-\infty, 0)$	$(0,1)$	$(1, \infty)$
Number Chosen	-1	$1/2$	2
Value of h	2	$-7/16$	14
Conclusion	Positive	Negative	Positive

$f(x) \le g(x)$ if $0 \le x \le 1$. That is, on the interval $[0,1]$.



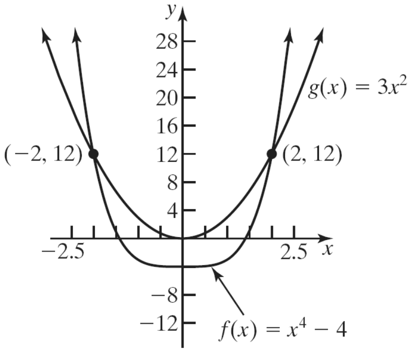
75.

$$f(x) \le g(x)$$
$$x^4 - 4 \le 3x^2$$
$$x^4 - 3x^2 - 4 \le 0$$
$$(x^2 - 4)(x^2 + 1) \le 0$$
$$(x-2)(x+2)(x^2+1) \le 0$$
$$h(x) = (x-2)(x+2)(x^2+1)$$

$x = -2$ and $x = 2$ are the zeros of *h*.

Interval	$(-\infty, -2)$	$(-2, 2)$	$(2, \infty)$
Number Chosen	−3	0	3
Value of *h*	50	−4	50
Conclusion	Positive	Negative	Positive

$f(x) \le g(x)$ if $-2 \le x \le 2$. That is, on the interval $[-2, 2]$.



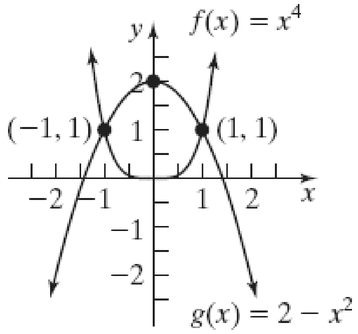
76.

$$f(x) \le g(x)$$
$$x^4 \le 2 - x^2$$
$$x^4 + x^2 - 2 \le 0$$
$$(x^2 + 2)(x^2 - 1) \le 0$$
$$(x^2+2)(x-1)(x+1) \le 0$$
$$h(x) = (x^2+2)(x-1)(x+1)$$

$x = -1$ and $x = 1$ are the zeros of *h*.

Interval	$(-\infty, -1)$	$(-1, 1)$	$(1, \infty)$
Number Chosen	−2	0	2
Value of *h*	18	−2	18
Conclusion	Positive	Negative	Positive

$f(x) \le g(x)$ if $-1 \le x \le 1$. That is, on the interval $[-1, 1]$.



77. We need to solve $\bar{C}(x) \le 100$.

$$\frac{80x + 5000}{x} \le 100$$
$$\frac{80x + 5000}{x} - \frac{100x}{x} \le 0$$
$$\frac{5000 - 20x}{x} \le 0$$
$$\frac{20(250 - x)}{x} \le 0$$
$$f(x) = \frac{20(250 - x)}{x}$$

The zeros and values where the expression is undefined are $x = 0$ and $x = 250$.

Interval	$(-\infty, 0)$	$(0, 250)$	$(250, \infty)$
Number Chosen	−1	1	260
Value of *f*	−5020	4980	−10/13
Conclusion	Negative	Positive	Negative

The number of bicycles produced cannot be negative, so the solution is $\{x \mid x \geq 250\}$ or, using interval notation, $[250, \infty)$. The company must produce at least 250 bicycles each day to keep average costs to no more than \$100.

78. We need to solve $\bar{C}(x) \leq 100$.

$$\frac{80x+6000}{x} \leq 100$$

$$\frac{80x+6000}{x} - \frac{100x}{x} \leq 0$$

$$\frac{6000-20x}{x} \leq 0$$

$$\frac{20(300-x)}{x} \leq 0$$

$$f(x) = \frac{20(300-x)}{x}$$

The zeros and values where the expression is undefined are $x = 0$ and $x = 300$.

Interval	$(-\infty, 0)$	$(0, 300)$	$(300, \infty)$
Number Chosen	-1	1	310
Value of *f*	-6020	5980	$-20/31$
Conclusion	Negative	Positive	Negative

The number of bicycles produced cannot be negative, so the solution is $\{x \mid x \geq 300\}$ or, using interval notation, $[300, \infty)$. The company must produce at least 300 bicycles each day to keep average costs to no more than \$100.

79. a.

$$K \geq 16$$

$$\frac{2(150)(S+42)}{S^2} \geq 16$$

$$\frac{300S+12{,}600}{S^2} \geq 16$$

$$\frac{300S+12{,}600}{S^2} - 16 \geq 0$$

$$\frac{300S+12{,}600-16S^2}{S^2} \geq 0$$

Solve $-16S^2 + 300S + 12{,}600 = 0$ and $S^2 = 0$. The zeros and values where the left-hand side is undefined are $S = 0$, $S \approx 39$, $S \approx -20$. Since the stretch cannot be negative, we only consider cases where $S \geq 0$.

Interval	$(0, 39)$	$(39, \infty)$
Number Chosen	1	40
Value of left side	12884	-0.625
Conclusion	Positive	Negative

The cord will stretch less than 39 feet.

b. The safe height is determined by the minimum clearance (3 feet), the free length of the cord (42 feet), and the stretch in the cord (39 feet). Therefore, the platform must be at least $3 + 42 + 39 = 84$ feet above the ground for a 150-pound jumper.

80. Let *r* = the distance between Earth and the object in kilometers. Then $384{,}400 - r$ = the distance between the object and the moon. We want

$$G\frac{m_{moon}m_{obj}}{(384{,}400-r)^2} > G\frac{m_{earth}m_{obj}}{r^2}$$

$$\frac{m_{moon}}{(384{,}400-r)^2} > \frac{m_{earth}}{r^2}$$

$$\frac{m_{moon}}{(384{,}400-r)^2} - \frac{m_{earth}}{r^2} > 0$$

$$\frac{r^2 m_{moon} - (384{,}400-r)^2 m_{earth}}{r^2(384{,}400-r)^2} > 0$$

The zeros and values where the left-hand side is undefined are $r = 0$, $r \approx 432{,}353$, $r \approx 346{,}022$, and $r = 384{,}400$. Since the distance from Earth to the object will be greater than 0 but less than the distance to the moon, we can exclude some of these values.

Interval	$(0, 346022)$	$(346022, 384400)$
Number Chosen	100,000	350,000
Value of left side	-6×10^{14}	1.3×10^{13}
Conclusion	Negative	Positive

The gravitational force on the object due to the moon will be greater than the force due to the Earth when the object is more than 346,022 kilometers from Earth.

81. Let *x* represent the number of student that attend the play. Then the discounted price per ticket is $40 - 0.20x$. Each student's share of the bus cost is $\frac{500}{x}$. Thus, each student's total cost will be

$$C(x) = 40 - 0.20x + \frac{500}{x}.$$

We need to solve $C(x) \le 40$.

$$40 - 0.20x + \frac{500}{x} \le 40$$

$$-0.20x + \frac{500}{x} \le 0$$

$$0.20x - \frac{500}{x} \ge 0$$

$$\frac{0.20x^2 - 500)}{x} \ge 0$$

$$\frac{0.20(x^2 - 2500)}{x} \ge 0$$

$$\frac{0.20(x+50)(x-50)}{x} \ge 0$$

$$f(x) = \frac{0.20(x+50)(x-50)}{x}$$

The zeros and values where *f* is undefined are $x = -50$, $x = 0$ and $x = 50$. Since the number of students cannot be negative, we only consider cases where $x \ge 0$.

Interval	(0, 50)	(50, ∞)
Number Chosen	25	100
Value of *f*	−15	15
Conclusion	Negative	Positive

We are looking for where $f(x) \ge 0$. Thus, the solution is $\{x \mid x \ge 50\}$ or, using interval notation, $[50, \infty)$. If at least 50 students attend the play, the price per student will be at or below $40.

82. Answers will vary, for example, $x^2 < 0$ has no real solution and $x^2 \le 0$ has exactly one real solution.

83. $x^4 + 1 < -5$ has no solution because the quantity $x^4 + 1$ is never negative. ($x^4 + 1 \ge 1$)

84. No, the student is not correct. For example, $x = -5$ is in the solution set, but does not satisfy the original inequality.

$$\frac{-5+4}{-5-3} = \frac{-1}{-8} = \frac{1}{8} \not\le 0$$

When multiplying both sides of an inequality by a negative, we must switch the direction of the inequality. Since we do not know the sign of $x+3$, we cannot multiply both sides of the inequality by this quantity.

85. Answers will vary. One example:

$$\frac{x-5}{x+3} \le 0$$

Chapter 4 Review Exercises

1. $f(x)=4x^5-3x^2+5x-2$ is a polynomial of degree 5.

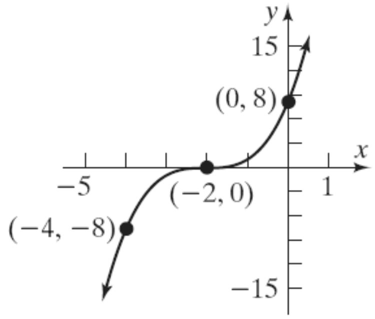
2. $f(x)=\dfrac{3x^5}{2x+1}$ is a rational function. It is not a polynomial because there are variables in the denominator.

3. $f(x)=3x^2+5x^{1/2}-1$ is not a polynomial because the variable x is raised to the $\frac{1}{2}$ power, which is not a nonnegative integer.

4. $f(x)=3$ is a polynomial of degree 0.

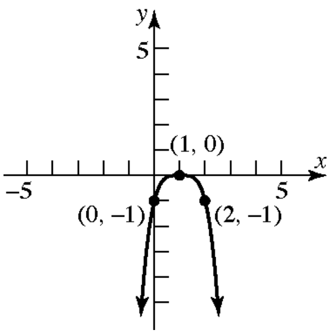
5. $f(x)=(x+2)^3$

Using the graph of $y=x^3$, shift left 2 units.



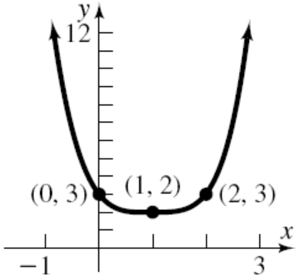
6. $f(x)=-(x-1)^4$

Using the graph of $y=x^4$, shift right 1 unit, then reflect about the x-axis.



7. $f(x)=(x-1)^4+2$

Using the graph of $y=x^4$, shift right 1 unit, then shift up 2 units.



8. $f(x)=x(x+2)(x+4)$

Step 1: Degree is 3. The function resembles $y=x^3$ for large values of $|x|$.

Step 2: y-intercept: $f(0)=0(0+2)(0+4)=0$

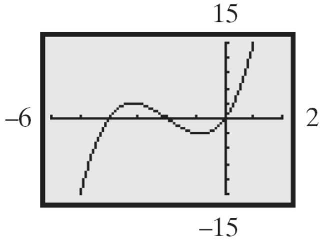
x-intercepts: solve $f(x)=0$

$x(x+2)(x+4)=0$

$x=0$ or $x=-2$ or $x=-4$

Step 3: Real zeros: -4 with multiplicity one, -2 with multiplicity one, 0 with multiplicity one. The graph crosses the x-axis at $x=-4$, $x=-2$, and $x=0$.

Step 4: Graphing utility:

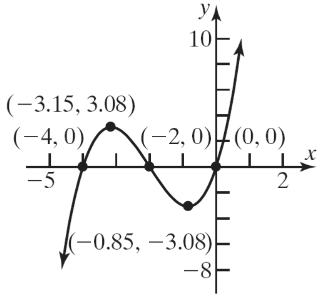


Step 5: 2 turning points;

local maximum: $(-3.15, 3.08)$;

local minimum: $(-0.85, -3.08)$

Step 6: Graphing by hand:



Step 7: Domain: $(-\infty,\infty)$; Range: $(-\infty,\infty)$

Step 8: Increasing on $(-\infty,-3.15)$ and $(-0.85,\infty)$; decreasing on $(-3.15,-0.85)$

9. $f(x) = (x-2)^2(x+4)$

Step 1: Degree is 3. The function resembles $y = x^3$ for large values of $|x|$.

Step 2: y-intercept: $f(0) = (0-2)^2(0+4) = 16$

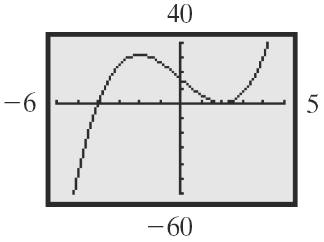
x-intercepts: solve $f(x) = 0$

$(x-2)^2(x+4) = 0$

$x = 2$ or $x = -4$

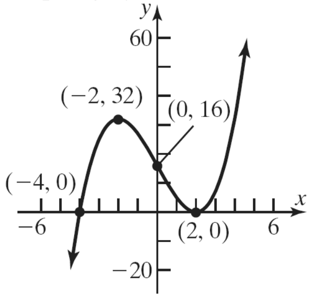
Step 3: Real zeros: -4 with multiplicity one, 2 with multiplicity two. The graph crosses the x-axis at $x = -4$ and touches it at $x = 2$.

Step 4: Graphing utility:



Step 5: 2 turning points; local maximum: $(-2, 32)$; local minimum: $(2, 0)$

Step 6: Graphing by hand:



Step 7: Domain: $(-\infty, \infty)$; Range: $(-\infty, \infty)$

Step 8: Increasing on $(-\infty, -2)$ and $(2, \infty)$; decreasing on $(-2, 2)$

10. $f(x) = -2x^3 + 4x^2$

Step 1: Degree is 3. The function resembles $y = -2x^3$ for large values of $|x|$.

Step 2: y-intercept: $f(0) = -2(0)^3 + 4(0)^2 = 0$

x-intercepts: solve $f(x) = 0$

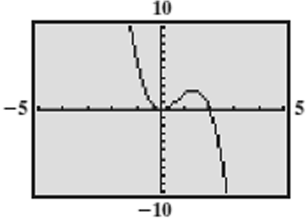
$-2x^3 + 4x^2 = 0$

$-2x^2(x-2) = 0$

$x = 0$ or $x = 2$

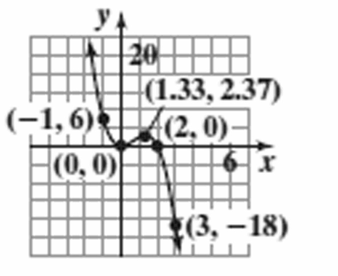
Step 3: Real zeros: 0 with multiplicity two, 2 with multiplicity one. The graph touches the x-axis at $x = 0$ and crosses it at $x = 2$.

Step 4: Graphing utility:



Step 5: 2 turning points; local minimum: $(0, 0)$; local maximum: $(1.33, 2.37)$

Step 6: Graphing by hand:



Step 7: Domain: $(-\infty, \infty)$; Range: $(-\infty, \infty)$

Step 8: Decreasing on $(-\infty, 0)$ and $(1.33, \infty)$; increasing on $(0, 1.33)$

11. $f(x) = (x-1)^2(x+3)(x+1)$

Step 1: Degree is 4. The function resembles $y = x^4$ for large values of $|x|$.

Step 2: y-intercept:

$f(0) = (0-1)^2(0+3)(0+1) = 3$

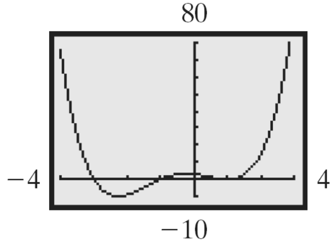
x-intercepts: solve $f(x) = 0$

$(x-1)^2(x+3)(x+1) = 0$

$x = 1$ or $x = -3$ or $x = -1$

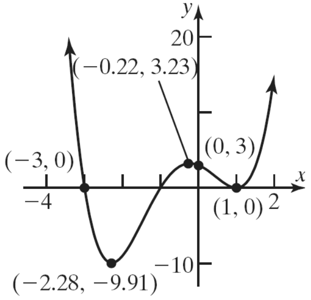
Step 3: Real zeros: -3 with multiplicity one, -1 with multiplicity one, 1 with multiplicity two. The graph crosses the x-axis at $x = -3$ and $x = -1$, and touches it at $x = 1$.

Step 4: Graphing utility:



Step 5: 3 turning points;
local maximum: $(-0.22, 3.23)$;
local minima: $(-2.28, -9.91)$ and $(1, 0)$

Step 6: Graphing by hand:



Step 7: Domain: $(-\infty, \infty)$; Range: $[-9.91, \infty)$

Step 8: Increasing on $(-2.28, -0.22)$ and $(1, \infty)$;
decreasing on $(-\infty, -2.28)$ and $(-0.22, 1)$

12. $f(x) = 8x^3 - 3x^2 + x + 4$
Since $g(x) = x - 1$ then $c = 1$. From the Remainder Theorem, the remainder R when $f(x)$ is divided by $g(x)$ is $f(c)$:

$$\begin{aligned} f(1) &= 8(1)^3 - 3(1)^2 + 1 + 4 \\ &= 8 - 3 + 1 + 4 \\ &= 10 \end{aligned}$$

So $R = 10$ and g is not a factor of f.

13. $f(x) = x^4 - 2x^3 + 15x - 2$
Since $g(x) = x + 2$ then $c = -2$. From the Remainder Theorem, the remainder R when $f(x)$ is divided by $g(x)$ is $f(c)$:

$$\begin{aligned} f(-2) &= (-2)^4 - 2(-2)^3 + 15(-2) - 2 \\ &= 16 - 2(-8) - 30 - 2 \\ &= 0 \end{aligned}$$

So $R = 0$ and g is a factor of f.

14.

$$\begin{array}{r|rrrrrrr} 4 & 12 & 0 & -8 & 0 & 0 & 0 & 1 \\ & & 48 & 192 & 736 & 2944 & 11{,}776 & 47{,}104 \\ \hline & 12 & 48 & 184 & 736 & 2944 & 11{,}776 & 47{,}105 \end{array}$$

$f(4) = 47{,}105$

15. $a_0 = -3$, $a_8 = 12$
$p = \pm 1, \pm 3$
$q = \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$

$$\frac{p}{q} = \pm 1, \pm 3, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}, \pm\frac{3}{4}, \pm\frac{1}{6}, \pm\frac{1}{12}$$

16. $f(x) = x^3 - 3x^2 - 6x + 8$

Possible rational zeros:
$p = \pm 1, \pm 2, \pm 4, \pm 8$; $q = \pm 1$;

$$\frac{p}{q} = \pm 1, \pm 2, \pm 4, \pm 8$$

Using synthetic division:
We try $x + 2$:

$$\begin{array}{r|rrrr} -2 & 1 & -3 & -6 & 8 \\ & & -2 & 10 & -8 \\ \hline & 1 & -5 & 4 & 0 \end{array}$$

$x + 2$ is a factor. The other factor is the quotient: $x^2 - 5x + 4$.

Thus, $f(x) = (x+2)\left(x^2 - 5x + 4\right)$.
$= (x+2)(x-1)(x-4)$

The zeros are –2, 1, and 4, each of multiplicity 1.

17. $f(x) = 4x^3 + 4x^2 - 7x + 2$

Possible rational zeros:
$p = \pm 1, \pm 2$; $q = \pm 1, \pm 2, \pm 4$;

$$\frac{p}{q} = \pm 1, \pm 2, \pm\frac{1}{2}, \pm\frac{1}{4}$$

Using synthetic division:
We try $x + 2$:

$$\begin{array}{r|rrrr} -2 & 4 & 4 & -7 & 2 \\ & & -8 & 8 & -2 \\ \hline & 4 & -4 & 1 & 0 \end{array}$$

$x + 2$ is a factor. The other factor is the quotient: $4x^2 - 4x + 1$.

Thus, $f(x) = (x+2)\left(4x^2 - 4x + 1\right)$.
$= (x+2)(2x-1)(2x-1)$

The zeros are –2, of multiplicity 1 and $\frac{1}{2}$, of multiplicity 2.

18. $f(x) = x^4 - 4x^3 + 9x^2 - 20x + 20$

Possible rational zeros:
$p = \pm 1, \pm 2, \pm 4, \pm 5, \pm 10, \pm 20; \quad q = \pm 1;$

$\frac{p}{q} = \pm 1, \pm 2, \pm 4, \pm 5, \pm 10, \pm 20$

Using synthetic division:
We try $x - 2$:

$$\begin{array}{r|rrrrr} 2 & 1 & -4 & 9 & -20 & 20 \\ & & 2 & -4 & 10 & -20 \\ \hline & 1 & -2 & 5 & -10 & 0 \end{array}$$

$x - 2$ is a factor and the quotient is

$$\begin{aligned} x^3 - 2x^2 + 5x - 10 &= x^2(x-2) + 5(x-2) \\ &= (x-2)(x^2+5) \end{aligned}$$

$$\text{Thus, } \begin{aligned} f(x) &= (x-2)(x-2)(x^2+5) \\ &= (x-2)^2(x^2+5) \end{aligned}$$

Since $x^2 + 5 = 0$ has no real solutions, the only zero is 2, of multiplicity 2.

19. $2x^4 + 2x^3 - 11x^2 + x - 6 = 0$
The solutions of the equation are the zeros of $f(x) = 2x^4 + 2x^3 - 11x^2 + x - 6$.
Possible rational zeros:
$p = \pm 1, \pm 2, \pm 3, \pm 6; \quad q = \pm 1, \pm 2;$

$\frac{p}{q} = \pm 1, \pm 2, \pm 3, \pm 6, \pm \frac{1}{2}, \pm \frac{3}{2}$

Using synthetic division:
We try $x + 3$:

$$\begin{array}{r|rrrrr} -3 & 2 & 2 & -11 & 1 & -6 \\ & & -6 & 12 & -3 & 6 \\ \hline & 2 & -4 & 1 & -2 & 0 \end{array}$$

$x + 3$ is a factor and the quotient is

$$\begin{aligned} 2x^3 - 4x^2 + x - 2 &= 2x^2(x-2) + 1(x-2) \\ &= (x-2)(2x^2+1) \end{aligned}$$

Thus, $f(x) = (x+3)(x-2)(2x^2+1)$.

Since $2x^2 + 1 = 0$ has no real solutions, the solution set is $\{-3, 2\}$.

20. $2x^4 + 7x^3 + x^2 - 7x - 3 = 0$
The solutions of the equation are the zeros of $f(x) = 2x^4 + 7x^3 + x^2 - 7x - 3$.
Possible rational zeros:
$p = \pm 1, \pm 3; \quad q = \pm 1, \pm 2;$

$\frac{p}{q} = \pm 1, \pm 3, \pm \frac{1}{2}, \pm \frac{3}{2}$

Using synthetic division:
We try $x + 3$:

$$\begin{array}{r|rrrrr} -3 & 2 & 7 & 1 & -7 & -3 \\ & & -6 & -3 & 6 & 3 \\ \hline & 2 & 1 & -2 & -1 & 0 \end{array}$$

$x + 3$ is a factor and the quotient is

$$\begin{aligned} 2x^3 + x^2 - 2x - 1 &= x^2(2x+1) - 1(2x+1). \\ &= (2x+1)(x^2-1) \end{aligned}$$

$$\text{Thus, } \begin{aligned} f(x) &= (x+3)(2x+1)(x^2-1) \\ &= (x+3)(2x+1)(x-1)(x+1) \end{aligned}$$

The solution set is $\left\{-3, \ -1, \ -\frac{1}{2}, \ 1\right\}$.

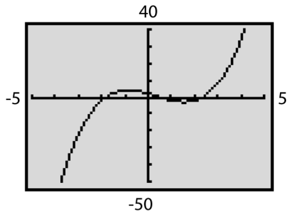
21. $f(x) = x^3 - x^2 - 4x + 2$

$a_2 = -1, \quad a_1 = -4, \quad a_0 = 2$

$\text{Max } \{1, |2| + |-4| + |-1|\} = \text{Max } \{1, 7\} = 7$

$1 + \text{Max } \{|2|, |-4|, |-1|\} = 1 + 4 = 5$

The smaller of the two numbers is 5, so every real zero of f lies between –5 and 5.



22. $f(x) = 2x^3 - 7x^2 - 10x + 35$

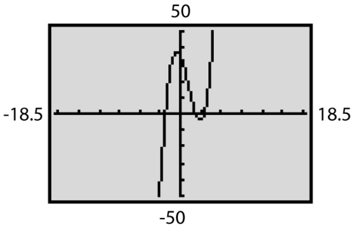
$$= 2\left(x^3 - \frac{7}{2}x^2 - 5x + \frac{35}{2}\right)$$

$a_2 = -\frac{7}{2}, \quad a_1 = -5, \quad a_0 = \frac{35}{2}$

$$\text{Max } \left\{1, \left|\frac{35}{2}\right| + |-5| + \left|-\frac{7}{2}\right|\right\} = \text{Max } \{1, 26\}$$
$$= 26$$

$$1 + \text{Max } \left\{\left|\frac{35}{2}\right|, |-5|, \left|-\frac{7}{2}\right|\right\} = 1 + \frac{35}{2}$$
$$= \frac{37}{2} = 18.5$$

The smaller of the two numbers is 18.5, so every real zero of f lies between –18.5 and 18.5.



23. $f(x) = 3x^3 - x - 1; \quad [0, 1]$

$f(0) = -1 < 0$ and $f(1) = 1 > 0$

The value of the function is positive at one endpoint and negative at the other. Since the function is continuous, the Intermediate Value Theorem guarantees at least one zero in the given interval.

24. $f(x) = 8x^4 - 4x^3 - 2x - 1; \quad [0, 1]$

$f(0) = -1 < 0$ and $f(1) = 1 > 0$

The value of the function is positive at one endpoint and negative at the other. Since the function is continuous, the Intermediate Value Theorem guarantees at least one zero in the given interval.

25. Since complex zeros appear in conjugate pairs, $4 - i$, the conjugate of $4 + i$, is the remaining zero of f.

$$f(x) = (x-6)(x-4-i)(x-4+i)$$
$$= x^3 - 14x^2 + 65x - 102$$

26. Since complex zeros appear in conjugate pairs, $-i$, the conjugate of i, and $1 - i$, the conjugate of $1 + i$, are the remaining zeros of f.

$$f(x) = (x-i)(x+i)(x-1-i)(x-1+i)$$
$$= x^4 - 2x^3 + 3x^2 - 2x + 2$$

27. $f(x) = x^3 - 3x^2 - 6x + 8$.

Possible rational zeros:
$p = \pm 1, \pm 2, \pm 4, \pm 8; \quad q = \pm 1;$

$\frac{p}{q} = \pm 1, \pm 2, \pm 4, \pm 8$

Using synthetic division:
We try $x - 1$:

$$\begin{array}{r|rrrr} 1 & 1 & -3 & -6 & 8 \\ & & 1 & -2 & -8 \\ \hline & 1 & -2 & -8 & 0 \end{array}$$

$x - 1$ is a factor and the quotient is $x^2 - 2x - 8$

Thus,

$f(x) = (x-1)\left(x^2 - 2x - 8\right) = (x-1)(x-4)(x+2)$.

The complex zeros are 1, 4, and –2, each of multiplicity 1.

28. $f(x) = 4x^3 + 4x^2 - 7x + 2$.

Possible rational zeros:
$p = \pm 1, \pm 2; \quad q = \pm 1, \pm 2, \pm 4;$

$\frac{p}{q} = \pm 1, \pm \frac{1}{2}, \pm \frac{1}{4}, \pm 2$

Using synthetic division:
We try $x + 2$:

$$\begin{array}{r|rrrr} -2 & 4 & 4 & -7 & 2 \\ & & -8 & 8 & -2 \\ \hline & 4 & -4 & 1 & 0 \end{array}$$

$x + 2$ is a factor and the quotient is $4x^2 - 4x + 1$.

Thus,

$$f(x) = (x+2)\left(4x^2 - 4x + 1\right) \qquad .$$
$$= (x+2)(2x-1)(2x-1)$$
$$= (x+2)(2x-1)^2 = 4(x+2)\left(x + \tfrac{1}{2}\right)^2$$

The complex zeros are –2, of multiplicity 1, and $\frac{1}{2}$, of multiplicity 2.

29. $f(x) = x^4 - 4x^3 + 9x^2 - 20x + 20$.

Possible rational zeros:
$p = \pm 1, \pm 2, \pm 4, \pm 5, \pm 10, \pm 20; \quad q = \pm 1;$

$\frac{p}{q} = \pm 1, \pm 2, \pm 4, \pm 5, \pm 10, \pm 20$

Using synthetic division:
We try $x-2$:

$$\begin{array}{r|rrrrr} 2 & 1 & -4 & 9 & -20 & 20 \\ & & 2 & -4 & 10 & -20 \\ \hline & 1 & -2 & 5 & -10 & 0 \end{array}$$

$x-2$ is a factor and the quotient is $x^3-2x^2+5x-10$.

Thus, $f(x)=(x-2)\left(x^3-2x^2+5x-10\right)$.

We can factor $x^3-2x^2+5x-10$ by grouping.

$$\begin{aligned} x^3-2x^2+5x-10 &= x^2(x-2)+5(x-2) \\ &= (x-2)\left(x^2+5\right) \\ &= (x-2)\left(x+\sqrt{5}i\right)\left(x-\sqrt{5}i\right) \end{aligned}$$

$f(x)=(x-2)^2\left(x+\sqrt{5}i\right)\left(x-\sqrt{5}i\right)$

The complex zeros are 2, of multiplicity 2, and $\sqrt{5}i$ and $-\sqrt{5}i$, each of multiplicity 1.

30. $f(x)=2x^4+2x^3-11x^2+x-6$.

Possible rational zeros:
$p=\pm1, \pm2, \pm3, \pm6;\quad q=\pm1, \pm2;$

$\frac{p}{q}=\pm1, \pm\frac{1}{2}, \pm2, \pm3, \pm\frac{3}{2}\pm6$

Using synthetic division:
We try $x-2$:

$$\begin{array}{r|rrrrr} 2 & 2 & 2 & -11 & 1 & -6 \\ & & 4 & 12 & 2 & 6 \\ \hline & 2 & 6 & 1 & 3 & 0 \end{array}$$

$x-2$ is a factor and the quotient is $2x^3+6x^2+x+3$.

Thus, $f(x)=(x-2)\left(2x^3+6x^2+x+3\right)$.

We can factor $2x^3+6x^2+x+3$ by grouping.

$$\begin{aligned} 2x^3+6x^2+x+3 &= 2x^2(x+3)+(x+3) \\ &= (x+3)\left(2x^2+1\right) \\ &= (x+3)\left(\sqrt{2}x+i\right)\left(\sqrt{2}x-i\right) \end{aligned}$$

$$\begin{aligned} f(x) &= (x-2)(x+3)\left(\sqrt{2}x+i\right)\left(\sqrt{2}x-i\right) \\ &= 2(x-2)(x+3)\left(x+\tfrac{\sqrt{2}}{2}i\right)\left(x-\tfrac{\sqrt{2}}{2}i\right) \end{aligned}$$

The complex zeros are 2, –3, $-\frac{\sqrt{2}}{2}i$, and $\frac{\sqrt{2}}{2}i$, each of multiplicity 1.

31. $R(x)=\frac{x+2}{x^2-9}=\frac{x+2}{(x+3)(x-3)}$ is in lowest terms. The denominator has zeros at –3 and 3. Thus, the domain is $\{x \mid x\neq-3, x\neq3\}$. The degree of the numerator, $p(x)=x+2$, is $n=1$. The degree of the denominator $q(x)=x^2-9$, is $m=2$. Since $n<m$, the line $y=0$ is a horizontal asymptote. Since the denominator is zero at –3 and 3, $x=-3$ and $x=3$ are vertical asymptotes.

32. $R(x)=\frac{x^2+3x+2}{(x+2)^2}=\frac{(x+2)(x+1)}{(x+2)^2}=\frac{x+1}{x+2}$ is in lowest terms. The denominator has a zero at –2. Thus, the domain is $\{x \mid x\neq-2\}$. The degree of the numerator, $p(x)=x^2+3x+2$, is $n=2$. The degree of the denominator, $q(x)=(x+2)^2=x^2+4x+4$, is $m=2$. Since $n=m$, the line $y=\frac{1}{1}=1$ is a horizontal asymptote. Since the denominator of $y=\frac{x+1}{x+2}$ is zero at -2, $x=-2$ is a vertical asymptote.

33. $R(x)=\frac{2x-6}{x}\quad p(x)=2x-6;\ q(x)=x;\ n=1;\ m=1$

Step 1: Domain: $\{x \mid x\neq0\}$
There is no y-intercept because 0 is not in the domain.

Step 2: $R(x)=\dfrac{2x-6}{x}=\dfrac{2(x-3)}{x}$ is in lowest terms.

Step 3: The x-intercept is the zero of $p(x)$: 3

Near 3: $R(x)\approx\dfrac{2}{3}(x-3)$. Plot the point $(3,0)$ and show a line with positive slope there.

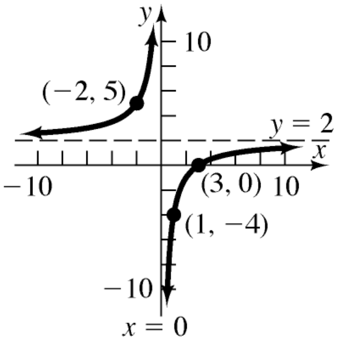
Step 4: $R(x)=\dfrac{2x-6}{x}=\dfrac{2(x-3)}{x}$ is in lowest terms. The vertical asymptote is the zero of $q(x)$: $x=0$. Graph this asymptote using a dashed line.

Step 5: Since $n=m$, the line $y=\dfrac{2}{1}=2$ is the horizontal asymptote. Solve to find intersection points:

$$\frac{2x-6}{x}=2$$
$$2x-6=2x$$
$$-6\neq 0$$

$R(x)$ does not intersect $y=2$. Plot the line $y=2$ with dashes.

Steps 6 & 7: Graphing:



34. $H(x)=\dfrac{x+2}{x(x-2)}$ $p(x)=x+2;\ q(x)=x(x-2)=x^2-2x;\ n=1;\ m=2$

Step 1: Domain: $\{x \mid x\neq 0,\ x\neq 2\}$.

Step 2: $H(x)=\dfrac{x+2}{x(x-2)}$ is in lowest terms.

Step 3: There is no y-intercept because 0 is not in the domain.

The x-intercept is the zero of $p(x)$: -2

Near -2: $H(x)\approx\dfrac{1}{8}(x+2)$. Plot the point $(-2,0)$ and show a line with positive slope there.

Step 4: $H(x)=\dfrac{x+2}{x(x-2)}$ is in lowest terms. The vertical asymptotes are the zeros of $q(x)$: $x=0$ and $x=2$.

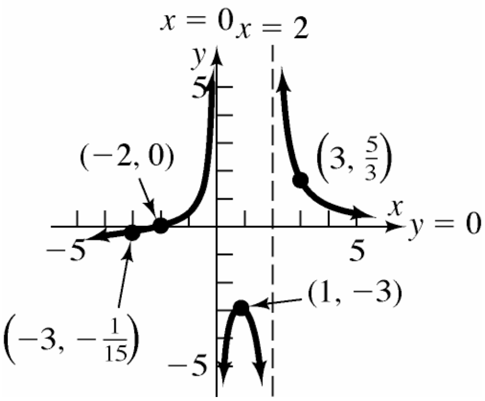
Graph these asymptotes using dashed lines.

Step 5: Since $n < m$, the line $y = 0$ is the horizontal asymptote. Solve to find intersection points:

$$\frac{x+2}{x(x-2)} = 0$$
$$x+2 = 0$$
$$x = -2$$

$H(x)$ intersects $y = 0$ at (–2, 0). Plot the line $y = 0$ using dashes.

Steps 6 & 7: Graphing:



35. $R(x) = \dfrac{x^2+x-6}{x^2-x-6} = \dfrac{(x+3)(x-2)}{(x-3)(x+2)}$ $\quad p(x) = x^2+x-6;\ q(x) = x^2-x-6;$

Step 1: Domain: $\{x \mid x \neq -2, x \neq 3\}$.

Step 2: $R(x) = \dfrac{x^2+x-6}{x^2-x-6}$ is in lowest terms.

Step 3: The y-intercept is $R(0) = \dfrac{0^2+0-6}{0^2-0-6} = \dfrac{-6}{-6} = 1$. Plot the point $(0,1)$.

The x-intercepts are the zeros of $p(x)$: –3 and 2.

Near -3: $R(x) \approx -\dfrac{5}{6}(x+3)$. Plot the point $(-3,0)$ and show a line with negative slope there.

Near 2: $R(x) \approx -\dfrac{5}{4}(x-2)$. Plot the point $(2,0)$ and show a line with negative slope there.

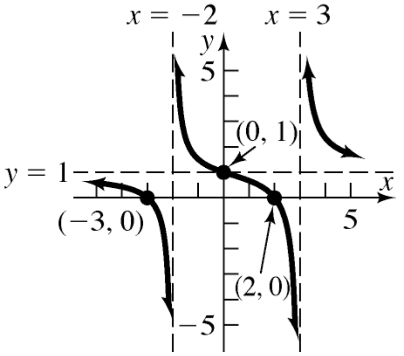
Step 4: $R(x) = \dfrac{x^2+x-6}{x^2-x-6}$ is in lowest terms. The vertical asymptotes are the zeros of $q(x)$: $x = -2$ and $x = 3$. Graph these asymptotes with dashed lines.

Step 5: Since $n = m$, the line $y = \frac{1}{1} = 1$ is the horizontal asymptote. Solve to find intersection points:

$$\frac{x^2 + x - 6}{x^2 - x - 6} = 1$$
$$x^2 + x - 6 = x^2 - x - 6$$
$$2x = 0$$
$$x = 0$$

$R(x)$ intersects $y = 1$ at (0, 1). Plot the line $y = 1$ using dashes.

Steps 6 & 7: Graphing:



36. $F(x) = \frac{x^3}{x^2 - 4} = \frac{x^3}{(x+2)(x-2)}$ $p(x) = x^3$; $q(x) = x^2 - 4$; $n = 3$; $m = 2$

Step 1: Domain: $\{x \mid x \neq -2, x \neq 2\}$.

Step 2: $F(x) = \frac{x^3}{x^2 - 4}$ is in lowest terms.

Step 3: The y-intercept is $F(0) = \frac{0^3}{0^2 - 4} = \frac{0}{-4} = 0$. Plot the point $(0,0)$.

The x-intercept is the zero of $p(x)$: 0.

Near 0: $F(x) \approx -\frac{1}{4}x^3$. Plot the point $(0,0)$ and indicate a cubic function there (left tail up and right tail down).

Step 4: $F(x) = \frac{x^3}{x^2 - 4}$ is in lowest terms. The vertical asymptotes are the zeros of $q(x)$: $x = -2$ and $x = 2$. Graph these asymptotes using dashed lines.

Step 5: Since $n = m + 1$, there is an oblique asymptote. Dividing:

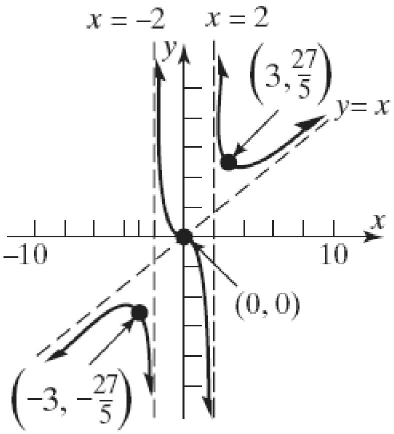
$$\begin{array}{r} x \qquad\quad \\ x^2-4\overline{)x^3 \qquad\quad} \\ \underline{x^3 \qquad -4x} \\ 4x \end{array} \qquad \frac{x^3}{x^2-4} = x + \frac{4x}{x^2-4}$$

The oblique asymptote is $y = x$. Solve to find intersection points:

$$\frac{x^3}{x^2-4} = x$$
$$x^3 = x^3 - 4x$$
$$4x = 0$$
$$x = 0$$

$F(x)$ intersects $y = x$ at (0, 0). Plot the line $y = x$ using dashed lines.

Steps 6 & 7: Graphing:



37. $R(x) = \dfrac{2x^4}{(x-1)^2}$ $\quad p(x) = 2x^4$; $q(x) = (x-1)^2$; $n = 4$; $m = 2$

Step 1: Domain: $\{x \mid x \neq 1\}$.

Step 2: $R(x) = \dfrac{2x^4}{(x-1)^2}$ is in lowest terms.

Step 3: The y-intercept is $R(0) = \dfrac{2(0)^4}{(0-1)^2} = \dfrac{0}{1} = 0$. Plot the point $(0, 0)$.

The x-intercept is the zero of $p(x)$: 0.

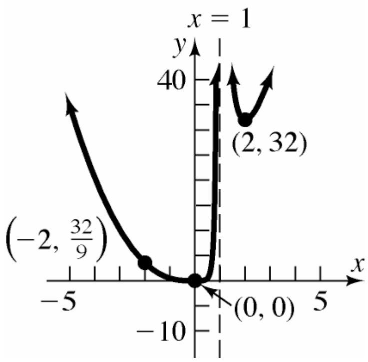
Near 0: $R(x) \approx 2x^4$. Plot the point $(0, 0)$ and show the graph of a quartic opening up there.

Step 4: $R(x)=\frac{2x^4}{(x-1)^2}$ is in lowest terms. The vertical asymptote is the zero of $q(x)$: $x=1$.

Graph this asymptote using a dashed line.

Step 5: Since $n>m+1$, there is no horizontal asymptote and no oblique asymptote.

Steps 6 & 7: Graphing:



38. $G(x)=\frac{x^2-4}{x^2-x-2}=\frac{(x+2)(x-2)}{(x-2)(x+1)}=\frac{x+2}{x+1}$ $p(x)=x^2-4$; $q(x)=x^2-x-2$;

Step 1: Domain: $\{x \mid x\neq -1, x\neq 2\}$.

Step 2: In lowest terms, $G(x)=\frac{x+2}{x+1}$, $x\neq 2$.

Step 3: The y-intercept is $G(0)=\frac{0^2-4}{0^2-0-2}=\frac{-4}{-2}=2$. Plot the point $(0,2)$.

The x-intercept is the zero of $y=x+2$: –2; Note: 2 is not a zero because reduced form must be used to find the zeros.

Near -2: $G(x)\approx -x-2$. Plot the point $(-2,0)$ and show a line with negative slope there.

Step 4: In lowest terms, $G(x)=\frac{x+2}{x+1}$, $x\neq 2$. The vertical asymptote is the zero of $f(x)=x+1$: $x=-1$;

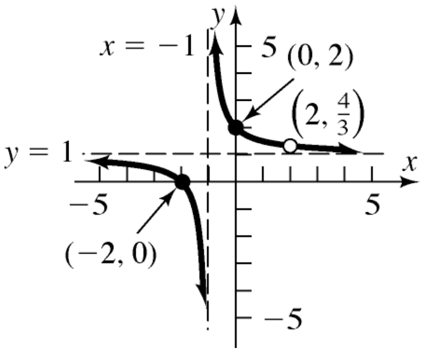
Graph this asymptote using a dashed line. Note: $x=2$ is not a vertical asymptote because reduced form must be used to find the asymptotes. The graph has a hole at $\left(2,\frac{4}{3}\right)$.

Step 5: Since $n=m$, the line $y=\frac{1}{1}=1$ is the horizontal asymptote. Solve to find intersection points:

$$\frac{x^2-4}{x^2-x-2}=1$$
$$x^2-4=x^2-x-2$$
$$x=2$$

$G(x)$ does not intersect $y=1$ because $G(x)$ is not defined at $x=2$. Plot the line $y=1$ using dashes.

Steps 6 & 7: Graphing:



39. The x-intercepts of the graph of f are -3 and 2.

a. $f(x)=0$ for $x=-3$ and 2.

b. The graph of f is above the x-axis (so f is positive) for $-3<x<2$ or $x>2$.

Therefore, the solution set is $\{x \mid -3<x<2 \text{ or } x>2\}$ or, using interval notation, $(-3,2)\cup(2,\infty)$.

c. The graph of f is below the x-axis (so f is negative) for $x<-3$. Since the inequality is not strict, we include -3 and 2 in the solution set. Therefore, the solution set is $\{x \mid x\le -3 \text{ or } x=2\}$ or, using interval notation $(-\infty,-3]\cup\{2\}$.

d. The graph crosses the x-axis at $x=-3$ and touches at $x=2$. Thus, -3 has odd multiplicity while 2 has an even multiplicity. Using one for the odd multiplicity and two for the even multiplicity, a possible function is $f(x)=a(x-2)^2(x+3)$**.** Since the y-intercept is 12, we know $f(0)=12$**.** Thus, *a=1*. Using $a=1$**,** the function is $f(x)=(x-2)^2(x+3)$**.**

40. The x-intercepts of the graph of f are -3 and -1.

a. The horizontal asymptote is $y=0.25$.

b. The vertical asymptotes are $x=-2$ and $x=2$

c. The graph of f is below the x-axis (so f is negative) for $-3<x<-2$ or $-1<x<2$. Therefore, the solution set is $\{x \mid -3<x<-2 \text{ or } -1<x<2\}$ or, using interval notation, $(-3,-2)\cup(-1,2)$.

d. The graph of f is above the x-axis (so f is positive) for $-\infty<x<-3$ or $-2<x<-1$ or $x>2$. Since the inequality is not strict, we include -3 and -1 in the solution set. Therefore, the solution set is $\{x \mid -\infty<x<-3 \text{ or } -2<x<-1 \text{ or } x>2\}$ or, using interval notation $(-\infty,-3]\cup(-2,-1]\cup(2,\infty)$.

e. One possibility: $R(x)=a\dfrac{(x+3)(x+1)}{(x+2)(x-2)}$

(Using the point $\left(0,-\dfrac{3}{16}\right)$ leads to $a=1/4$.) Thus,

$$R(x)=\frac{(x+3)(x+1)}{4(x+2)(x-2)}=\frac{x^2+4x+3}{4x^2-16}.$$

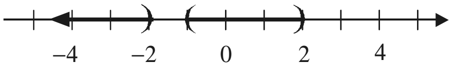
41.

$$x^3+x^2<4x+4$$
$$x^3+x^2-4x-4<0$$
$$x^2(x+1)-4(x+1)<0$$
$$(x^2-4)(x+1)<0$$
$$(x-2)(x+2)(x+1)<0$$
$$f(x)=(x-2)(x+2)(x+1)$$

$x=-2$, $x=-1$, and $x=2$ are the zeros of f.

Interval	$(-\infty,-2)$	$(-2,-1)$	$(-1, 2)$	$(2, \infty)$
Number Chosen	-3	$-3/2$	0	3
Value of f	-10	0.875	-4	20
Conclusion	Negative	Positive	Negative	Positive

The solution set is $\{x \mid x<-2 \text{ or } -1<x<2\}$, or, using interval notation, $(-\infty,-2)\ \cup\ (-1,2)$.



42. $\dfrac{6}{x+3}\ge 1$

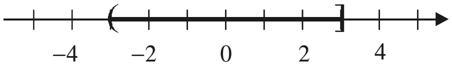
$$\frac{6}{x+3}-1\ge 0\Rightarrow\frac{6-1(x+3)}{x+3}\ge 0\Rightarrow\frac{-x+3}{x+3}\ge 0$$

$$f(x)=\frac{-(x-3)}{x+3}$$

The zeros and values where the expression is undefined are $x=3$ and $x=-3$.

Interval	$(-\infty,-3)$	$(-3,3)$	$(3,\infty)$
Number Chosen	–4	0	4
Value of *f*	–7	1	$-\frac{1}{7}$
Conclusion	Negative	Positive	Negative

The solution set is $\{\,x\,|\,-3<x\le 3\,\}$, or, using interval notation, $(-3,3]$.



43. $\dfrac{2x-6}{1-x}<2$

$$\frac{2x-6}{1-x}-2<0$$

$$\frac{2x-6-2(1-x)}{1-x}<0$$

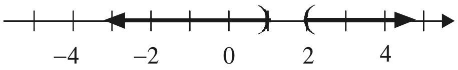
$$\frac{4x-8}{1-x}<0$$

$$f(x)=\frac{4(x-2)}{1-x}$$

The zeros and values where the expression is undefined are $x=1,$ and $x=2$.

Interval	$(-\infty,1)$	$(1,2)$	$(2,\infty)$
Number Chosen	0	1.5	3
Value of *f*	–8	4	–2
Conclusion	Negative	Positive	Negative

The solution set is $\{\,x\,|\,x<1 \text{ or } x>2\,\}$, or, using interval notation, $(-\infty,1)\ \cup\ (2,\infty)$.



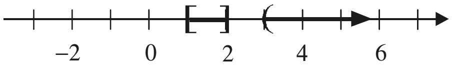
44. $\dfrac{(x-2)(x-1)}{x-3}\ge 0$

$$f(x)=\frac{(x-2)(x-1)}{x-3}$$

The zeros and values where the expression is undefined are $x=1$, $x=2$, and $x=3$.

Interval	$(-\infty,1)$	$(1,2)$	$(2,3)$	$(3,\infty)$
Number Chosen	0	1.5	2.5	4
Value of *f*	$-\frac{2}{3}$	$\frac{1}{6}$	$-\frac{3}{2}$	6
Conclusion	Negative	Positive	Negative	Positive

The solution set is $\{\,x\,|\,1\le x\le 2 \text{ or } x>3\,\}$, or, using interval notation, $[1,2]\ \cup\ (3,\infty)$.



45. $\dfrac{x^2-8x+12}{x^2-16}>0$

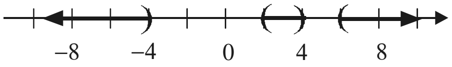
$$f(x)=\frac{x^2-8x+12}{x^2-16}$$

$$\frac{(x-2)(x-6)}{(x+4)(x-4)}>0$$

The zeros and values where the expression is undefined are $x=-4$, $x=2$, $x=4$, and $x=6$.

Interval	Number Chosen	Value of f	Conclusion
$(-\infty, -4)$	-5	$\frac{77}{9}$	Positive
$(-4, 2)$	0	$-\frac{3}{4}$	Negative
$(2, 4)$	3	$\frac{3}{7}$	Positive
$(4, 6)$	5	$-\frac{1}{3}$	Negative
$(6, \infty)$	7	$\frac{5}{33}$	Positive

The solution set is $\{x \mid x < -4 \text{ or } 2 < x < 4 \text{ or } x > 6\}$, or, using interval notation, $(-\infty, -4) \cup (2, 4) \cup (6, \infty)$.



46. a. $250 = \pi r^2 h \Rightarrow h = \frac{250}{\pi r^2}$;

$$A(r) = 2\pi r^2 + 2\pi rh = 2\pi r^2 + 2\pi r\left(\frac{250}{\pi r^2}\right)$$

$$= 2\pi r^2 + \frac{500}{r}$$

b. $A(3) = 2\pi \cdot 3^2 + \frac{500}{3}$

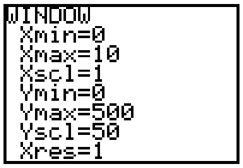
$$= 18\pi + \frac{500}{3} \approx 223.22 \text{ square cm}$$

c. $A(5) = 2\pi \cdot 5^2 + \frac{500}{5}$

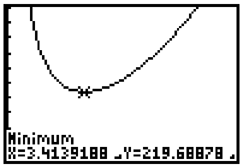
$$= 50\pi + 100 \approx 257.08 \text{ square cm}$$

d. Use MINIMUM on the graph of

$$y_1 = 2\pi x^2 + \frac{500}{x}$$

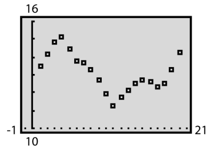


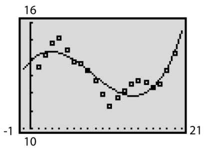
```
WINDOW
 Xmin=0
 Xmax=10
 Xscl=1
 Ymin=0
 Ymax=500
 Yscl=50
 Xres=1
```



```
Minimum
X=3.4139188 Y=219.68878
```

The area is smallest when the radius is approximately 3.41 cm.

47. a. 

c. 

b. $P(t) = 0.0033t^3 - 0.0856t^2 + 0.4013t$
$+ 13.8330$

$P(21) = 0.0033(21)^3 - 0.0856(21)^2$
$+ 0.4013(21) + 13.8330 = 15.1\%$

48 - 49. Answers will vary.

50. a. Since the graph points upward for large values of $|x|$, the degree is even.

b. Likewise, the leading coefficient is positive.

c. The graph is symmetric about the y-axis, so the function is even.

d. The graph touches the x-axis at $x = 0$, which means 0 is a zero of the function with even multiplicity. Thus, x^2 must be a factor.

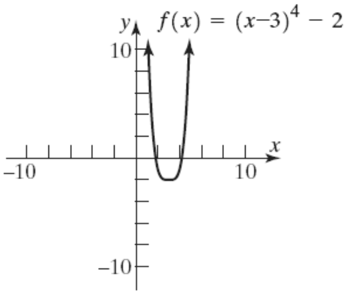
e. The graph has 7 turning points, so the degree of the polynomial must be at least 8.

f. Answers will vary.

Chapter 4 Test

1. $f(x) = (x-3)^4 - 2$

Using the graph of $y = x^4$, shift right 3 units, then shift down 2 units.



2. a. The maximum number of real zeros is the degree, $n = 3$.

b. First we write the polynomial so that the leading coefficient is 1.

$$g(x) = 2\left(x^3 + \frac{5}{2}x^2 - 14x - \frac{15}{2}\right)$$

For the expression in parentheses, we have

$a_2 = \frac{5}{2}$, $a_1 = -14$, and $a_0 = -\frac{15}{2}$.

$$\max\{1, |a_0| + |a_1| + |a_2|\}$$
$$= \max\left\{1, \left|-\tfrac{15}{2}\right| + |-14| + \left|\tfrac{5}{2}\right|\right\} = \max\{1, 24\} = 24$$
$$1 + \max\{|a_0|, |a_1|, |a_2|\}$$
$$= 1 + \max\left\{\left|-\tfrac{15}{2}\right|, |-14|, \left|\tfrac{5}{2}\right|\right\}$$
$$= 1 + 14 = 15$$

The smaller of the two numbers, 15, is the bound. Therefore, every zero of *g* lies between -15 and 15.

c. $g(x) = 2x^3 + 5x^2 - 28x - 15$

We list all integers *p* that are factors of $a_0 = -15$ and all the integers *q* that are factors of $a_3 = 2$.

$p: \pm 1, \pm 3, \pm 5, \pm 15$

$q: \ \pm 1, \pm 2$

Now we form all possible ratios $\frac{p}{q}$:

$$\frac{p}{q}: \ \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \pm\frac{5}{2}, \pm 3, \pm 5, \pm\frac{15}{2}, \pm 15$$

If *g* has a rational zero, it must be one of the 16 possibilities listed.

d. We can find the rational zeros by using the fact that if *r*is a zero of *g*, then $g(r) = 0$. That is, we evaluate the function for different values from our list of rational zeros. If we get $g(r) = 0$, we have a zero. Then we use long division to reduce the polynomial and start again on the reduced polynomial.

We will start with the positive integers:

$$g(1) = 2(1)^3 + 5(1)^2 - 28(1) - 15,$$
$$= 2 + 5 - 28 - 15$$
$$= -36$$
$$g(3) = 2(3)^3 + 5(3)^2 - 28(3) - 15$$
$$= 54 + 45 - 84 - 15$$
$$= 0$$

So, we know that 3 is a zero. This means that $(x-3)$ must be a factor of *g*. Using long division we get

$$\begin{array}{r}
2x^2 + 11x + 5 \\
x - 3 \overline{\smash{\big)}\, 2x^3 + 5x^2 - 28x - 15} \\
\underline{-\left(2x^3 - 6x^2\right)} \qquad\qquad \\
11x^2 - 28x \qquad \\
\underline{-\left(11x^2 - 33x\right)} \qquad \\
5x - 15 \\
\underline{-\left(5x - 15\right)} \\
0
\end{array}$$

Thus, we can now write

$g(x) = (x-3)\left(2x^2 + 11x + 5\right)$

The quadratic factor can be factored so we get:

$g(x) = (x-3)(2x+1)(x+5)$

To find the remaining zeros of *g*, we set the last two factors equal to 0 and solve.

$2x + 1 = 0 \qquad x + 5 = 0$

$2x = -1 \qquad x = -5$

$x = -\frac{1}{2}$

Therefore, the zeros are -5, $-\frac{1}{2}$, and 3.

Notice how these rational zeros were all in the list of potential rational zeros.

e. The x-intercepts of a graph are the same as the zeros of the function. In the previous part, we found the zeros to be -5, $-\frac{1}{2}$, and 3. Therefore, the x-intercepts are -5, $-\frac{1}{2}$, and 3.

To find the *y*-intercept, we simply find $g(0)$.

$g(0) = 2(0)^3 + 5(0)^2 - 28(0) - 15 = -15$

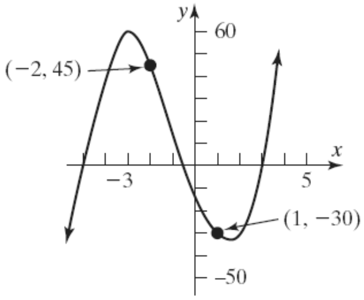
So, the y-intercept is -15.

f. Whether the graph crosses or touches at an x-intercept is determined by the multiplicity. Each factor of the polynomial occurs once, so the multiplicity of each zero is 1. For odd multiplicity, the graph will cross the x-axis at the zero. Thus, the graph crosses the x-axis at each of the three x-intercepts.

g. The power function that the graph of g resembles for large values of $|x|$ is given by the term with the highest power of x. In this case, the power function is $y = 2x^3$. So, the graph of g will resemble the graph of $y = 2x^3$ for large values of $|x|$.

h. 2 turning points;
local maximum: $(-3.15, 60.30)$;
local minima: $(1.48, -39.00)$

i. We could first evaluate the function at several values for x to help determine the scale.
Putting all this information together, we obtain the following graph:



3. $x^3 - 4x^2 + 25x - 100 = 0$

$x^2(x-4) + 25(x-4) = 0$

$(x-4)(x^2+25) = 0$

$x - 4 = 0$ or $x^2 + 25 = 0$

$x = 4$ $\quad x^2 = -25$

$x = \pm\sqrt{-25}$

$x = \pm 5i$

The solution set is $\{4, -5i, 5i\}$.

4. $3x^3 + 2x - 1 = 8x^2 - 4$

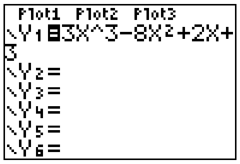
$3x^3 - 8x^2 + 2x + 3 = 0$

If we let the left side of the equation be $f(x)$, then we are simply finding the zeros of f.
We list all integers p that are factors of $a_0 = 3$ and all the integers q that are factors of $a_3 = 3$.

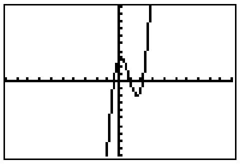
$p: \pm1, \pm3$; $\quad q: \pm1, \pm3$

Now we form all possible ratios $\frac{p}{q}$:

$\frac{p}{q}: \ \pm\frac{1}{3}, \pm1, \pm3$



```
Plot1 Plot2 Plot3
\Y1■3X^3-8X²+2X+
3
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
```



It appears that there is a zero near $x = 1$.

$f(1) = 3(1)^3 - 8(1)^2 + 2(1) + 3 = 0$

Therefore, x=1 is a zero and $(x-1)$ is a factor of $f(x)$. We can reduce the polynomial expression by using synthetic division.

$$\begin{array}{r|rrrr} 1 & 3 & -8 & 2 & 3 \\ & & 3 & -5 & -3 \\ \hline & 3 & -5 & -3 & 0 \end{array}$$

Thus, $f(x) = (x-1)(3x^2 - 5x - 3)$. We can find the remaining zeros by using the quadratic formula.

$3x^2 - 5x - 3 = 0$

$a = 3, b = -5, c = -3$

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(3)(-3)}}{2(3)}$$

$$= \frac{5 \pm \sqrt{25+36}}{6} = \frac{5 \pm \sqrt{61}}{6}$$

Thus, the solution set is $\left\{1, \frac{5-\sqrt{61}}{6}, \frac{5+\sqrt{61}}{6}\right\}$.

5. We start by factoring the numerator and denominator.

$$g(x) = \frac{2x^2 - 14x + 24}{x^2 + 6x - 40} = \frac{2(x-3)(x-4)}{(x+10)(x-4)}$$

The domain of f is $\{x \mid x \neq -10, x \neq 4\}$.

In lowest terms, $g(x) = \frac{2(x-3)}{x+10}$ with $x \neq 4$.

The graph has one vertical asymptote, $x = -10$, since $x + 10$ is the only factor of the denominator of g in lowest terms. The graph is still undefined at $x = 4$, but there is a hole in the graph there instead of an asymptote.

Since the degree of the numerator is the same as the degree of the denominator, the graph has a horizontal asymptote equal to the quotient of the leading coefficients. The leading coefficient in the numerator is 2 and the leading coefficient in the denominator is 1. Therefore, the graph has the horizontal asymptote $y = \frac{2}{1} = 2$.

6. $r(x) = \dfrac{x^2 + 2x - 3}{x+1}$

Start by factoring the numerator.

$r(x) = \dfrac{(x+3)(x-1)}{x+1}$

The domain of the function is $\{x \mid x \neq -1\}$.

Asymptotes:

Since the function is in lowest terms, the graph has one vertical asymptote, $x = -1$.

The degree of the numerator is one more than the degree of the denominator so the graph will have an oblique asymptote. To find it, we need to use long division (note: we could also use synthetic division in this case because the dividend is linear).

$$\begin{array}{r} x+1 \\ x+1\overline{\smash{\big)}x^2+2x-3} \\ \underline{-(x^2+x)} \\ x-3 \\ \underline{-(x+1)} \\ -4 \end{array}$$

The oblique asymptote is $y = x + 1$.

7. From problem 6 we know that the domain is $\{x \mid x \neq -1\}$ and that the graph has one vertical asymptote, $x = -1$, and one oblique asymptote, $y = x + 1$.

x-intercepts:

To find the *x*-intercepts, we need to set the numerator equal to 0 and solve the resulting equation.

$(x+3)(x-1) = 0$

$x + 3 = 0 \quad \text{or} \quad x - 1 = 0$

$x = -3 \qquad\qquad x = 1$

The *x*-intercepts are -3 and 1.

The points $(-3, 0)$ and $(1, 0)$ are on the graph.

y-intercept:

$r(0) = \dfrac{0^2 + 2(0) - 3}{0+1} = -3$

The *y*-intercept is -3. The point $(0, -3)$ is on the graph.

Test for symmetry:

$r(-x) = \dfrac{(-x)^2 + 2(-x) - 3}{(-x)+1} = \dfrac{x^2 - 2x - 3}{-x+1}$

Since $r(-x) \neq r(x)$, the graph is not symmetric with respect to the y-axis.

Since $r(-x) \neq -r(x)$, the graph is not symmetric with respect to the origin.

Behavior near the asymptotes:

To determine if the graph crosses the oblique asymptote, we solve the equation

$$r(x) = x + 1$$

$$\frac{x^2 + 2x - 3}{x+1} = x + 1, \quad x \neq -1$$

$$x^2 + 2x - 3 = x^2 + 2x + 1$$

$$-3 = 1 \text{ false}$$

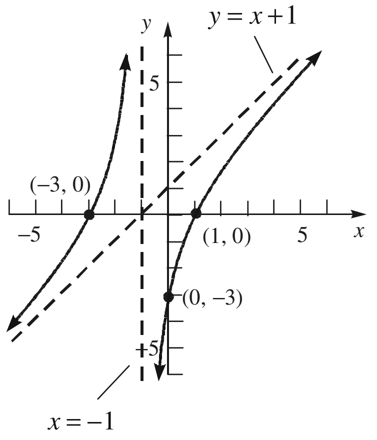
The result is a contradiction so the graph does not cross the oblique asymptote.

The zeros of the numerator and denominator, -3, -1, and 1, divide the x-axis into four subintervals.

$(-\infty, -3), (-3, -1), (-1, 1), (1, \infty)$

We can check a point in each subinterval to determine if the graph is above or below the x-axis.

Interval	$(-\infty,-3)$	$(-3,-1)$	$(-1,1)$	$(1,\infty)$
Number	–5	–2	0	3
Value of *r*	–3	3	–3	3
Location	below	above	below	above
Point	$(-5,-3)$	$(-2,3)$	$(0,-3)$	$(3,3)$



8. Since the polynomial has real coefficients, we can apply the Conjugate Pairs Theorem to find the remaining zero. If $3+i$ is a zero, then its conjugate, $3-i$, must also be a zero. Thus, the four zeros are -2, 0, $3-i$, and $3+i$. The Factor Theorem says that if $f(c)=0$, then $(x-c)$ is a factor of the polynomial. This allows us to write the following function:
$$f(x) = a(x-(-2))(x-0)(x-(3-i))(x-(3+i))$$
where a is any real number. If we let $a=1$, we get
$$\begin{aligned} f(x) &= (x+2)(x)(x-3+i)(x-3-i) \\ &= (x^2+2x)(x-3+i)(x-3-i) \\ &= (x^2+2x)(x^2-6x+10) \\ &= x^4-6x^3+10x^2+2x^3-12x^2+20x \\ &= x^4-4x^3-2x^2+20x \end{aligned}$$

9. Since the domain excludes 4 and 9, the denominator must contain the factors $(x-4)$ and $(x-9)$. However, because there is only one vertical asymptote, $x=4$, the numerator must also contain the factor $(x-9)$.

The horizontal asymptote, $y=2$, indicates that the degree of the numerator must be the same as the degree of the denominator and that the ratio of the leading coefficients needs to be 2. We can accomplish this by including another factor in the numerator, $(x-a)$, where $a \neq 4$, along with a factor of 2.

Therefore, we have $r(x) = \dfrac{2(x-9)(x-a)}{(x-4)(x-9)}$.

If we let $a=1$, we get
$$r(x) = \frac{2(x-9)(x-1)}{(x-4)(x-9)} = \frac{2x^2-20x+18}{x^2-13x+36}.$$

10. Since we have a polynomial function and polynomials are continuous, we simply need to show that $f(a)$ and $f(b)$ have opposite signs (where a and b are the endpoints of the interval).
$$f(0) = -2(0)^2-3(0)+8 = 8$$
$$f(4) = -2(4)^2-3(4)+8 = -36$$
Since $f(0)=8>0$ and $f(4)=-36<0$, the Intermediate Value Theorem guarantees that there is at least one real zero between 0 and 4.

11. $\dfrac{x+2}{x-3} < 2$

We note that the domain of the variable consists of all real numbers except 3.
Rearrange the terms so that the right side is 0.
$$\frac{x+2}{x-3} - 2 < 0$$
For $f(x) = \dfrac{x+2}{x-3} - 2$, we find the zeros of f and the values of x at which f is undefined. To do this, we need to write f as a single rational expression.
$$\begin{aligned} f(x) &= \frac{x+2}{x-3} - 2 \\ &= \frac{x+2}{x-3} - 2 \cdot \frac{x-3}{x-3} \\ &= \frac{x+2-2x+6}{x-3} \\ &= \frac{-x+8}{x-3} \end{aligned}$$
The zero of f is $x=8$ and f is undefined at $x=3$. We use these two values to divide the real number line into three subintervals.

Interval	$(-\infty,3)$	$(3,8)$	$(8,\infty)$
Num. chosen	0	4	9
Value of f	$-\frac{8}{3}$	4	$-\frac{1}{6}$
Conclusion	negative	positive	negative

Since we want to know where $f(x)$ is negative, we conclude that values of x for which $x < 3$ or $x > 8$ are solutions. The inequality is strict so the solution set is $\{x \mid x < 3 \text{ or } x > 8\}$. In interval notation we write $(-\infty, 3)$ or $(8, \infty)$.

Chapter 4 Cumulative Review

1. $P = (1,3)$, $Q = (-4,2)$

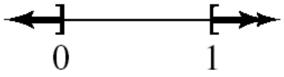
$$d_{P,Q} = \sqrt{(-4-1)^2 + (2-3)^2}$$
$$= \sqrt{(-5)^2 + (-1)^2} = \sqrt{25+1}$$
$$= \sqrt{26}$$

2.
$$x^2 \geq x$$
$$x^2 - x \geq 0$$
$$x(x-1) \geq 0$$
$$f(x) = x^2 - x$$

$x = 0, x = 1$ are the zeros of f.

Interval	$(-\infty, 0)$	$(0, 1)$	$(1, \infty)$
Number Chosen	-1	0.5	2
Value of f	2	-0.25	2
Conclusion	Positive	Negative	Positive

The solution set is $\{x \mid x \leq 0 \text{ or } x \geq 1\}$ or $(-\infty, 0]$ or $[1, \infty)$ in interval notation.

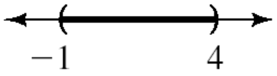


3.
$$x^2 - 3x < 4$$
$$x^2 - 3x - 4 < 0$$
$$(x-4)(x+1) < 0$$
$$f(x) = x^2 - 3x - 4$$

$x = -1, x = 4$ are the zeros of f.

Interval	$(-\infty, -1)$	$(-1, 4)$	$(4, \infty)$
Number Chosen	-2	0	5
Value of f	6	-4	6
Conclusion	Positive	Negative	Positive

The solution set is $\{x \mid -1 < x < 4\}$ or $(-1, 4)$ in interval notation.

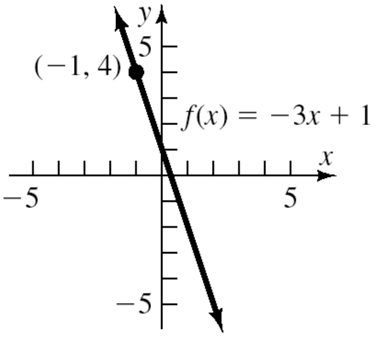


4. Slope -3, Containing the point $(-1, 4)$

Using the point-slope formula yields:

$$y - y_1 = m(x - x_1)$$
$$y - 4 = -3(x - (-1))$$
$$y - 4 = -3x - 3$$
$$y = -3x + 1$$

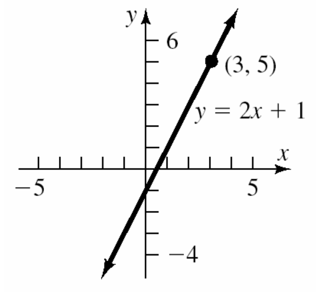
Thus, $f(x) = -3x + 1$.



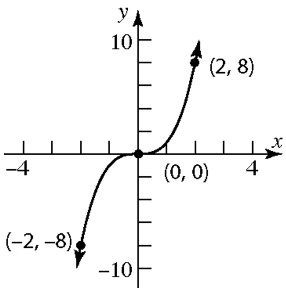
5. Parallel to $y = 2x + 1$; Slope 2, Containing the point (3, 5)

Using the point-slope formula yields:

$$y - y_1 = m(x - x_1)$$
$$y - 5 = 2(x - 3)$$
$$y - 5 = 2x - 6$$
$$y = 2x - 1$$



6. $y = x^3$



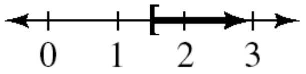
7. This relation is not a function because the ordered pairs (3, 6) and (3, 8) have the same first element, but different second elements.

8.
$$\begin{aligned} x^3 - 6x^2 + 8x &= 0 \\ x\left(x^2 - 6x + 8\right) &= 0 \\ x(x-4)(x-2) &= 0 \\ x = 0 \text{ or } x = 4 &\text{ or } x = 2 \end{aligned}$$

The solution set is $\{0, 2, 4\}$.

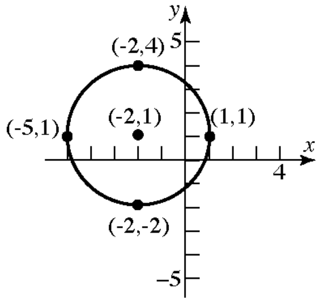
9. $3x + 2 \le 5x - 1$
$$\begin{aligned} 3 &\le 2x \\ \frac{3}{2} &\le x \\ x &\ge \frac{3}{2} \end{aligned}$$

The solution set is $\left\{x \middle| x \ge \frac{3}{2}\right\}$ or $\left[\frac{3}{2}, \infty\right)$ in interval notation.



10.
$$\begin{aligned} x^2 + 4x + y^2 - 2y - 4 &= 0 \\ (x^2 + 4x + 4) + (y^2 - 2y + 1) &= 4 + 4 + 1 \\ (x+2)^2 + (y-1)^2 &= 9 \\ (x+2)^2 + (y-1)^2 &= 3^2 \end{aligned}$$

Center: (–2, 1)
Radius 3



11. $y = x^3 - 9x$

x-intercepts:
$$\begin{aligned} 0 &= x^3 - 9x \\ 0 &= x\left(x^2 - 9\right) \\ 0 &= x(x+3)(x-3) \\ x &= 0, -3, \text{ and } 3 \\ &(0,0),\ (-3,0),\ (3,0) \end{aligned}$$

y-intercepts: $y = 0^3 - 9(0) = 0 \Rightarrow (0,0)$

Test for symmetry:

x-axis: Replace y by $-y$: $-y = x^3 - 9x$, which is not equivalent to $y = x^3 - 9x$.

y-axis: Replace x by $-x$:
$$\begin{aligned} y &= (-x)^3 - 9(-x) \\ &= -x^3 + 9x \end{aligned}$$

which is not equivalent to $y = x^3 - 9x$.

Origin: Replace x by $-x$ and y by $-y$:
$$\begin{aligned} -y &= (-x)^3 - 9(-x) \\ y &= -x^3 + 9x \end{aligned}$$
which is equivalent to $y = x^3 - 9x$. Therefore, the graph is symmetric with respect to origin.

12. $3x-2y=7$

$$-2y=-3x+7$$

$$y=\frac{3}{2}x-\frac{7}{2}$$

The given line has slope $\frac{3}{2}$. Every line that is perpendicular to the given line will have slope $-\frac{2}{3}$. Using the point $(1,5)$ and the point-slope formula yields:

$$y-y_1=m\left(x-x_1\right)$$

$$y-5=-\frac{2}{3}(x-1)$$

$$y-5=-\frac{2}{3}x+\frac{2}{3}$$

$$y=-\frac{2}{3}x+\frac{17}{3}$$

13. Not a function, since the graph fails the Vertical Line Test, for example, when $x=0$.

14. $f(x)=x^2+5x-2$

a. $f(3)=3^2+5(3)-2=9+15-2=22$

b. $f(-x)=(-x)^2+5(-x)-2=x^2-5x-2$

c. $-f(x)=-\left(x^2+5x-2\right)=-x^2-5x+2$

d. $f(3x)=(3x)^2+5(3x)-2=9x^2+15x-2$

e. $\frac{f(x+h)-f(x)}{h}$

$$=\frac{(x+h)^2+5(x+h)-2-\left(x^2+5x-2\right)}{h}$$

$$=\frac{x^2+2xh+h^2+5x+5h-2-x^2-5x+2}{h}$$

$$=\frac{2xh+h^2+5h}{h}$$

$$=2x+h+5$$

15. $f(x)=\frac{x+5}{x-1}$

a. Domain $\{x\,|\,x\neq 1\}$.

b. $f(2)=\frac{2+5}{2-1}=\frac{7}{1}=7\neq 6;$

$(2,6)$ is not on the graph of f.

The point $(2,7)$ is on the graph.

c. $f(3)=\frac{3+5}{3-1}=\frac{8}{2}=4;$

$(3,4)$ is on the graph of f.

d. Solve for x

$$\frac{x+5}{x-1}=9$$

$$x+5=9(x-1)$$

$$x+5=9x-9$$

$$14=8x$$

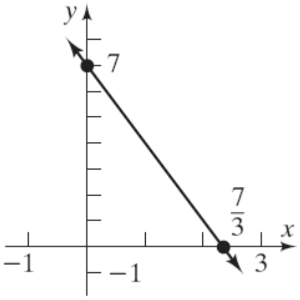
$$x=\frac{14}{8}=\frac{7}{4}$$

Therefore, $\left(\frac{7}{4},9\right)$ is on the graph of f.

e. $f(x)$ is a rational function since it is in the form $\frac{p(x)}{q(x)}$.

16. $f(x)=-3x+7$

The graph is a line with slope –3 and *y*-intercept (0, 7).



17. $f(x)=2x^2-4x+1$

$a=2,\ b=-4,\ c=1.$ Since $a=2>0,$ the graph opens up.

The *x*-coordinate of the vertex is $x=-\frac{b}{2a}=-\frac{-4}{2(2)}=1$.

The y-coordinate of the vertex is
$$f\left(-\frac{b}{2a}\right)=f(1)=2(1)^2-4(1)+1=-1.$$

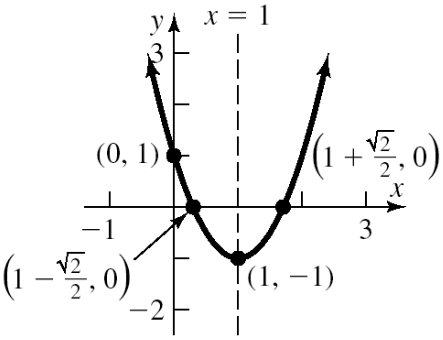
Thus, the vertex is (1, –1).
The axis of symmetry is the line $x=1$.

The discriminant is:
$b^2-4ac=(-4)^2-4(2)(1)=8>0$, so the graph has two x-intercepts.

The x-intercepts are found by solving:
$$2x^2-4x+1=0$$
$$x=\frac{-(-4)\pm\sqrt{8}}{2(2)}$$
$$=\frac{4\pm2\sqrt{2}}{4}=\frac{2\pm\sqrt{2}}{2}$$

The x-intercepts are $\frac{2-\sqrt{2}}{2}$ and $\frac{2+\sqrt{2}}{2}$.

The y-intercept is $f(0)=1$.



18. $f(x)=x^2+3x+1$

average rate of change of f from 1 to 2:
$$\frac{f(2)-f(1)}{2-1}=\frac{11-5}{1}=6=m_{\text{sec}}$$

$f(2)=11$ so the point $(2,11)$ is on the graph.
Using this point and the slope $m=6$, we can obtain the equation of the secant line:
$$y-y_1=m(x-x_1)$$
$$y-11=6(x-2)$$
$$y-11=6x-12$$
$$y=6x-1$$

19. **a.** x-intercepts: $(-5,0);(-1,0);(5,0)$;
y-intercept: $(0,-3)$

b. The graph is not symmetric with respect to the origin, x-axis or y-axis.

c. The function is neither even nor odd.

d. f is increasing on $(-\infty,-3)$ and $(2,\infty)$; f is decreasing on $(-3,2)$;

e. f has a local maximum at $x=-3$, and the local maximum is $f(-3)=5$.

f. f has a local minimum at $x=2$, and the local minimum is $f(2)=-6$.

20. $f(x)=\dfrac{5x}{x^2-9}$

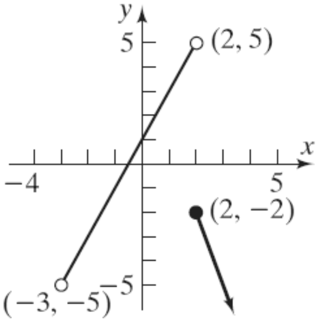
$f(-x)=\dfrac{5(-x)}{(-x)^2-9}=\dfrac{-5x}{x^2-9}=-f(x)$, therefore f is an odd function.

21. $f(x)=\begin{cases}2x+1 & \text{if } -3<x<2\\ -3x+4 & \text{if } x\ge 2\end{cases}$

a. Domain: $\{x \mid x>-3\}$ or $(-3,\infty)$

b. x-intercept: $\left(-\frac{1}{2},0\right)$
y-intercept: $(0,1)$

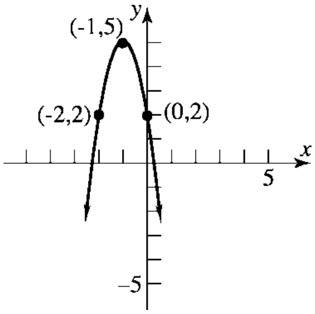
c.



d. Range: $\{y \mid y<5\}$ or $(-\infty,5)$

22. $f(x)=-3(x+1)^2+5$

Using the graph of $y = x^2$, shift left 1 unit, vertically stretch by a factor of 3, reflect about the x-axis, then shift up 5 units.



23. $f(x) = x^2 - 5x + 1 \qquad g(x) = -4x - 7$

a. $(f+g)(x) = x^2 - 5x + 1 + (-4x - 7)$
$= x^2 - 9x - 6$
The domain is: $\{x \mid x \text{ is a real number}\}$.

b. $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)} = \dfrac{x^2 - 5x + 1}{-4x - 7}$

The domain is: $\left\{x \,\middle|\, x \neq -\dfrac{7}{4}\right\}$.

24. a. $R(x) = x \cdot p$
$= x\left(-\dfrac{1}{10}x + 150\right)$
$= -\dfrac{1}{10}x^2 + 150x$

b. $R(100) = -\dfrac{1}{10}(100)^2 + 150(100)$
$= -1000 + 15,000$
$= \$14,000$

c. Since $R(x) = -\dfrac{1}{10}x^2 + 150x$ is a quadratic function with $a = -\dfrac{1}{10} < 0$, the vertex will be a maximum point. The vertex occurs when $x = -\dfrac{b}{2a} = -\dfrac{150}{2(-1/10)} = 750$.
Thus, the revenue is maximized when $x = 750$ units sold.

The maximum revenue is given by

$R(750) = -\dfrac{1}{10}(750)^2 + 150(750)$.
$= -56,250 + 112,500$
$= \$56,250$

d. $p = -\dfrac{1}{10}(750) + 150 = -75 + 150 = \$75$ is the selling price that maximizes the revenue.

Chapter 4 Projects

Project I – Internet-based Project

Answers will vary

Project II

a. $x^2 + 8x - 9 = 0$
$(x+9)(x-1) = 0$
sum $= -9 + 1 = -8$, product $= (-9)(1) = -9$
$x = -9$ or $x = 1$

b. $x^2 + bx + c = 0$
$(x - r_1)(x - r_2) = 0$
$x^2 - r_1x - r_2x + r_1r_2 = 0$
$x^2 - (r_1 + r_2)x + r_1r_2 = 0$
$b = -(r_1 + r_2)$
$c = r_1r_2$

c. $f(x) = x^3 - x^2 - 10x - 8$
$f(x) = (x+2)(x^2 - 3x - 4)$
$f(x) = (x+2)(x-4)(x+1)$
zeros: $-2 . 4. -1$
sum $= -2 + 4 - 1 = 1$, product $= (-2)(4)(-1) = 8$
sum of double products
$= -2(4) + (-2)(-1) + 4(-1) = -8 + 2 - 4 = -10$

The coefficient of x^2 is the negative sum. The coefficient of x is the sum of the double products. The constant term is the negative product.

d. $f(x) = x^3 + bx^2 + cx + d$

$$f(x) = (x - r_1)(x - r_2)(x - r_3)$$
$$f(x) = (x^2 - (r_1 + r_2)x + r_1r_2)(x - r_3)$$
$$f(x) = x^3 - (r_1 + r_2 + r_3)x^2 + (r_1r_2 + r_1r_3 + r_2r_3)x - r_1r_2r_3$$
$$b = -(r_1 + r_2 + r_3)$$
$$c = r_1r_2 + r_1r_3 + r_2r_3$$
$$d = -r_1r_2r_3$$

e. $f(x) = x^4 + bx^3 + cx^2 + dx + e$

$$f(x) = (x - r_1)(x - r_2)(x - r_3)(x - r_4)$$
$$f(x) = (x^3 - (r_1 + r_2 + r_3)x^2 + (r_1r_2 + r_1r_3 + r_2r_3)x - r_1r_2r_3)(x - r_3)$$
$$f(x) = x^4 - (r_1 + r_2 + r_3 + r_4)x^3 + \left(r_1r_2 + r_1r_3 + r_2r_3 + r_1r_4 + r_2r_4 + r_3r_4\right)x^2 - (r_1r_2r_4 + r_1r_3r_4 + r_2r_3r_4 + r_1r_2r_3)x + r_1r_2r_3r_4$$

$$b = -(r_1 + r_2 + r_3 + r_4)$$
$$c = r_1r_2 + r_1r_3 + r_2r_3 + r_1r_4 + r_2r_4 + r_3r_4$$
$$d = -(r_1r_2r_4 + r_1r_3r_4 + r_2r_3r_4 + r_1r_2r_3)$$
$$e = r_1r_2r_3r_4$$

f. The coefficients are sums, products, or sums of products of the zeros.

If $f(x) = x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1x + a_0$, then:

a_{n-1} will be the negative of the sum of the zeros.

a_{n-2} will be the sum of the double products.

a_1 will be the negative (if n is even) or positive (if n is odd) of the sum of (n-1) products.

a_0 will be the negative (if n is odd) or positive (if n is even) product of the zeros.

These will always hold. These would be useful if you needed to multiply a number of binomials in $x - c$ form together and you did not want to have to do the multiplication out. These formulas would help same time.