Date ________

5.3 Notes Exponential Functions

Properties of Exponents:

Product prop. $a^m \cdot a^n = a^{m+n}$

Quotient prop. $\dfrac{a^m}{a^n} = a^{m-n}$

Power of zero prop. $a^0 = 1$

Power of a negative exponent prop. $a^{-m} = \dfrac{1}{a^m}$ $\dfrac{1}{a^{-m}} = a^m$

Power of a power prop. $(a^m)^n = a^{mn}$

Power of a quotient prop. $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$

Power of a product prop. $(ab)^m = a^m b^m$

Rational Exponents $a^{\frac{1}{n}} = \sqrt[n]{a}$

$a^{\frac{m}{n}} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}$

Examples: Simplify without using your TI.

1. $7^{\frac{1}{4}} \cdot 7^{\frac{7}{4}} = 7^{8/4} = 7^2 = \boxed{49}$

2. $\left(5^{\frac{3}{4}}\right)^4 = 5^{12/4} = 5^3 = \boxed{125}$

3. $\left(1{,}376^{-75}\right)^0 = \boxed{1}$

4. $4x^2(4x)^{-2} = \dfrac{4\cancel{x^2}}{16\cancel{x^2}} = \boxed{\dfrac{1}{4}}$

5. $(8)^{-\frac{2}{3}} = \dfrac{1}{8^{2/3}} = \dfrac{1}{(\sqrt[3]{8})^2} = \dfrac{1}{2^2} = \boxed{\dfrac{1}{4}}$

Exponential Functions (the variable is the exponent.)

of the form $f(x) = a^x$ and $a > 0, a \neq 1$

Domain: $(-\infty, \infty)$ Property: Exponential functions are continuous

Note: Exponential functions are either strictly increasing or strictly decreasing $\Rightarrow$ exp. functions are one-to-one. This means an exponential function will have an inverse.

Two types: Exponential growth $(a > 1)$ and Exponential decay $(0 < a < 1)$.

Describe the transformations from $h(x) = 3^x$.

1. $g(x) = 4(3^x) - 5$

- vert stretch by 4
vert down 5

2. $f(x) = -3^{6x}$

horiz compression by 1/6
reflect over x-axis

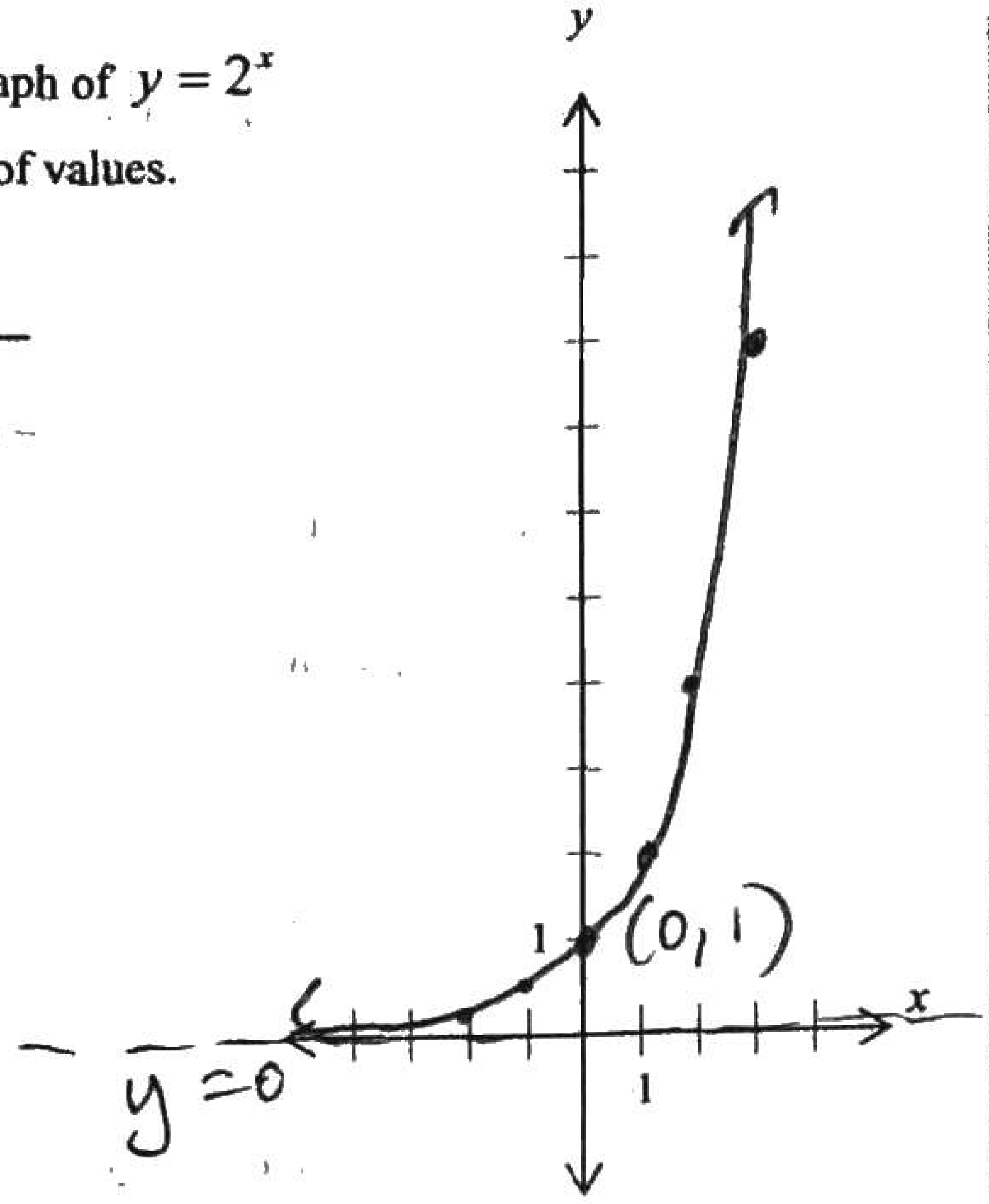
3. $k(x) = 3^{5-x}$
$= 3^{-(x-5)}$

reflect over y-axis
horiz rt 5

Know the basic shapes of exponential graphs. Always label key values: y-intercept and asymptotic behavior.

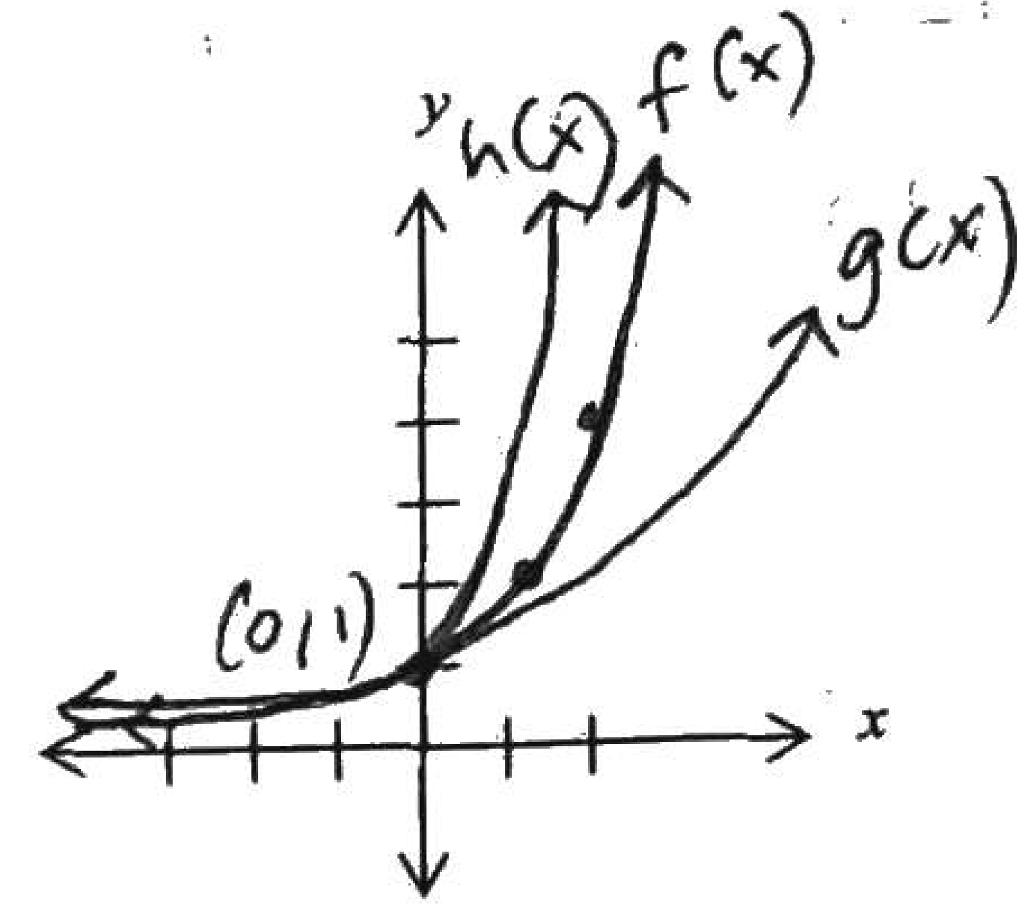
Sketch the graph of $y = 2^x$ using a table of values.

x	y
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4
3	8



Graph w/ your TI: Use three colors for the graphs of f, g, and h

$f(x) = 2^x$, $g(x) = 1.2^x$, $h(x) = 3^x$.



Conclusions: Exp growth, $a > 0$

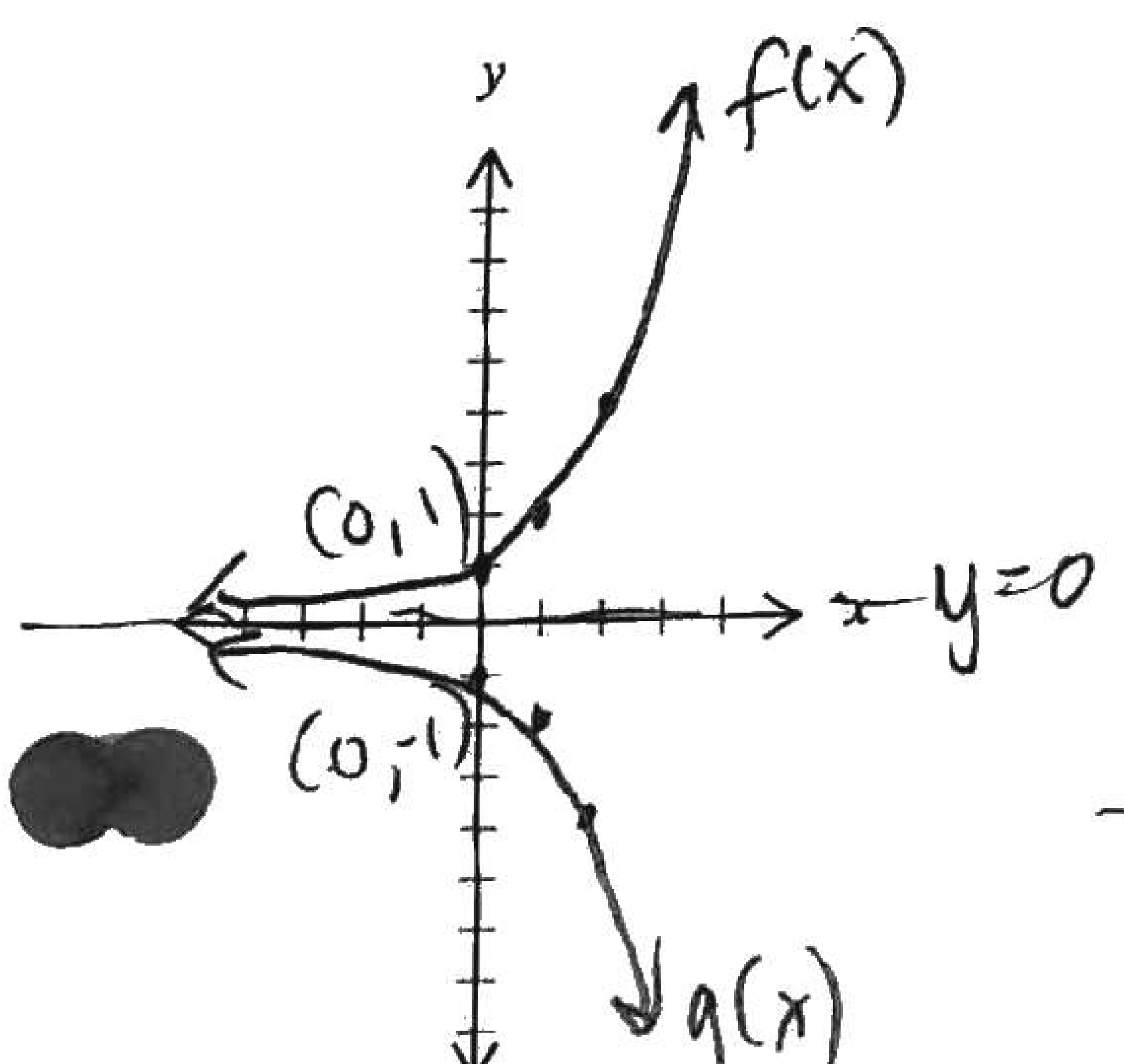
1. Graph is above x-axis
2. y-intercept: (0, 1)
3. Increasing at increasing rate
4. As the base increases $\Rightarrow$ the growth increases.
5. End behavior: $\lim_{x \to -\infty} f(x) =$ 0 $\Rightarrow$

$f(x)$ has an HA with equation $y = 0$ on left.

Describe the transformation from $f(x) = 2^x$. Sketch $f(x) = 2^x$ in pencil and the transformed graph with a color. Label each graph. Label the y-intercept (or its corresponding point) and the equation of the horizontal asymptote.

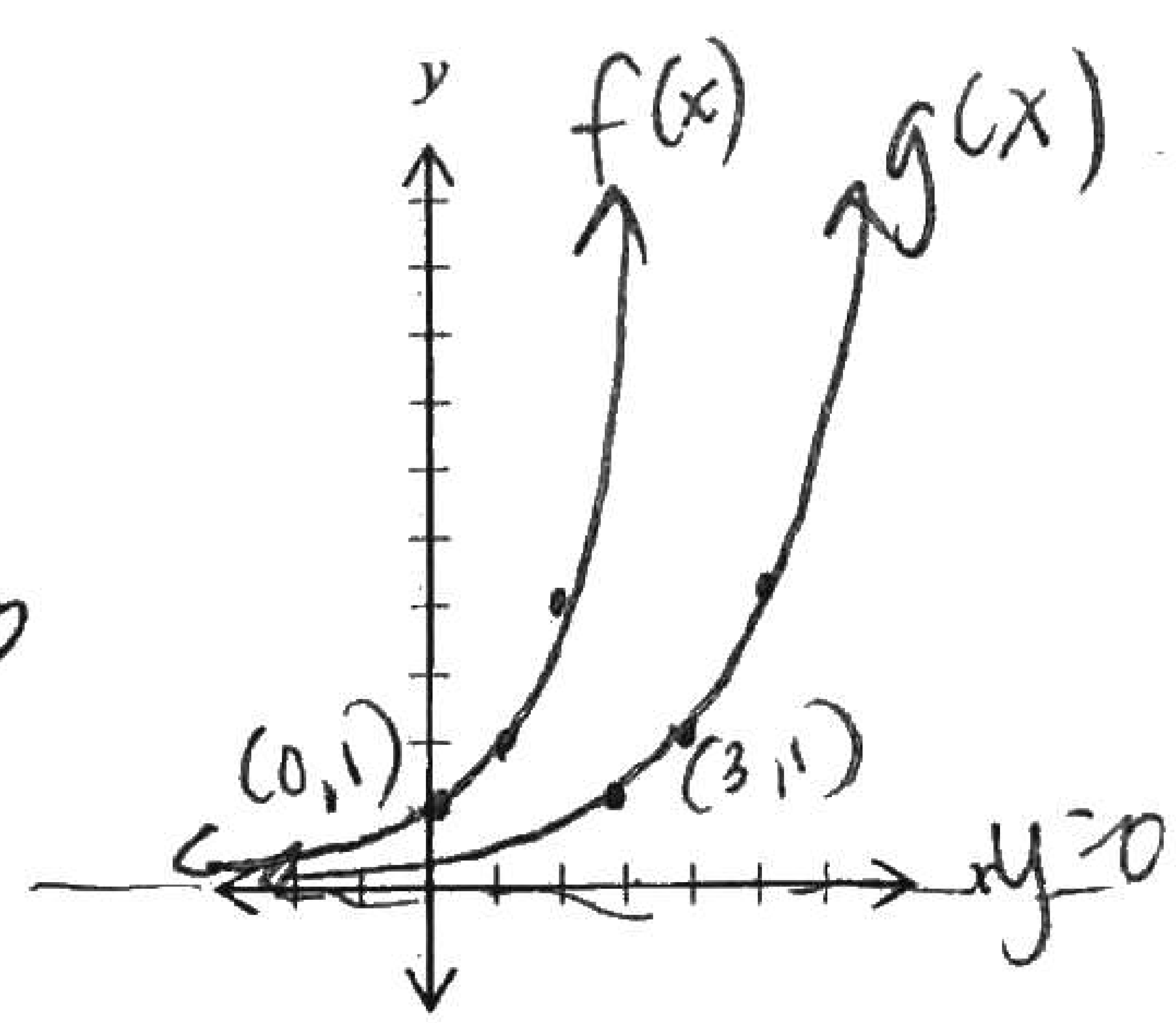
4. $g(x) = -2^x$

reflect over X-axis



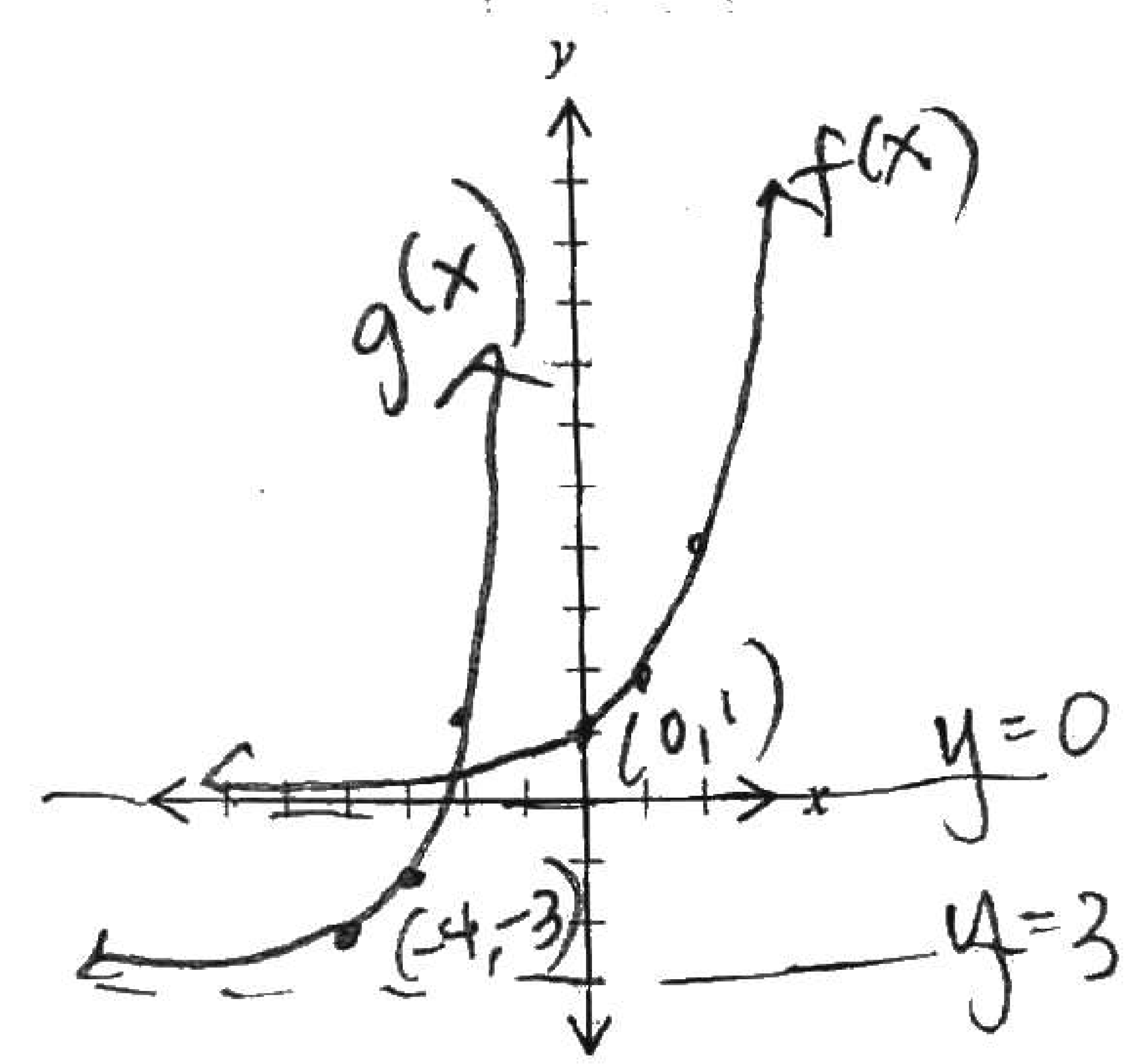
5. $h(x) = 2^{x-3}$

horiz rt 3



6. $k(x) = 2^{x+4} - 3$

horiz left 4
down 3



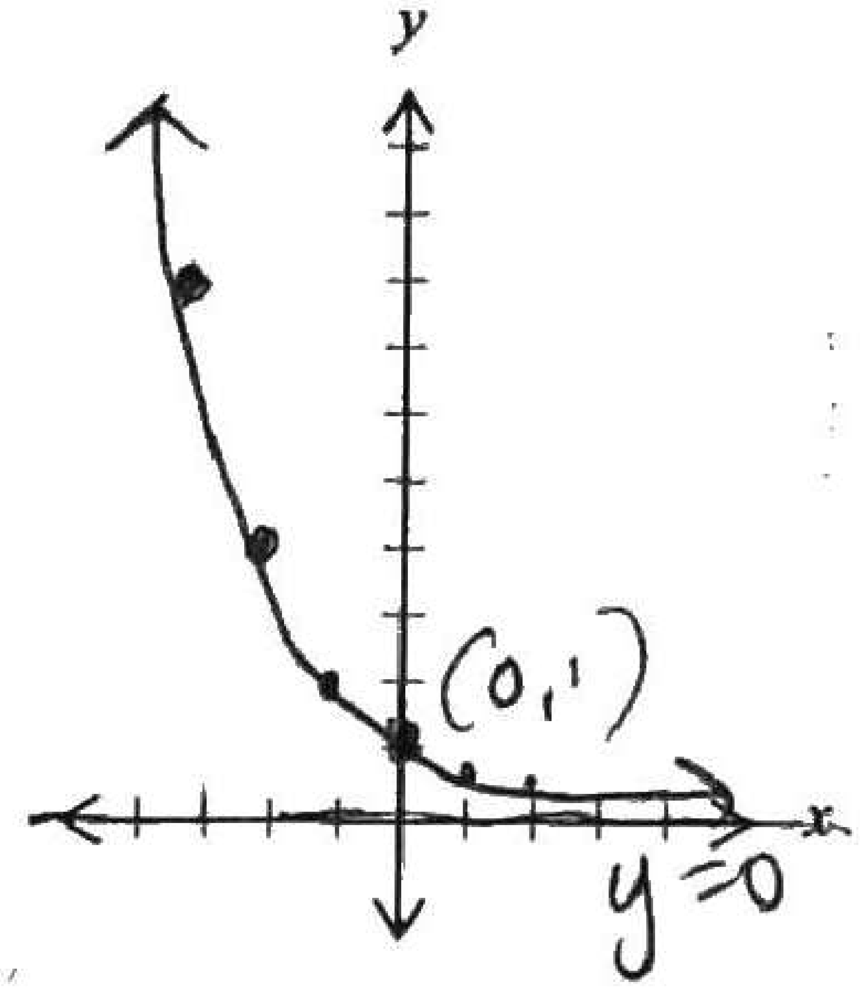
7. $y = 2^{-x}$

$y = 2^{-x}$

$= \frac{1}{2^x}$

$y = \left(\frac{1}{2}\right)^x$

x	y
-3	8
-2	4
-1	2
0	1
1	$\frac{1}{2}$
2	$\frac{1}{4}$



Called Exp decay, $0 < a < 1$

1. Graph is above x-axis.
2. y-intercept: $(0, 1)$
3. Decreasing at increasing rate.
4. As the base decreases $\Rightarrow$ the growth decreases
5. End behavior: $\lim_{x \to +\infty} f(x) = 0 \Rightarrow$

$f(x)$ has an HA with equation $y = 0$ on rt.

$4^{-2} \; \frac{1}{4^2} \left(\frac{1}{4}\right)^2 \; \frac{1}{4^{x-2}}$

Use properties of exponents to prove: ~~determine which functions (if any) are the same~~.

7. $f(x) = 3^{x-2} = 3^x \cdot 3^{-2} = 3^x \cdot \frac{1}{9}$

~~$g(x) = 3^x - 9$~~

$h(x) = \frac{1}{9}\left(3^x\right)$ $\boxed{f(x) = h(x)}$

8. $f(x) = 16\left(4^{-x}\right) = 4^2 4^{-x} = 4^{2-x} = 4^{-(x-2)} = \boxed{\left(\frac{1}{4}\right)^{x-2}}$

$g(x) = \left(\frac{1}{4}\right)^{x-2}$

$h(x) = 16\left(2^{-2x}\right) = 2^4 \cdot 2^{-2x} = 2^{4-2x} \; 2^{-2(x-2)} = \boxed{\left(\frac{1}{4}\right)^{x-2}}$

$\boxed{f(x) = g(x) = h(x)}$

Review from Algebra II: (also covered in section 5.7)

Compound Interest Formula:

If P dollars is invested at rate r (expressed as a decimal) for time period t, compounded n times per year, then the account has a value of

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

Example: Penny Wise is going to invest \$1250 @ 8.5% annual interest, compounded quarterly. Find the balance after 3 years.

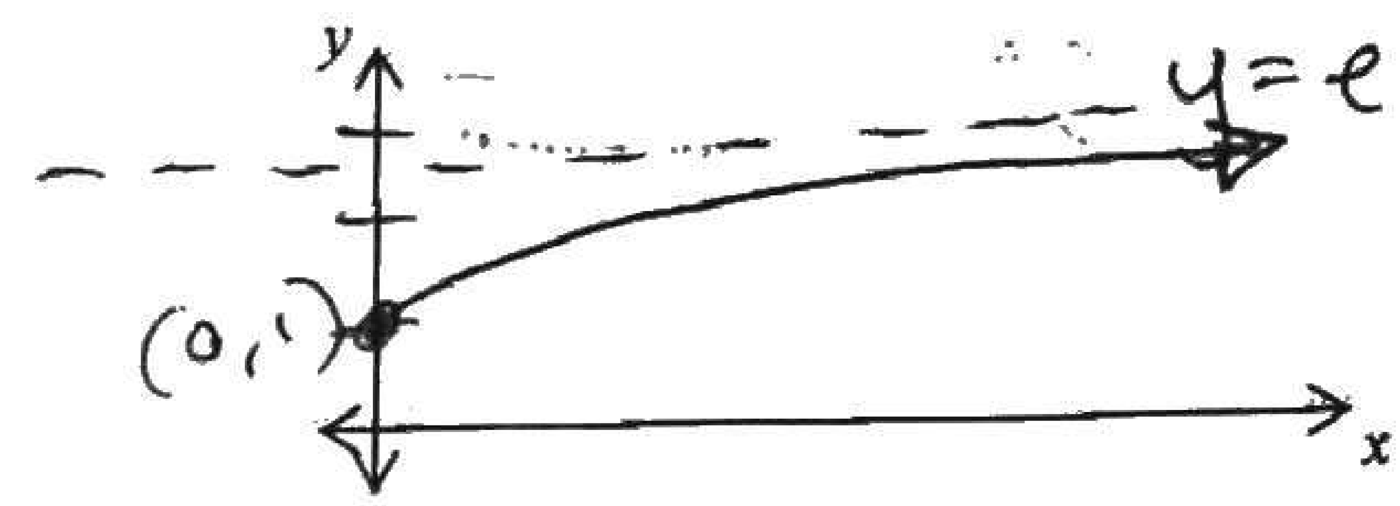
$A = 1250\left(1 + \frac{.085}{4}\right)^{4 \cdot 3} = \boxed{\$1,608.77}$

It is also possible to compound interest *continuously*. This is based on a special irrational number, *e*. Complete the chart for \$1 investment that earns 100% annual interest (r = 1) over 1 year.

Compounding Schedule	n	$1\left(1 + \frac{1}{n}\right)^n$	Value, A
Annually	1	$1\left(1 + \frac{1}{1}\right)^1 =$	2
Semi-annually	2	$1\left(1 + \frac{1}{2}\right)^2 =$	2.25
Quarterly	4	$1\left(1 + \frac{1}{4}\right)^4 =$	2.613
Monthly	12	$1\left(1 + \frac{1}{12}\right)^{12} =$	2.714
Daily	365	$1\left(1 + \frac{1}{365}\right)^{365}$	2.718
hourly	$365 \cdot 24$	⋮	2.718
minutely	$365 \cdot 24 \cdot 60$	⋮	2.718

Conclusion:

$\lim\limits_{n\to\infty} 1\left(1+\frac{1}{n}\right)^n =$ 2.718, which is the special irrational #, e. Check *graphically* $y=\left(1+\frac{1}{x}\right)^x$



Continuous Compound Interest Formula:

If P dollars is invested at rate r (expressed as a decimal) for time period t, compounded continuously, then the account has a value of

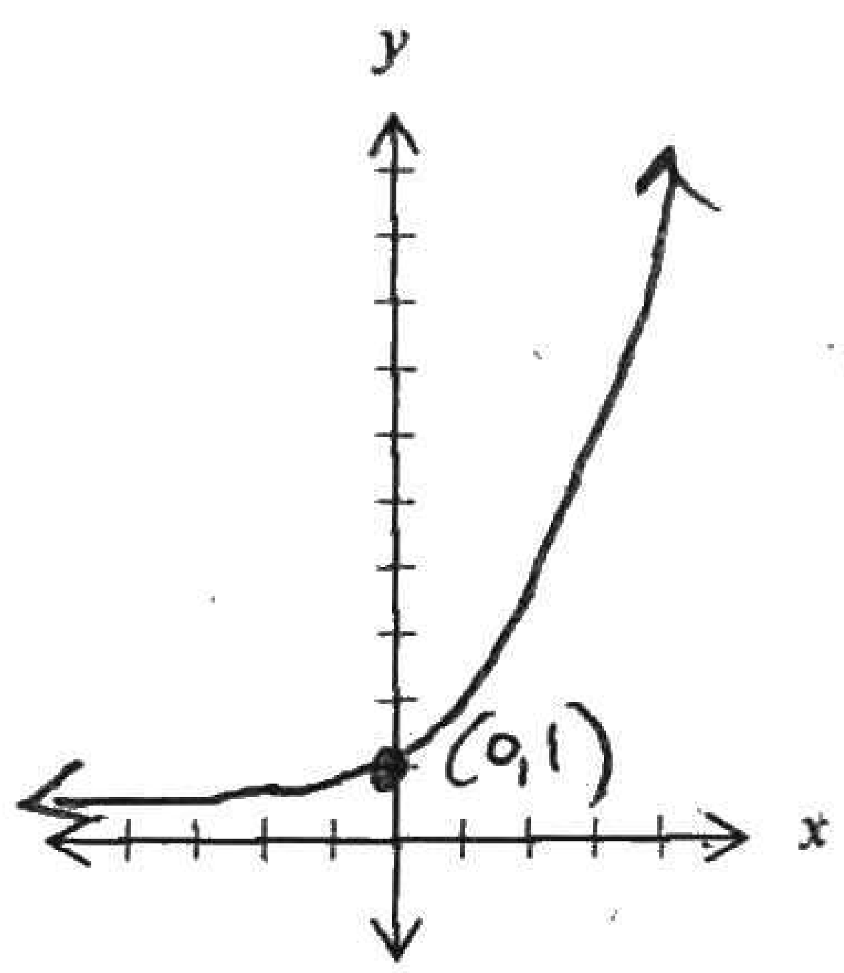
$$A = Pe^{rt}$$

Ex: Homer plans to invest \$2575 @ 6.25% annual interest, compounded *continuously*. Find the balance after $\boxed{\text{5 years 3 months.}}$ $5\frac{1}{4}$ yrs

$A = 2575e^{(.0625)(5.25)}$

$\boxed{A \approx \$3575.03}$

Sketch the graph of $y = e^x$



Use your TI to approximate to 3 decimal places:

$e^{2.1} \approx \boxed{8.166}$ $\qquad e^{-5} \approx \frac{1}{e^5} \approx \boxed{.0067}$

Exponential functions are one-to-one $\Rightarrow a^x = a^y$ if and only if $x = y$.

This property allows us to solve exponential equations.

Solve without using your TI.

1. $2a^{4/5} = 32$

$(a^{4/5})^{5/4} = (16)^{5/4}$

$a = (\sqrt[4]{16})^5$

$a = 2^5 = \boxed{32}$

2. $7^x = \frac{1}{49}$

$7^{x} = 7^{-2}$

$\boxed{x = -2}$

3. $2^x \cdot 4^{x+3} = 8^{5-x}$

$2^x \cdot 2^{2(x+3)} = 2^{3(5-x)}$

$2^x \cdot 2^{2x+6} = 2^{15-3x}$

$x + 2x + 6 = 15 - 3x$

$3x + 6 = 15 - 3x$

$6x = 9$

$\boxed{x = 3/2}$

Use your TI to approximate the 3 decimal places.

$(a^{5/2})^{2/5} = (7)^{2/5}$

$a = (\sqrt[5]{7})^2$

$\boxed{a \approx 2.178}$

5. $\sqrt[3]{x^5} = 75$

$(x^{5/3})^{3/5} = (75)^{3/5}$

$\boxed{x \approx 13.336}$

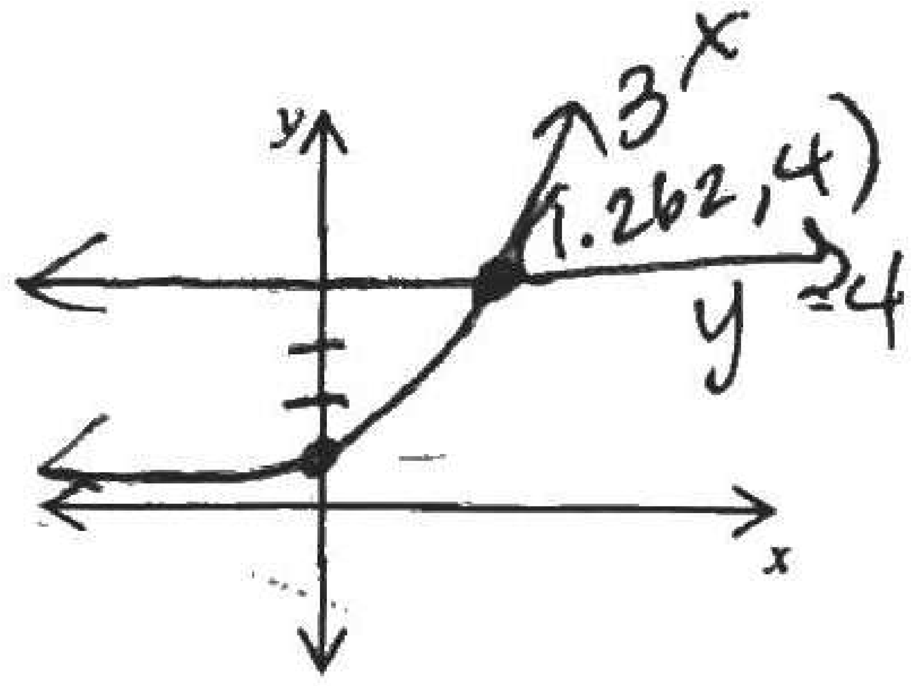
Solve graphically.

6. $3^x = 4$

Let

$y_1 = 3^x$

$y_2 = 4$



$\boxed{x \approx 1.262}$

7. Use variable substitution and your TI to solve:

$e^{2x} - 2e^x - 3 = 0$

$u = e^x$

$u^2 - 2u - 3 = 0$

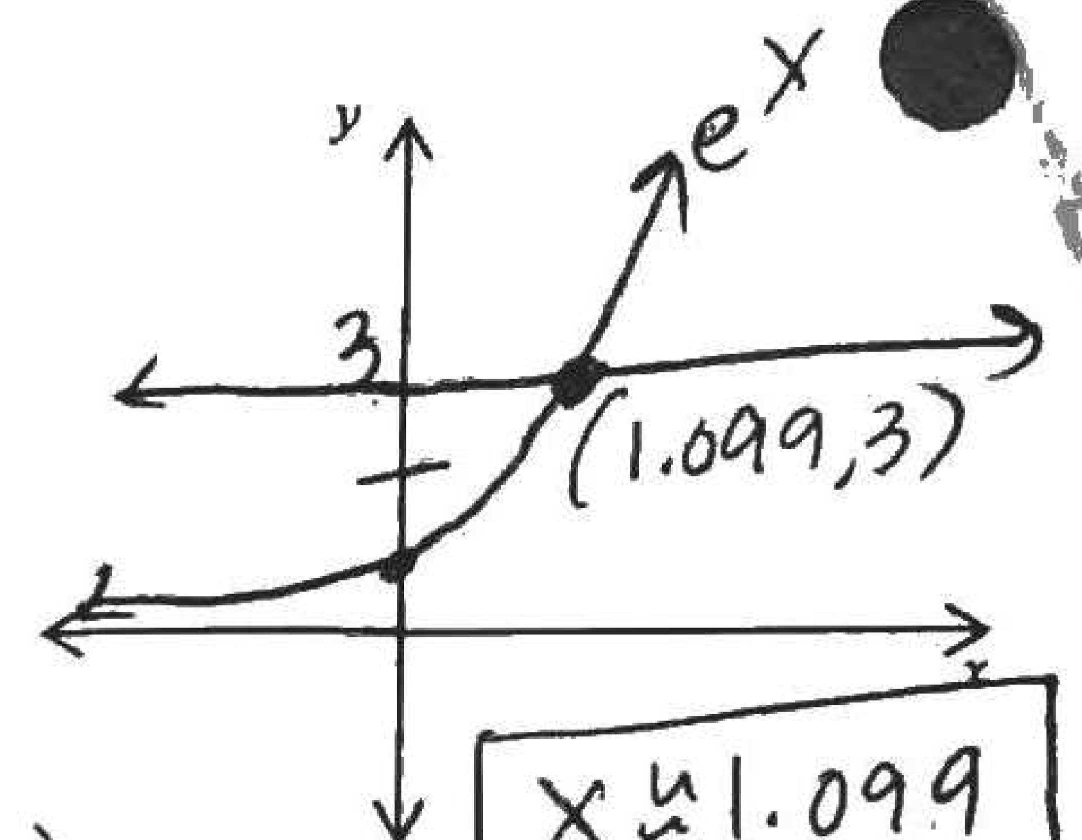
$(u-3)(u+1) = 0$

$u = 3 \quad u = -1$

$\boxed{e^x = 3} \quad \cancel{e^x = -1}$ (not below x-axis)

$y_1 = e^x$

$y_2 = 3$

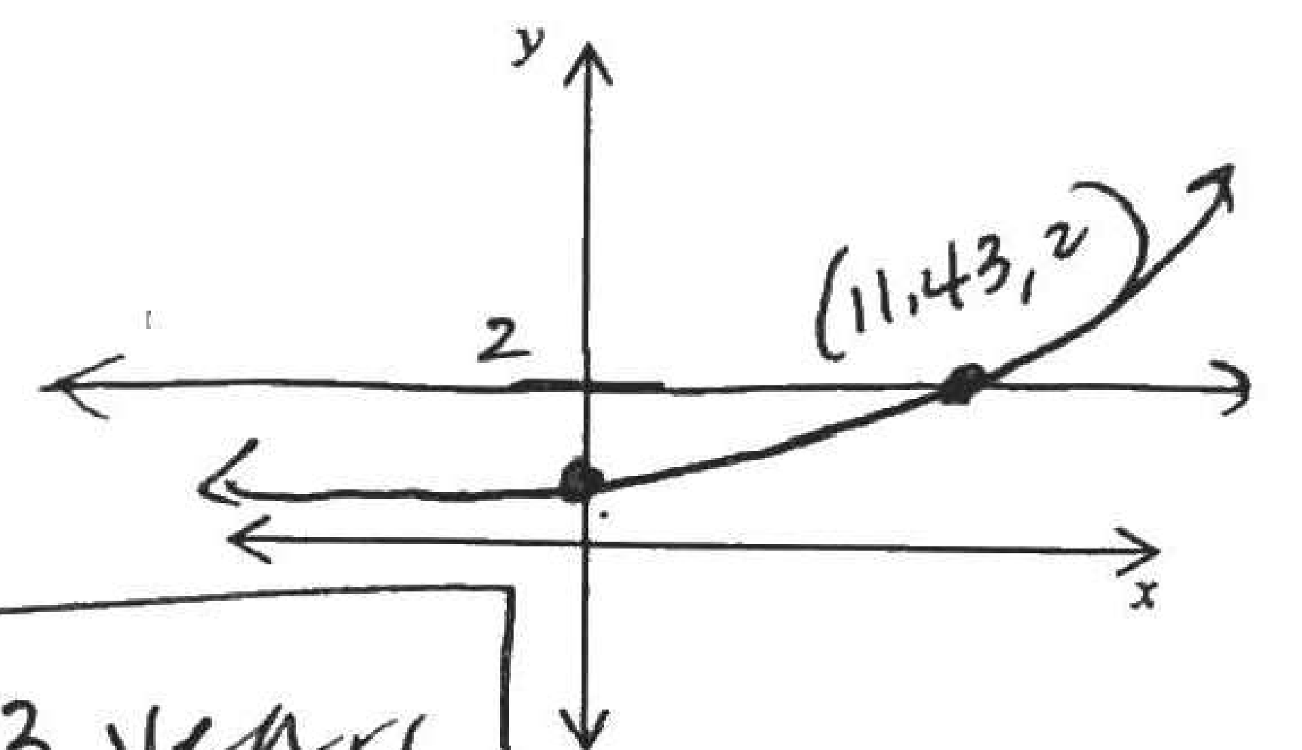


$\boxed{x \approx 1.099}$

8. How long will it take for an investment to double if it is invested at 6.25% and compounded continuously? Solve graphically.

$y_1 = e^{.0625x}$

$y_2 = 2$



$A = Pe^{(.0625)t}$

$200 = 100e^{(.0625)t}$

$2 = e^{(.0625t)}$

$\boxed{x \approx 11.43 \text{ years}}$