Date: ____________ 5.2 Notes One-to-one functions/Inverses

Review from Algebra II:

Find the inverse, $f^{-1}(x)$, algebraically

$f(x) = 2x + 2$.

$y = 2x + 2$

$x = 2y + 2$

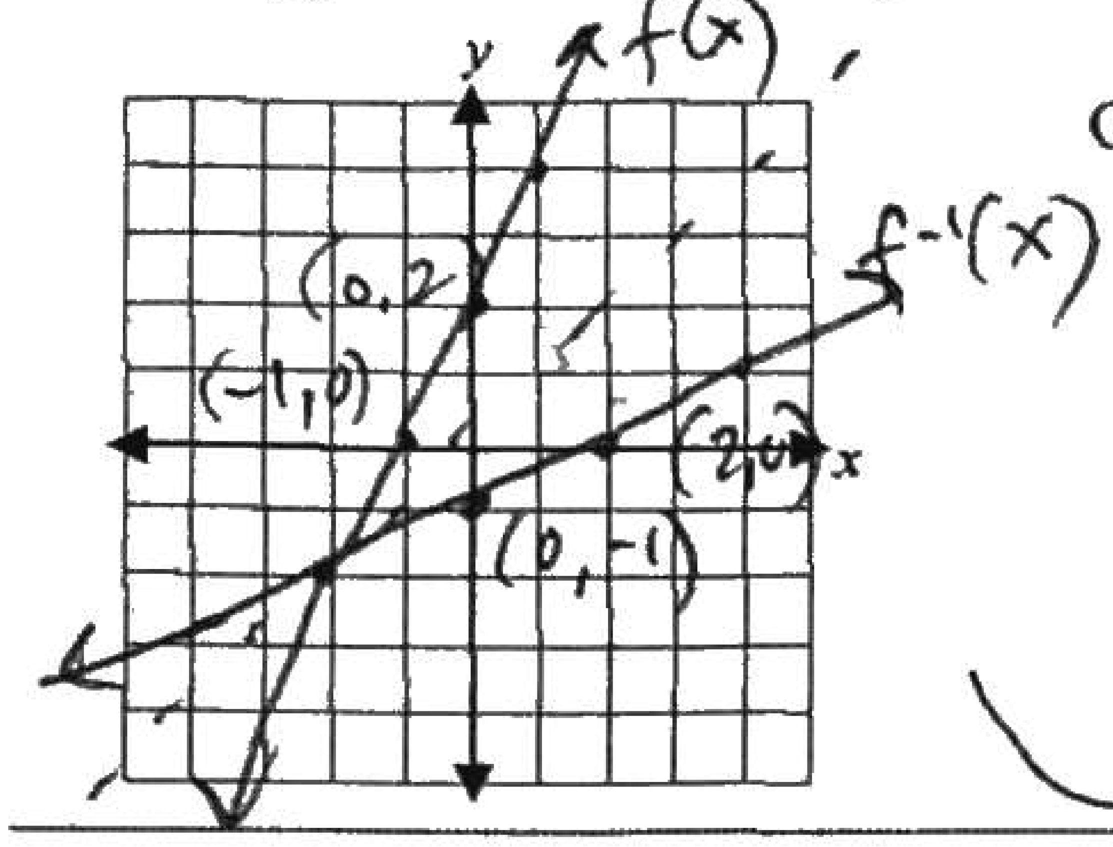
$\frac{x-2}{2} = \frac{2y}{2}$

$y = \frac{1}{2}x - 1$

$\boxed{f^{-1}(x) = \frac{1}{2}x - 1}$

Steps:

1. change $f(x)$ to y
2. switch x : y
3. solve for y
4. If a funct., use $f^{-1}(x)$



Graph $f(x)$ and $f^{-1}(x)$. Label a few key points. Draw the line of symmetry (use color).

Functions and their inverses are symmetric over the line $y = x$.

This means if $f(x)$ contains (a, b) then $f^{-1}(x)$ will contain (b, a).

Domain of the function = the range of its inverse.

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There is an important **Property of Inverses:**

$f(g(x)) = g(f(x)) = x$

$\Leftrightarrow f$: g are inverses

Demonstrate this property for $f(x)$ and $f^{-1}(x)$.

$f(\frac{1}{2}x - 1)$

$= 2(\frac{1}{2}x - 1) + 2$

$= x \cancel{-2+2}$

$= \boxed{x}$

$f^{-1}(2x+2)$

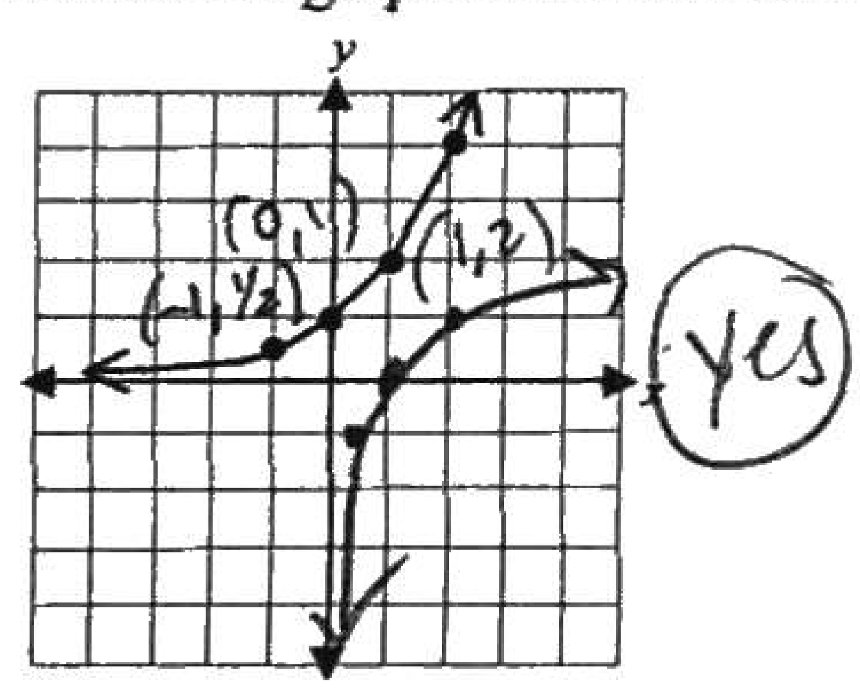
$= \frac{1}{2}(2x+2) - 1$

$= x \cancel{+1-1}$

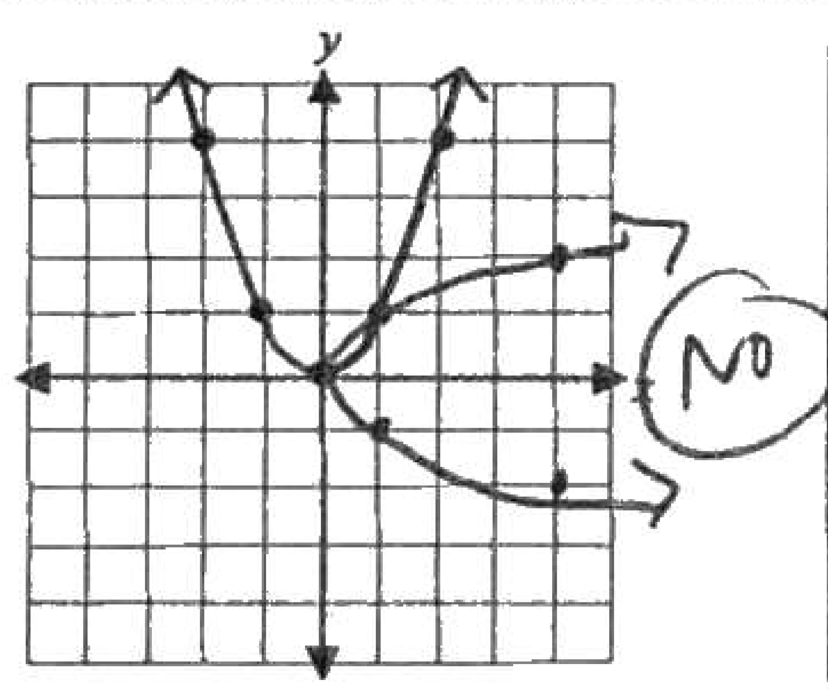
$= \boxed{x}$

More on graphs of inverses:

Sketch the graph of the inverse and determine if the inverse is a function.



Yes

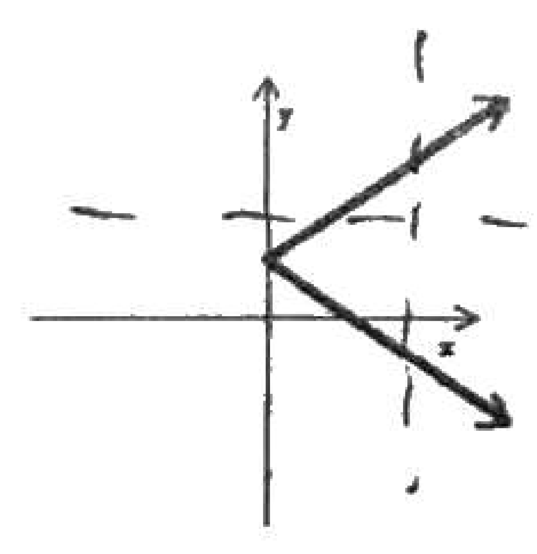


No

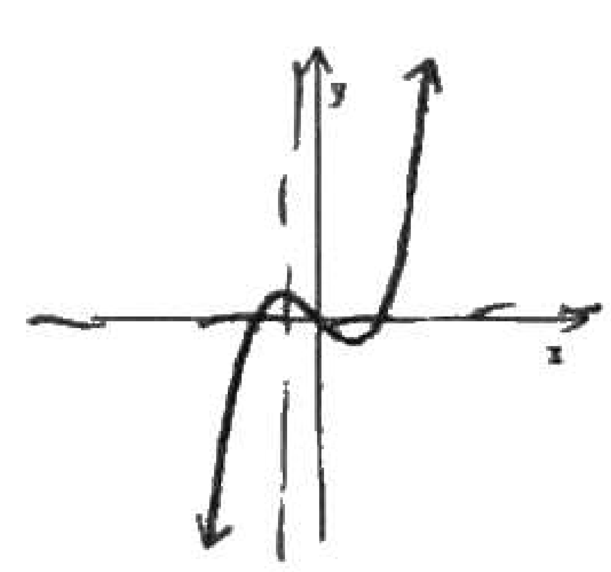
The vertical line test determines whether a relation is a function. What kind of line will determine if the ***inverse of a relation*** is a function?

horizontal line

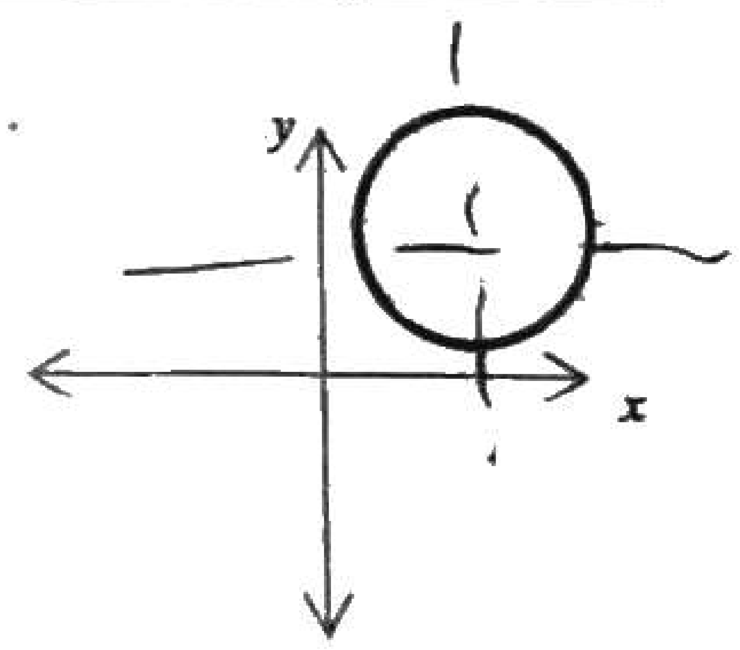
Use the VLT to determine if the relation is a function and the HLT to determine if the inverse of the relation is a function.



Func: No Inv: Yes



Func: Yes Inv. No



Func: No Inv: No

Definition: A function is one to one $\Leftrightarrow$ its inverse is also a function. This means that each input has exactly one output and each output came from exactly one input. Graphically, a one-to-one function must pass the horiz. line test.

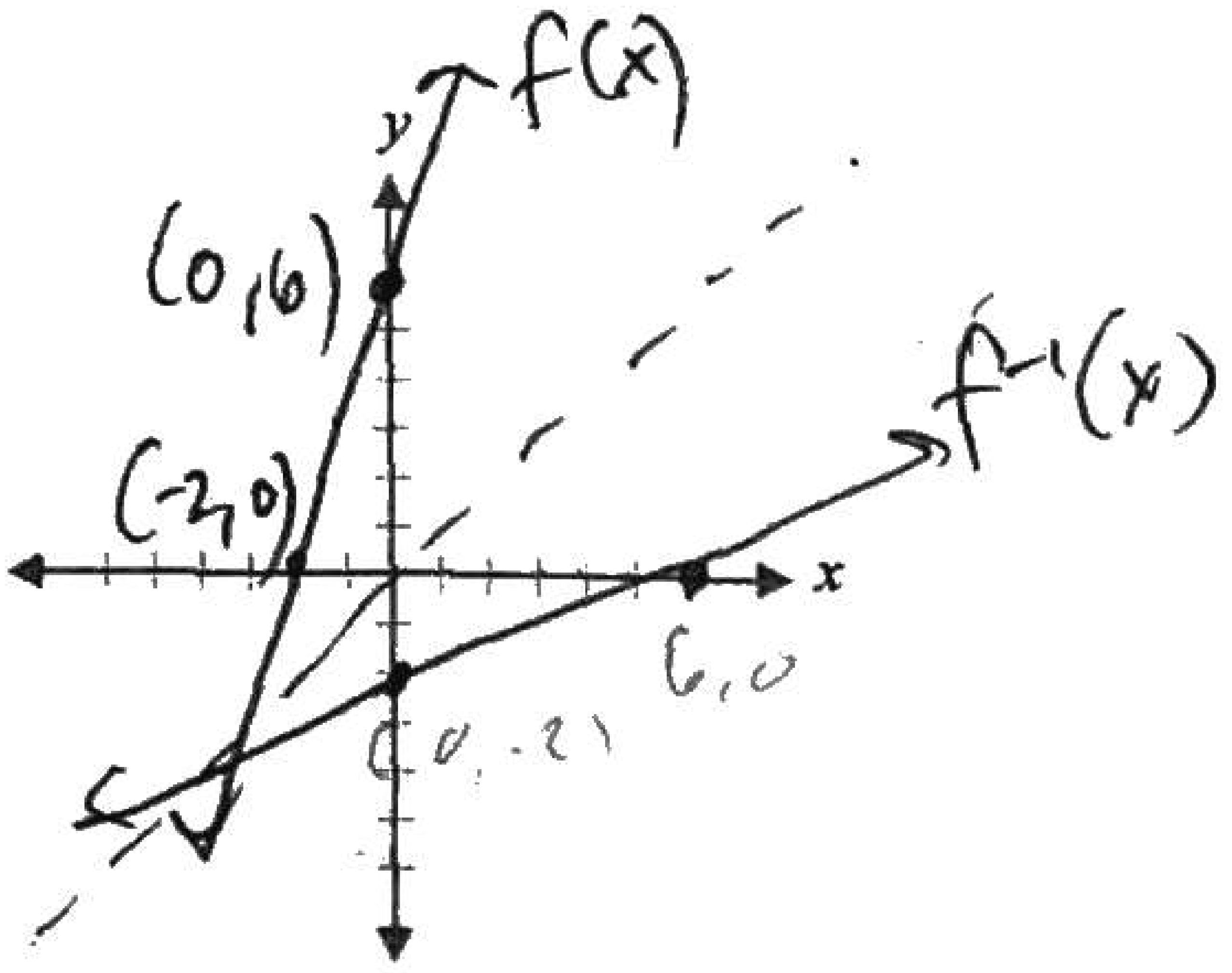
~~Testing for~~ One-to-One Functions:

1. $f(x)=3x+6$

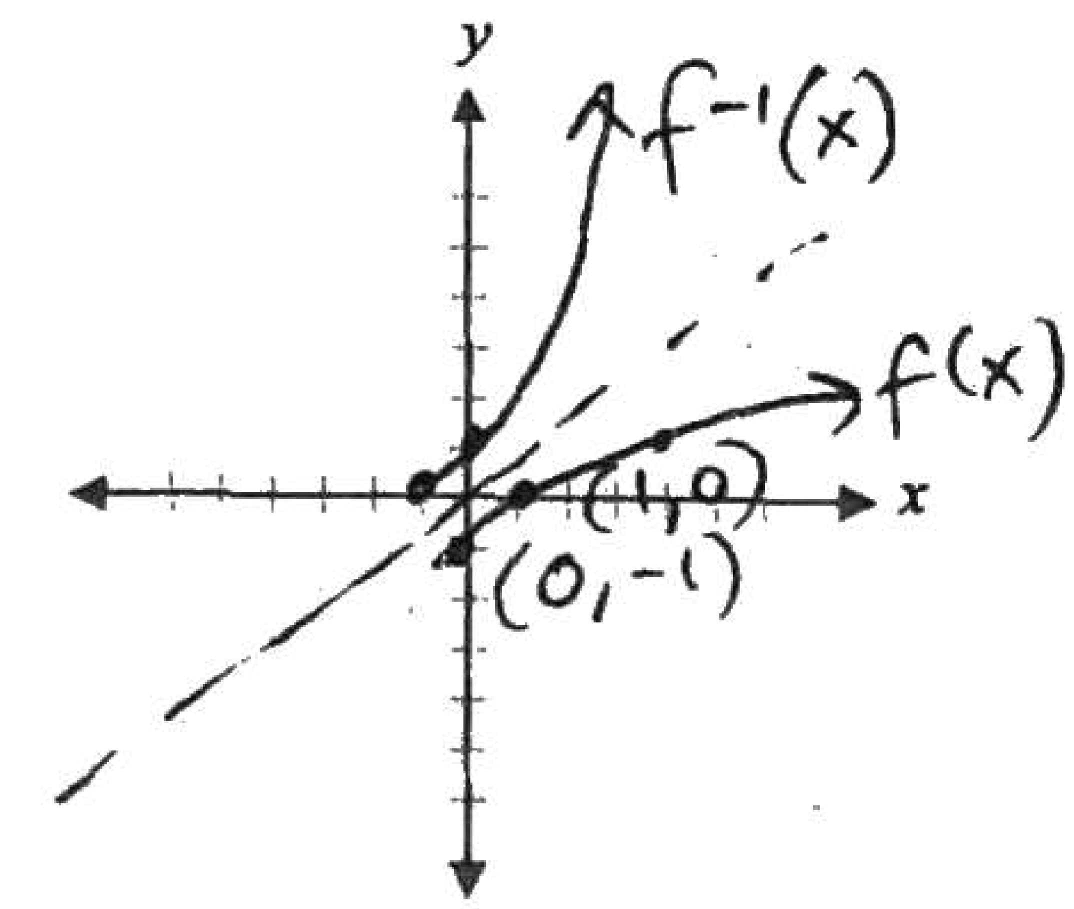
2. $f(x)=\sqrt{x}-1$

3. $f(x)=x^2+1$

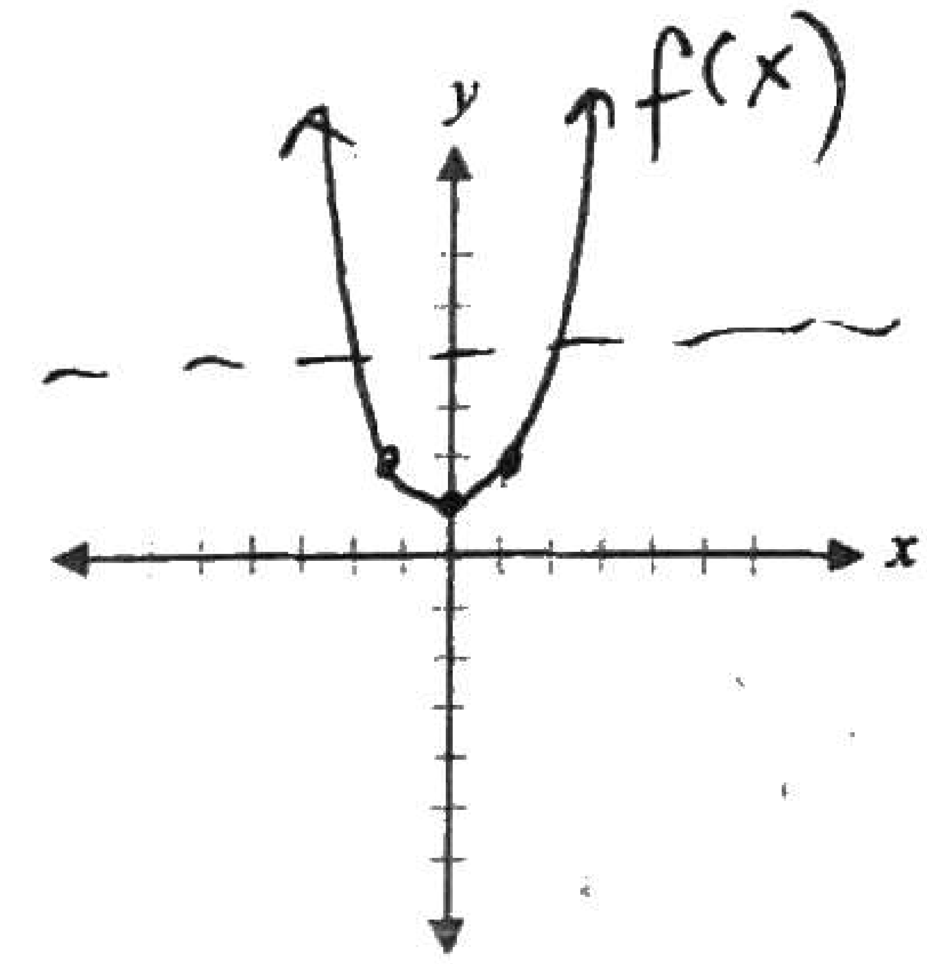
Sketch $f(x)=3x+6$. Label the x- and y-intercepts. Use the intercepts to sketch $f^{-1}(x)$. Use color.



Sketch $f(x)=\sqrt{x}-1$. Label the x- and y-intercepts. Use the intercepts to sketch $f^{-1}(x)$. Use color.



Sketch $f(x)=x^2+1$. Give an example Where $f(x)$ does not pass the HLT.



Use the **Property of Inverses** to show that f and g are inverses: $f(x)=\dfrac{-3}{2x+5}$ and $g(x)=\dfrac{-3-5x}{2x}$.

$(g\circ f)(x)=g(f(x))$

$$g\left(\frac{-3}{2x+5}\right)=\frac{-3-5\left(\frac{-3}{2x+5}\right)}{2\left(\frac{-3}{2x+5}\right)}$$

Bell rang! poop

$$=\frac{-3+\frac{15}{2x+5}}{\frac{-6}{2x+5}}\cdot\frac{2x+5}{2x+5}$$

$$=\frac{-6x-\cancel{15}+\cancel{15}}{-6}=\frac{-\cancel{6}x}{-\cancel{6}}=\boxed{x}$$

$(f\circ g)(x)=f(g(x))$

I finished $(f\circ g)(x)$

$$f\left(\frac{-3-5x}{2x}\right)=\frac{-3}{2\left(\frac{-3-5x}{2x}\right)+5}$$

$$=\frac{-3}{\frac{-3-5x}{x}+5}\cdot\frac{x}{x}=\frac{-3x}{-3-\cancel{5x}+\cancel{5x}}$$

$$=\frac{-\cancel{3}x}{-\cancel{3}}=\boxed{x}$$

In conclusion, if a function does not pass the horizontal line test, we know it is not one-to-one

Visually, a one-to-one function must be strictly increasing or strictly decreasing to pass both VLT and HLT.

Remember a function and its inverse have a special relationship:

1. Symmetric across the line $y=x$. This means if (a,b) is a point on f then (b,a) must lie on f^{-1}.
2. The domain and range switch between a function and its inverse.
3. The property of inverses says the composition of a function and its inverse is always the identity function or $f^{-1}\circ f=f\circ f^{-1}=x$

Examples.

1. Find $f(x)$ and $g(x)$ so that the function can be described as $y = f(g(x))$. $y = \dfrac{1}{(x+2)^3}$
Do NOT use the identity function for f or g.

$\boxed{f(x) = \dfrac{1}{x^3}}$ $\boxed{g(x) = x+2}$

2. Find $f \circ g$ if $f(x) = \dfrac{1}{x+1}$, $g(x) = \dfrac{1-x}{x}$; Give the domain of the composite function.

$D: g(x) \to (-\infty, 0)$ $(0, \infty)$

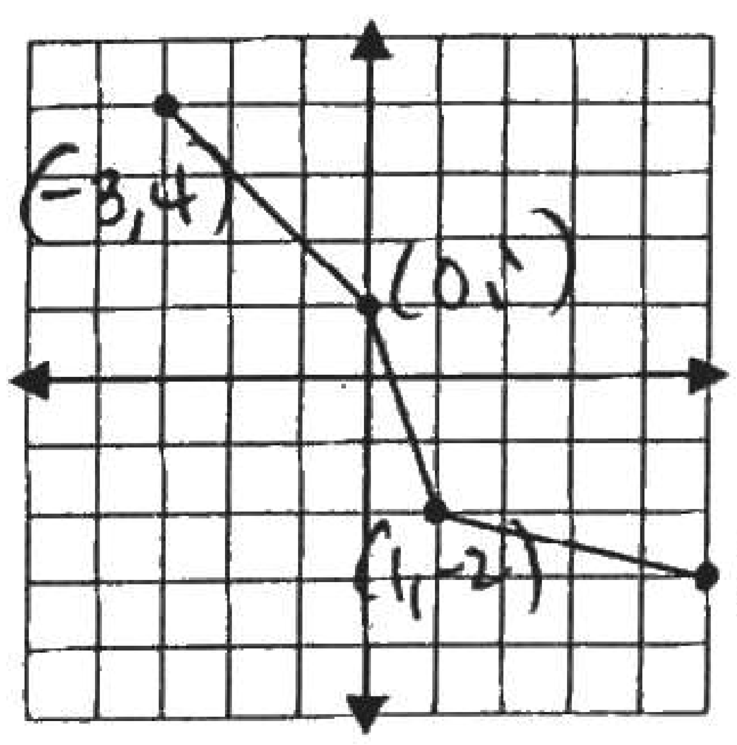
$$f\left(\frac{1-x}{x}\right) = \frac{1}{\frac{1-x}{x}+1} \cdot \frac{x}{x} = \frac{x}{1\cancel{-x}\cancel{+x}} = \boxed{x}$$

$D: (-\infty, \infty)$ → overlap

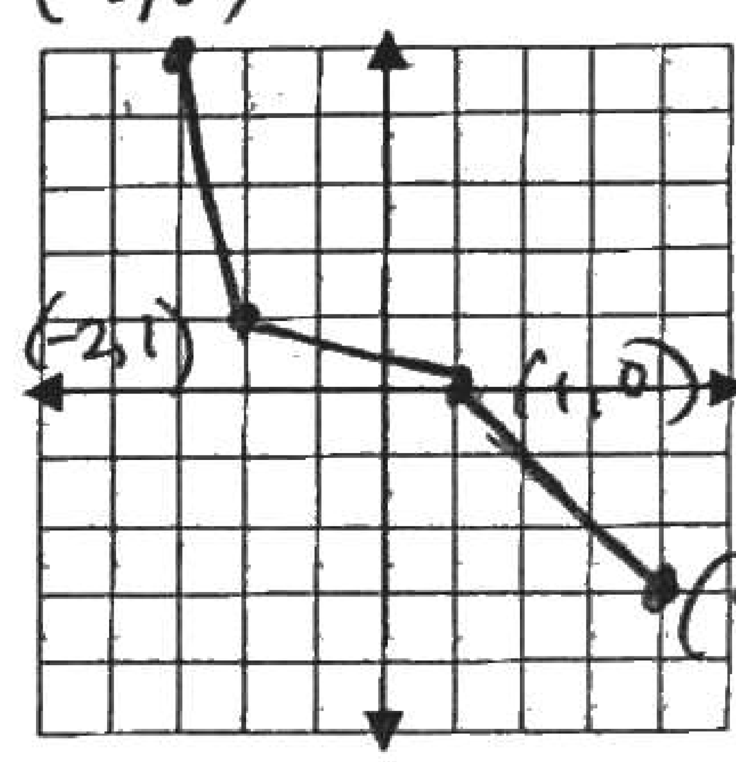
$\boxed{D = (-\infty, 0)(0, \infty)}$

3. Given the graph of $f(x)$, sketch its inverse. *Use color and label key points.*

$f(x)$



$f^{-1}(x)$



4. Find the inverse of $f(x) = \dfrac{x^3 - 1}{3 - x^3}$ *algebraically*. If the inverse is a function, write your final answer using correct function notation.

$$y = \frac{x^3 - 1}{3 - x^3}$$

$$x = \frac{y^3 - 1}{3 - y^3}$$

$$x(3 - y^3) = y^3 - 1$$

$$3x - xy^3 = y^3 + 1$$

$$3x - 1 = y^3 + xy^3$$

$$\frac{3x-1}{1+x} = \frac{y^3(1+\cancel{x})}{\cancel{1+x}}$$

$$\sqrt[3]{\frac{3x-1}{1+x}} = \sqrt[3]{y^3}$$

$$\boxed{f^{-1}(x) = \sqrt[3]{\frac{3x-1}{1+x}}}$$

$1+x$

$3x - xy^3 = y^3 + 1$

$3x - 1 = y^3 + xy^3$

$3x - 1 = y^3(1+x)$

5. If a function is one-to-one then it must pass the HLT.