Notes for Arithmetic Series

When the famous mathematician Karl Friederich Gauss(1777-1855) was 9 years old, his teacher asked the class to find the sum of the natural numbers from 1 to 100. Historical accounts indicate the teacher was hoping to take a break from his students and expected the students to add the terms one by one. (Remember there were no calculators at this time!)

Gauss had this ingenious solution:

$$
\begin{array}{r}
1 + 2 + 3 + 4 + \ldots 98 + 99 + 100 \\
+\ 100 + 99 + 98 + 97 + \ldots\ 3 + \ 2 + \ 1 \\
\hline
101 + 101 + 101 + 101 + \ldots 101 + 101 + 101
\end{array}
$$

$$= \frac{100(101)}{2} = 5{,}050$$

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So, using Gauss's ingenious idea, let's find the following:

2 of series

1) 2 + 5 + 8 + 11 + 14

$14+11+8+5+2$

$16+16+16+16+16$

$\frac{16(5)}{2} = \boxed{40}$

2) The first 100 terms of $\boxed{1+3+5+7+...}$ one series

arith. $d = 2$ $a_n = a_1 + d(n-1)$

we need $a_{100} = 1 + 2(100-1)$

$= 1 + 2(99)$

$= 1 + 198$

$= 199$

2 series

$1+3+5+7+ \ldots 199$

$199+197+ \ldots \ldots + 1$

$200+200+ \ldots \ldots +200$

$\frac{200(100)}{2} = \boxed{10000}$

Could we generalize this and write a formula?

$S_n = \frac{(1\text{st in series}) + (\text{last one in series})}{2}$ # terms

arithmetic series

$$\boxed{S_n = \frac{(a_1 + a_n)n}{2}}$$

Notes for Arithmetic Series

Examples

Find the sum of each arithmetic series.

$d=5$

1. $\underbrace{3+8}_{5}+\underbrace{8+13}_{5}$ ... 3 + 8 + 13 + 18 +...+ 128

We need to find n.

$a_n = a_1 + d(n-1)$

$128 = 3 + 5(n-1)$

$128 = 3 + 5n - 5$

$128 = 5n - 2$

$\frac{130}{5} = \frac{5n}{5}$

$\boxed{n=26}$

$$S_n = \frac{(a_1 + a_n)\cdot n}{2}$$

$$S_{26} = \frac{(3+128)26}{2} = \boxed{1{,}703}$$

2. $\sum_{k=1}^{75}(2k-1)$

$a_1 = 2(1) - 1 = 1$

$a_2 = 2(2) - 1 = 3$

$a_3 = 2(3) - 1 = 5$

$a_{75} = 2(75) - 1 = 149$

$$S_n = \frac{(a_1 + a_n)n}{2}$$

$$S_{75} = \frac{(1+149)\cdot 75}{2} = \boxed{5{,}625}$$

3. The first 100 terms of 2+4+6+8+...

$d = 2$ arithmetic

$$S_n = \frac{(a_1 + a_n)\cdot n}{2}$$

need to find $a_n = a_{100}$

$a_n = a_1 + d(n-1)$

$a_{100} = 2 + 2(100-1)$

$= 2 + 198$

$= 200$

$$S_{100} = \frac{(2 + a_{100})100}{2}$$

$$S_{100} = \frac{(2+200)100}{2}$$

$$= \frac{(202)(100)}{2} = \boxed{10{,}100}$$

Notes for Arithmetic Series

Examples continued

Use the values of a_1 and S_n to find the value of a_n.

4. $a_1 = 6$ and $S_{50}=7{,}650$; a_{50}

$$S_n = \frac{(a_1 + a_n)\cdot n}{2}$$

$$2 \cdot 7650 = \frac{(6 + a_{50})50}{\cancel{2}} \cdot \cancel{2}$$

$$\frac{15{,}300}{50} = \frac{(6 + a_{50})\cancel{50}}{\cancel{50}}$$

$$306 = 6 + a_{50}$$
$$-6 \quad -6$$

$$\boxed{300 = a_{50}}$$

Find a_1 for each arithmetic series

5. $S_{30}=1890$ and $d = 4$.

Find a_{30}

$$a_n = a_1 + d(n-1)$$
$$a_{30} = a_1 + 4(30-1)$$
$$a_{30} = a_1 + 116$$

$$S_n = \frac{(a_1 + a_n)\cdot n}{2}$$

$$S_{30} = \frac{(a_1 + a_{30})30}{2}$$

$$1890 = \frac{(a_1 + a_{30})30}{2}$$

$$1890 = \frac{(a_1 + a_1 + 116)30}{2}$$

$$\frac{1890}{15} = \frac{(2a_1 + 116)\cdot \cancel{15}}{\cancel{15}}$$

$$126 = 2a_1 + 116$$
$$-116 \quad -116$$

$$\frac{10}{2} = \frac{2a_1}{2}$$

$$5 = a_1 \quad \text{so} \quad \boxed{a_1 = 5}$$